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THE SPILL-OVER PHENOMENON IN QUADRATIC MODEL UPDATING

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Abstract. Model updating concerns the modification of an existing but inaccurate model with measured data. For models characterized by quadratic pencils, the measured data usually involve incomplete knowledge of natural frequencies, mode shapes, or other spectral information. In conducting the updating, it is often desirable to match only the part of observed data without tampering with the other part of unmeasured or unknown eigenstructure inherent in the original model. Such an updating, if possible, is said to have no spill-over. This paper studies the spill-over phenomenon in the updating of quadratic pencils. In particular, it is shown that an updating with no spill-over is always possible for undamped quadratic pencils whereas spill-over for damped quadratic pencils is generally unpreventable.

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Key words. quadratic pencil, inverse eigenvalue problem, model updating, eigenstructure assignment, parametric representation, spill-over

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Nomenclature.

$A,\ B$ = parameter matrices; Eq.(3.12)
C = damping matrix in a pencil
C_0 = initial damping matrix in a pencil
D = diagonal matrix; Eq.(2.21)
ΔC = correction of damping matrix C
ΔK = correction of stiffness matrix K
ΔM = correction of mass matrix M
$\Delta Q(\lambda)$ = incremental pencil; Eq.(4.3)
$f(t)$ = external force
Γ = (diagonal) parameter matrix
$H,\ \hat{H}$ = intermediate matrices
k_{max} = maximal allowable number of prescribed eigenpairs; Eq.(3.3)
K = stiffness matrix in a pencil
K_0 = initial stiffness matrix in a pencil
λ = eigenvalue
$\{\lambda_i, \mathbf{u}_i\}_{i=1}^k$ = initial eigenpairs to be updated
$\lambda_j^{[2]}$ = 2×2 real-valued matrix with complex eigenvalues $\alpha_j \pm \beta_j \imath$; Eq.(3.13)
Λ = (diagonal) eigenvalue matrix
$\widehat{\Lambda}$ = expanded matrix of Λ; Eq.(3.20)
Λ_2 = (diagonal) eigenvalue matrix of inert eigenvalues
μ = $-\lambda^2$
M = mass matrix in a pencil
M_0 = initial mass matrix in a pencil
Ξ = parameter matrix; Eq.(3.28)
Φ = parameter matrix; Eq.(2.19)
$Q(\lambda)$ = quadratic pencil in λ
$\widehat{Q}$ = orthogonal matrix in the QR decomposition
$\widehat{R}$ = upper triangular matrix in the QR decomposition
$\mathbb{R}^n$ = n dimensional Euclidean space over real numbers
$\mathbb{R}^{n\times n}$ = vector space of all $n \times n$ real-valued matrices
$\{\sigma_i, \mathbf{y}_i\}_{i=1}^k$ = newly measured eigenpairs
Σ = (diagonal) eigenvalue matrix with eigenvalues $\{\sigma_i\}_{i=1}^k$
$S,\ T,\ U$ = parameter matrices; Eq.(3.8)
t = time variable
$\Upsilon_1,\ \Upsilon_2$ = intermediate matrices
$\mathbf{v}$ = eigenvector
V = orthogonal matrix; Eq.(2.21)
$\mathbf{x}$ = state variable
X = eigenvector matrix
$\widehat{X}$ = extended matrix of X; Eq.(3.19)
X_1 = eigenvector matrix with eigenvectors $\{\mathbf{u}_i\}_{i=1}^k$
X_2 = eigenvector matrix of inert eigenvectors
$\dot{\mathbf{x}}$ = derivative of $\mathbf{x}$ with respect to time t
Y = eigenvector matrix with eigenvectors $\{\mathbf{y}_i\}_{i=1}^k$
$\Psi,\ \Omega$ = intermediate matrices

1. Introduction. Modeling is one of the most fundamental tools that we use to simulate the complex world. The goal of modeling is to come up with a representation that is simple enough for mathematical manipulation yet powerful enough for describing, inducing, and reasoning complicated phenomena. Partially due to the inevitable disturbances to the measuring devices of an observation and partial due to the insufficient representation of the true attributes of a physical system, precise mathematical models are rarely available in practice. With gradual confidence built on improved technologies or repeated experiments, we often regard the measured data as more realistic to the true natural phenomena than the predicted value from an existing model. It thus becomes necessary, when compared with realistic data, to *update* a primitive model to attain consistency with empirical results. This procedure of updating or revising an existing model is an essential step toward establishing an effective model. This paper concerns the model updating of quadratic pencils to reflect measured spectral information [16, 23, 24].

Quadratic pencils arise from the study the second order differential system

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = f(t), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and M, C, $K \in \mathbb{R}^{n\times n}$. Such a differential system has a wide scope of important applications, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, signal processing, and finite element discretization of PDEs. In most applications involving (1.1), specifications of the underlying physical system are embedded in the matrix coefficients M, C and K. If a fundamental solution to (1.1) is represented by

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t},$$

then the scalar λ and the vector $\mathbf{v}$ must solve the *quadratic eigenvalue problem* (QEP)

$$(\lambda^2 M + \lambda C + K)\mathbf{v} = 0. \tag{1.2}$$

In this way, the bearing of the dynamical system (1.1) can be largely interpreted via the eigenvalues and eigenvectors of the algebraic system (1.2). Because of this connection and applications to other disciplines, considerable efforts have be devoted to the QEP in the literature. A list of applications, mathematical properties, and a variety of numerical techniques for the QEP can be found in the survey treatise by Tisseur and Meerbergen [25]. For convenience, we shall identify henceforth the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ by the triplet (M, C, K) of matrices.

The eigenvalue problem associated with the model (1.1) can be studied from two different aspects. The process of analyzing and deriving the spectral information and, hence, inferring the dynamical behavior of a system from *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a *direct* problem. The *inverse* problem, in contrast, is to validate, determine, or estimate the parameters of the system according to its observed or expected behavior. In other words, the concern in the direct problem is to express the behavior in terms of the parameters whereas in the inverse problem the concern is to express the parameters in term of the behavior. The inverse problem is just as important as the direct problem in applications. The model updating problem considered in this paper is a special case of the inverse eigenvalue problem.

The inverse eigenvalue problem is a diverse area full of research interests and activities. See the newly revised book by Gladwell [18], the review article [7], and the recently completed monograph by Chu and Golub [8] in which more than 460 references are collected. At present, the theory and algorithms for quadratic inverse eigenvalue problem (QIEP), that is, finding (M, C, K) from given eigeninformation, are far from being complete. Conceivably, the quadratic problem is more challenging than the linear problems with many unanswered questions.

There are various ways to formulate a QIEP, differing mainly in the desirable structure of the matrix coefficients M, C and K and the available eigeninformation. In this paper, we shall limit ourselves to the updating of self-adjoint QIEPs, that is, all matrix coefficients are symmetric. We caution that this is just an important first step toward more sophisticate structures where, for example, in vibration modeling often the mass matrix M is diagonal, both the damping matrix C and the stiffness matrix K are symmetric and banded, M is positive definite and K is positive semi-definite. We shall also limit ourselves to the specific scenario where only partial spectral information is available. ***That is, we will assume that only a few eigenvalue and athe corresponding eigenvectors (measured at full degreee of freedom) are available. There are multiple reasons why such a scenario is justifiable — In vibration industries, including aerospace, automobile, and manufacturing, through vibration tests where the excitation and the response of the structure at selected points are measured experimentally, there are identification techniques to extract a portion of eigenpair information from the measurements. However, quantities related to high frequency terms in a finite-dimensional model generally are susceptible to measurement errors due to the finite bandwidth of measuring devices. It is simply unwise to use experimental values of high natural frequencies to reconstruct a model. In fact, in a large and complicated physical system, it is often impossible to acquire knowledge of the entire spectral information. While there is no reasonable analytical tool available to evaluate the entire spectral information, we can attain only partial information through experiments. Additionally, it is often demanded, especially in structural design, that certain eigenvectors should also satisfy some specific conditions. For these reasons, it might be more sensible to consider a model updating using only a few measured eigenvalues and eigenvectors [3, 15, 16]. ***However, in practice, the eigenvectors are measured only at limited degrees of freedom due to hardware limitations. There are ways to deal with incomplete measured data, such as model reduction and model expansion techniques (see the book by Friswell and Mottershead [16]. Recently some algorithmic approach has also been developed [6]. For the purpose of this paper, we will assume that the eigenvectors have been measured to the full degree of freedom or some measures have been taken so that a comparison with analytical eigenvectors are possible.

An added challenge, known as the *no spill-over phenomenon* in the engineering literature, is that in updating an existing model it is often desirable that the current vibration parameters not related to the newly measured parameters should remain invariant. No spill-over is required either because these parameters are proven to be acceptable in the previous model and engineers do not wish to introduce new vibrations via updating or because engineers simply do not know of any information about these parameters. The quadratic model updating problem with no spill-over therefore can be stated as follows:

(MUP) Given a structured quadratic pencil (M_0, C_0, K_0) and a few of its associated eigenpairs $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^{k}$ with $k << 2n$, assume that newly measured eigenpairs $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^{k}$ have been obtained. Update the pencil (M_0, C_0, K_0) to (M, C, K) *of the same structure* such that

1. The subset $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^{k}$ is replaced by $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^{k}$ as k eigenpairs of (M, C, K).
2. The remaining (unknown) $2n - k$ eigenpairs of (M, C, K) are the same as those of the original (M_0, C_0, K_0).

The MUP as stated above is of immense practical importance. Similar problems have been studied in the literature including the work by Friswell, Inman and Pilkey [17] where the model is updated by minimal adjustment of only the damping and the stiffness matrices; the consideration by Baruch [1], Bermann and Nagy [4], and Wei [26, 27] on only undamped systems; the finite element model correction by Minas and Inman [21, 22] with measured modal data; and the feedback control approach by Datta, Elhay, Ram, and Sarkissian [12, 13, 14]. Despite the many efforts, there does not seem to have satisfactory theory or techniques thus far even for the case where the required structure in the MUP is self-adjoint only. Existing methods have severe computational and engineering limitations, which restrict their usefulness in real applications. One of the main concerns is that these methods “cannot guarantee that extra, spurious modes are not introduced into the range of the frequency range of interest [16].” The purpose of this paper is to provide a systematic study toward this spill-over phenomenon.

Our main contribution in this paper is as follows. We offer a simple yet effective mathematical argument to draw a conclusion that perhaps is a rather surprising disappointment to engineering practitioners — at the end of our study we establish the fact that for a damped system the MUP as is described above generally is unsolvable. In other words, unless the newly measured eigenpairs $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^{k}$ satisfy some fairly stringent conditions, an updating of a damped quadratic pencils will surely cause spill-over. The characterization of those sufficient and necessary conditions for solvability is complicated enough that it warrants a separate paper [10] to address the details. This paper concentrates on the unsolvability, which we think offers a different view of the MUP.

For convenience, we adopt the notation that the diagonal matrix $\Lambda \in \mathbb{R}^{2n\times 2n}$ represents the “eigenvalue matrix” of the quadratic pencil (1.2) in the sense that Λ is in real diagonal form with 2×2 blocks along the diagonal replacing the complex-conjugate pairs of eigenvalues originally there. Similarly, let $X \in \mathbb{R}^{n\times 2n}$ represents the “eigenvector matrix” in the sense that each pair of column vectors associated with a 2×2 block in Λ retains the real and the imaginary part of the original complex eigenvector. It is clear that the relationship

$$MX\Lambda^2 + CX\Lambda + KX = 0_{n\times 2n} \tag{1.3}$$

holds. We partition X and Λ as

$$X = [Y, X_2], \quad \Lambda = \operatorname{diag}\{\Sigma, \Lambda_2\}, \tag{1.4}$$

where the pair $(Y, \Sigma) \in \mathbb{R}^{n\times k} \times \mathbb{R}^{k\times k}$ correspond to the portion of eigenstructure that has been modified and (X_2, Λ_2) corresponds to the inert portion of eigenstructure in the original model which should not be changed. To answer whether a self-adjoint quadratic pencil can be updated with no spill-over, a more fundamental question is whether a self-adjoint quadratic pencil can have *arbitrary* spectral structure (X, Λ).

2. Zero damping.

We first consider the self-adjoint pencil

$$\lambda^2 M_0 + K_0$$

where M_0 is assumed to be positive definite. It is known in this case that λ^2 is real. Thus, by defining $\mu := -\lambda^2$, we can rewrite the quadratic pencil as a linear pencil

$$\mu M_0 - K_0, \tag{2.1}$$

effectively reducing the number of eigenvalues for the system (2.1) to n. We shall continue using the same notation in (1.4) to indicate the partition of eigenstructure for (2.1), except that we know for sure in this case that $\Lambda \in \mathbb{R}^{n\times n}$ is truly diagonal and that no complex-valued eigenvectors are involved in $X \in \mathbb{R}^{n\times n}$. We shall make a practical assumption that all diagonal entries of Λ are distinct. Such an assumption can be deemed reasonable because multiple roots are sensitive to perturbations and, hence, are hardly observable in real applications.

2.1. ***Simultaneous Updating of Mass and Stiffness Matrices.

Given (X, Λ), ***this updating problem concerns finding symmetric matrices ΔM and ΔK such that the following equations hold simultaneously:

$$(M_0 + \Delta M) X_2 \Lambda_2 = (K_0 + \Delta K) X_2 \tag{2.2}$$

$$(M_0 + \Delta M) Y \Sigma = (K_0 + \Delta K) Y. \tag{2.3}$$

Note that each eigenpair gives rise to n equations and that ΔM and ΔK involve only $n(n+1)$ unknown entries. Since there are n^2 equations in $n(n+1)$ unknowns, it is likely that the system (2.2) and (2.3) is solvable for any given (X, Λ) where $X \in \mathbb{R}^{n\times n}$ is nonsingular and $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal. In other words, not only the updating with no spill-over is always possible, but there are n degrees of freedom in choosing the parameters. The question is how to find such a general solution.

Our answer comes from the observation that for the linear pencil $\mu M - K$ to have eigenstructure (X, Λ), it is necessary that

$$[-X^\top, \Lambda^\top X^\top] \begin{bmatrix} K^\top \\ M^\top \end{bmatrix} = 0 \tag{2.4}$$

On the other hand, it is trivial that

$$[-I_n, \Lambda^\top] \begin{bmatrix} \Lambda^\top S \\ S \end{bmatrix} = 0 \tag{2.5}$$

for any $S \in \mathbb{R}^{n\times n}$. We thus obtain a parametric representation,

$$\begin{aligned} M &= S^\top X^{-1}, \\ K &= S^\top \Lambda X^{-1}. \end{aligned}$$

We are interested in selecting S so as to construct self-adjoint pencils. For M to be symmetric, the matrix S must be such that

$$S^\top X^{-1} = X^{-\top} S,$$

implying that the matrix Γ defined by

$$\Gamma := SX = X^\top S^\top, \tag{2.6}$$

is symmetric. For K to be symmetric, the matrix S must also be such that

$$S^\top \Lambda X^{-1} = X^{-\top} \Lambda^\top S,$$

implying the equality,

$$\Lambda^\top \Gamma = \Gamma \Lambda. \tag{2.7}$$

Since Λ is of diagonal matrix with distinct entries, it follows that Γ is also a diagonal matrix containing exactly n free parameters. Upon **choosing an arbitrary** Γ, a substitution by $S^\top = X^{-\top}\Gamma$ concludes that the linear pencil $\mu M - K$ with

$$M = X^{-\top} \Gamma X^{-1}, \tag{2.8}$$

$$K = X^{-\top} \Gamma \Lambda X^{-1}, \tag{2.9}$$

is self-adjoint and has eigenstructure (X, Λ). This is the parametric solution to the inverse eigenvalue problem associated with (X, Λ). More importantly, if the parameter matrix Γ is positive definite, then so is the matrix M. We thus have proved the following fact.

THEOREM 2.1. *A self-adjoint linear pencil can have arbitrary eigenstructure with distinct eigenvalues and linearly independent eigenvectors. Indeed, given an eigenstructure* (X, Λ)*, the solutions* (M, K) *form a subspace of dimensionality* n *in the product space* $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ *and can be parameterized by the diagonal matrix* Γ *via the relationships (2.8) and (2.9).*

The task of modifying a partial eigenstructure from (X_1, Λ_1) to (Y, Σ) while maintaining the remaining eigenstructure (X_2, Λ_2) therefore is possible.

COROLLARY 2.2. *Given any* $k \leq n$*, assume that the observed eigenvalues* Σ *and the original eigenvalues* Λ_2 *are all distinct. Assume also that the corresponding observed eigenvectors* Y *and the original eigenvectors* X_2 *form an nonsingular matrix. Then the model updating of (2.1) with no spill-over is always possible.*

With the parametrization (2.8) and (2.9) in hand, we can further refine the model updating problem by demanding that the changes ΔM and ΔK be kept at minimum with respect to some measurement. For instance, the model updating problem could be modified to the problem of finding the optimal solution to the minimization problem [2],

$$\min_{\Gamma = \text{diagonal}} \|X^{-\top}\Gamma X^{-1} - M_0\|_F^2 + \|X^{-\top}\Gamma\Lambda X^{-1} - K_0\|_F^2. \tag{2.10}$$

We hastily point out that in the recipe (2.8) and (2.9) for constructing M and K, as well as in the minimal change formulation (2.10), knowledge of the full eigenstructure is required. This is precisely the scenario which we dismissed earlier as not feasible in practice. What we have proved is that the updating with no spill-over is possible in theory. It remains a problem of practical importance to construct M and K *without* any a priori knowledge of (X_2, Λ_2). ***The following discussion shows precisely how to construct M and K without any prior knowledge of $(X_{1,2})$. We emphasize that the constructions M and K given below are just theoretical and not suggested for practical computations.

***To demonstrate our point, recall that we may always assume without loss of generality that the eigenvectors $(X_1, X_2) \in \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times(n-k)}$ of the original pencil $\mu M_0 - K_0$ are normalized in such a way that

$$\begin{bmatrix} X_1^\top \\ X_2^\top \end{bmatrix} M_0[X_1, X_2] = I_n, \tag{2.11}$$

$$\begin{bmatrix} X_1^\top \\ X_2^\top \end{bmatrix} K_0[X_1, X_2] = \operatorname{diag}\{\Lambda_1, \Lambda_2\}. \tag{2.12}$$

Likewise, by choosing $\Gamma = I_n$, we see that

$$M = X^{-\top}X^{-1}, \tag{2.13}$$

$$K = X^{-\top}\Lambda X^{-1}. \tag{2.14}$$

If none of the diagonal entries is zero, then from the facts that

$$
\begin{aligned}
M^{-1} &= XX^{\top} = YY^{\top} + X_2X_2^{\top},\\
K^{-1} &= X\Lambda^{-1}X^{\top} = Y\Sigma^{-1}Y^{\top} + X_2\Lambda_2^{-1}X_2^{\top},\\
M_0^{-1} &= X_1X_1^{\top} + X_2X_2^{\top},\\
K_0^{-1} &= X_1\Lambda_1^{-1}X_1^{\top} + X_2\Lambda_2^{-1}X_2^{\top},
\end{aligned}
$$

we can express M^{-1} and K^{-1} as

$$M^{-1} = M_0^{-1} + YY^{\top} - X_1X_1^{\top}, \tag{2.15}$$

$$K^{-1} = K_0^{-1} + Y\Sigma^{-1}Y^{\top} - X_1\Lambda_1^{-1}X_1^{\top}, \tag{2.16}$$

showing the construction of M and K without the knowledge of (X_2, Λ_2). Note, however, the matrix coefficients M and K constructed in this way are uniquely determined and we no longer have the freedom in selecting other Γ. We do have in this case the inequalities,

$$
\begin{aligned}
\|M_0M^{-1} - I\| &\leq \|M_0\|\,\|X_1X_1^{\top} - YY^{\top}\|,\\
\|K_0K^{-1} - I\| &\leq \|K_0\|\,\|X_1\Lambda^{-1}X_1^{\top} - Y\Sigma^{-1}Y^{\top}\|,
\end{aligned}
$$

which might be useful for estimating an upper bound for (2.10).

2.2. ***Single-Step Updating . It might be worthy to briefly examine another scenario proposed in the dissertation by Carvalho [5]. The question is whether an updating can be accomplished by just modifying one single coefficient matrix, say, K_0. That is, instead of (2.2) and (2.3), can a symmetric matrix ΔK be found such that the equations,

$$M_0X_2\Lambda_2 = (K_0 + \Delta K)X_2 \tag{2.17}$$

$$M_0Y\Sigma = (K_0 + \Delta K)Y, \tag{2.18}$$

are satisfied? A quick count shows that there are n^2 equations in $\frac{n(n+1)}{2}$ unknowns. We think that the updating problem, as an over-determined system, cannot be solved unless the newly prescribed eigenstructure (Y, Σ) satisfies some consistency stipulations. To explore the necessary conditions, we first claim that any feasible candidate ΔK must be parameterized as follows:

Lemma 2.3. *Assume that the eigenvectors* (X_1, X_2) *of the pencil* $\mu M_0 - K_0$ *have been normalized as in (2.11) and (2.12). A symmetric matrix* ΔK *satisfies (2.17) if and only if there exists a symmetric matrix* $\Phi \in \mathbb{R}^{k \times k}$ *such that*

$$\Delta K = M_0 X_1 \Phi X_1^\top M_0. \tag{2.19}$$

Proof. By direction substitution, it is easy to see that

$$(K_0 + \Delta K)X_2 = K_0 X_2 + M_0 X_1 \Phi (X_1^\top M_0 X_2) = K_0 X_2 = M_0 X_2 \Lambda_2.$$

To see the converse, we write the QR decomposition of X_2 as

$$X_2 = [Q_1, Q_2] \begin{bmatrix} R_1 \\ 0_{k \times (n-k)} \end{bmatrix},$$

where $R_1 \in \mathbb{R}^{(n-k)\times(n-k)}$ is upper triangular and $Q = [Q_1, Q_2] \in \mathbb{R}^{n \times n}$ is orthogonal with its columns partitioned in conformal sizes. Let S denote the symmetric matrix defined by

$$S := Q^\top \Delta K Q = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^\top & S_{22} \end{bmatrix}.$$

Then in order to satisfy (2.17), it is necessary that $S_{11} = 0_{(n-k)\times(n-k)}$, $S_{12} = 0_{k \times (n-k)}$ and, hence, $\Delta K = Q_2 S_{22} Q_2^\top$. Observe further that $Q_2^\top X_2 = 0_{k\times(n-k)}$ and $X_1^\top M_0 X_2 = 0_{k \times (n-k)}$. Therefore, there must exist a matrix $W \in \mathbb{R}^{k\times k}$ such that $Q_2^\top = W X_1^\top M_0$. Together, we see that

$$\Delta K = M_0 X_1 W^\top S_{22} W X_1^\top M_0.$$

We may take $\Phi = W^\top S_{22} W$ which obviously is symmetric. ∎

With ΔK defined by (2.19), we now argue that the equation (2.18) holds only when a rather strict consistency condition is satisfied. The call made in [5] about updating (2.1) with no spill-over on the single matrix K_0 therefore can be achieved only when Y is of some very special form. We remark that a similar spirit holds for the damped problem, but the derivation of the corresponding form is much more complicated [10].

LEMMA 2.4. *There exists a symmetric matrix* $\Phi \in \mathbb{R}^{k\times k}$ *such that*

$$M_0 Y \Sigma = (K_0 + M_0 X_1 \Phi X_1^\top M_0) Y, \tag{2.20}$$

if and only if

$$Y = X_1 V D, \tag{2.21}$$

for some orthogonal matrix $V \in \mathbb{R}^{k\times k}$ *and some nonsingular diagonal matrix* $D \in \mathbb{R}^{k\times k}$.

Proof. The sufficiency is straightforward. Define

$$\Phi := V \Sigma V^\top - \Lambda_1.$$

By direct substitution, it is easy to see that

$$M_0 X_1 V D \Sigma = M_0 X_1 \left(\Lambda_1 V D + \Phi V D\right) = (K_0 + M_0 X_1 \Phi X_1^\top M_0) X_1 V D.$$

To prove the necessity, write the QR decomposition of $M_0 X_1$ as

$$M_0 X_1 = [\widehat{Q}_1, \widehat{Q}_2] \begin{bmatrix} \widehat{R}_1 \\ 0_{(n-k)\times k} \end{bmatrix},$$

where $\widehat{R}_1 \in \mathbb{R}^{k\times k}$ is upper triangular and $\widehat{Q} = [\widehat{Q}_1, \widehat{Q}_2] \in \mathbb{R}^{n\times n}$ is orthogonal with its columns partitioned in conformal sizes. Premultiplying both sides of (2.20) by $\widehat{Q}^\top$, we see that

$$\widehat{Q}_1^\top (M_0 Y \Sigma - K_0 Y) = \widehat{R}_1 \Phi X_1^\top M_0 Y, \tag{2.22}$$

$$\widehat{Q}_2^\top (M_0 Y \Sigma - K_0 Y) = 0. \tag{2.23}$$

The equation (2.23) implies that $M_0 Y \Sigma - K_0 Y$ is in the column space on $\widehat{Q}_1$. We denote this fact by

$$M_0 Y \Sigma - K_0 Y = \widehat{Q}_1 H = M_0 X_1 \widehat{R}_1^{-1} H \tag{2.24}$$

for some $H \in \mathbb{R}^{k \times k}$. Let $\mathbf{y}_j$ and $\mathbf{h}_j$, $j = 1, \ldots, k$, denote the j column of Y and H, respectively. It follows from (2.24) that

$$\mathbf{y}_j = (\sigma_j I_n - M_0^{-1} K_0)^{-1} X_1 \widehat{R}_1^{-1} \mathbf{h}_j = X_1 (\sigma_j I_k - \Lambda_1)^{-1} \widehat{R}_1^{-1} \mathbf{h}_j.$$

In other words, we can relate columns of Y to those of X_1 via

$$Y = X_1 \widehat{H}$$

for some nonsingular $\widehat{H} \in \mathbb{R}^{k \times k}$. To further characterize $\widehat{H}$, we rewrite (2.22) as

$$\widehat{Q}_1^\top (M_0 X_1 \widehat{H} \Sigma - K_0 X_1 \widehat{H}) = \widehat{R}_1 \Phi X_1^\top M_0 X_1 \widehat{H},$$

and deduce that

$$\widehat{R}_1 \widehat{H} \Sigma - \widehat{R}_1 \Lambda_1 \widehat{H} = \widehat{R}_1 \Phi \widehat{H}.$$

It follows that

$$\Phi = \widehat{H} \Sigma \widehat{H}^{-1} - \Lambda_1.$$

Since Λ_1 is diagonal and Φ is symmetric, the matrix $\widehat{H} \Sigma \widehat{H}^{-1}$ must also be symmetric. From the facts that

$$\Sigma \widehat{H}^\top \widehat{H} = \widehat{H}^\top \widehat{H} \Sigma$$

and that Σ is diagonal with distinct diagonal elements, we conclude that $\widehat{H}^\top \widehat{H}$ must be a diagonal matrix itself. We can rewrite $\widehat{H} = VD$ in terms of some orthogonal matrix V and nonsingular diagonal matrix D. The necessity is thus proved. $\square$

3. Quadratic inverse eigenvalue problem. We have seen in the preceding section that a self-adjoint pencil $\mu M - K$ in $\mathbb{R}^{n\times n}$ with positive definite M can have arbitrary n distinct real eigenvalues and n linearly independent real eigenvectors. ***** Consequently, the model updating with no spill-over for self-adjoint linear pencils is always possible**. Before we explore whether this result can be extended to quadratic pencils, it is natural to ask how much partial eigenpair information is allowable for constructing a self-adjoint quadratic pencil $\lambda^2 M + \lambda C + K$. A special case related to this question can be found in an earlier paper [9]. A more detailed analysis partially addressing this issue is given in the recent paper by Kuo, Lin and Xu [19]. Our main contribution in this section is a complete characterization of the general solution.

Without causing ambiguity, we shall use the same notation $(X, \Lambda) \in \mathbb{R}^{n\times k} \times \mathbb{R}^{k\times k}$ to denote k given eigenpairs. For the moment, k can be any integer between 1 and $2n$. We shall assume that Λ is closed under complex conjugation. Thus Λ is of diagonal form with 2×2 blocks along the diagonal wherever a complex-conjugate pair of eigenvalues appear in the prescribed spectrum.

Consider the algebraic system,

$$MX\Lambda^2 + CX\Lambda + KX = 0_{n\times k}, \tag{3.1}$$

for the triplet (M, C, K). There are nk equations in $\frac{3n(n+1)}{2}$ unknowns in this homogeneous equation. It is intuitively true that if the number k of prescribed eigenpair is capped by the bound,

$$k < \frac{3(n+1)}{2}, \tag{3.2}$$

then the system (3.1) is under-determined and the solutions form a subspace of dimensionality $\frac{3n(n+1)}{2} - nk$; otherwise, the algebraic system and, hence, the QIEP will have only a trivial solution. In what follows, we prove that this conjecture is indeed true. More importantly, in our proof we provide a parametric representation of the solution. For solvability, we thus see that the maximal allowable number k_{max} of prescribed eigenpairs is given by

$$k_{max} = \begin{cases} 3\ell + 1, & \text{if } n = 2\ell, \\ 3\ell + 2, & \text{if } n = 2\ell + 1. \end{cases} \tag{3.3}$$

These bounds of k_{max} have the consequence that, in contrast to the linear pencil, the remaining $2n - k_{max}$ eigenpairs of a quadratic pencil *cannot* be arbitrarily assigned anymore. In other words, when $n \geq 3$, there will be no room to maintain no spill-over if the updating intends to replace k_{max} original eigenpairs by newly measured data.

We first analyze the necessary condition for the self-adjoint quadratic pencil (M, C, K) to have eigenstructure $(X, \Lambda) \in \mathbb{R}^{n \times k} \times \mathbb{R}^{k \times k}$. For now, we have no restriction on k. Consider the matrix

$$\Omega := [I_k, \Lambda^\top, \Lambda^{2\top}] \in \mathbb{R}^{k \times 3k}, \tag{3.4}$$

and let a basis of its null space be partitioned into three blocks so that we can write

$$\Omega \begin{bmatrix} U \\ T \\ S \end{bmatrix} = 0_{k \times 2k}, \tag{3.5}$$

where S, T and U are matrices in $\mathbb{R}^{k \times 2k}$. Obviously,

$$U = -\Lambda^\top T - \Lambda^{2\top} S \tag{3.6}$$

is determined, once S and T are specified. Any triplet (M, C, K) satisfying (3.1) must be such that

$$\Omega \begin{bmatrix} X^\top K \\ X^\top C \\ X^\top M \end{bmatrix} = 0_{k \times n}. \tag{3.7}$$

There must exist a matrix $\Psi \in \mathbb{R}^{2k\times n}$ such that

$$\begin{bmatrix} U \\ T \\ S \end{bmatrix} \Psi = \begin{bmatrix} X^\top K \\ X^\top C \\ X^\top M \end{bmatrix}. \tag{3.8}$$

Since M, C and K are symmetric, the three matrices

$$A := S\Psi X, \tag{3.9}$$

$$B := T\Psi X, \tag{3.10}$$

$$F := U\Psi X \tag{3.11}$$

must also be symmetric in $\mathbb{R}^{k\times k}$. Substituting (3.6) into (3.11) and using the fact that $F = F^\top$, we obtain a critical relationship between A and B:

$$\Lambda^\top B - B\Lambda = A\Lambda^2 - \Lambda^{2\top} A. \tag{3.12}$$

Observe that the difference on either side of (3.12) is a skew-symmetric matrix.

The above necessary condition can also be used to construct a solutions (M, C, K) in terms of A and B. Because of the constraint (3.12), not all entries in A or B are free. We shall exploit those free parameters and establish a parametric relationship. For clarity, we divide our discussion into three cases.

3.1. The case $k = n$**.** This is the most important case which plays a pivot role in the other two cases. Suppose that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is given. Denote

$$\Theta := A\Lambda^2 - \Lambda^{2\top} A.$$

We need to see how B can be determined from the equation

$$\Lambda^\top B - B\Lambda = \Theta.$$

For convenience, we may assume without loss of generality that Λ is of the diagonal form,

$$\Lambda = \operatorname{diag}\{\lambda_1^{[2]}, \ldots, \lambda_\nu^{[2]}, \lambda_{\nu+1}, \ldots, \lambda_\ell\}, \tag{3.13}$$

where $\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $\beta_j \neq 0$, if $j = 1, \ldots, \nu$; $\lambda_j \in \mathbb{R}$ if $j = \nu+1, \ldots, \ell$, and $\ell + \nu = n$. Partition B into $\ell \times \ell$ blocks, denoted by $B = [B_{ij}]$, in such a way that $\operatorname{diag}\{B_{11}, \ldots, B_{\ell\ell}\}$ has exactly the same structure as Λ. Then the (i,j)-block of the skew-symmetric matrix $\Lambda^\top B - B\Lambda$ is given by one of the following three possibilities:

$$\begin{cases} \lambda_i^\top B_{ij} - B_{ij}\lambda_j, & \text{if } \nu+1 \leq i, j \leq \ell, \\ (\lambda_i^{[2]})^\top B_{ij} - B_{ij}\lambda_j, & \text{if } 1 \leq i \leq \nu \text{ and } \nu+1 \leq j \leq \ell, \\ (\lambda_i^{[2]})^\top B_{ij} - B_{ij}(\lambda_j^{[2]}), & \text{if } 1 \leq i, j \leq \nu. \end{cases} \tag{3.14}$$

In the first case, B_{ij} is a scalar. Upon comparing with the corresponding blocks in Θ, we find that B_{ij} is uniquely determined except that B_{ii} is free. Likewise, B_{ij} in the second case is a 2×1 block and all its entries are uniquely determined. In the third case, if we write

$$B_{ij} = \begin{bmatrix} x & y \\ y & z \end{bmatrix},$$

then

$$(\lambda_i^{[2]})^\top B_{ij} - B_{ij}(\lambda_j^{[2]}) = \begin{bmatrix} x(\alpha_i - \alpha_j) - y(\beta_i - \beta_j) & -z\beta_i - x\beta_j \\ x\beta_i + y(\alpha_i - \alpha_j) + z\beta_j & y(\beta_i - \beta_j) \end{bmatrix}.$$

It is clear that if $i = j$, then y is free and $x + z$ is a fixed constant, still giving rise to two degrees of freedom. If $i \neq j$, the all entries of B_{ij} are uniquely determined. In all, we conclude that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be totally arbitrary whereas B is determined up to n free parameters. We thus declare the following theorem.

Theorem 3.1. *Given n distinct eigenvalues Λ and n linearly independent eigenvectors X, both of which are closed under conjugation, let $A \in \mathbb{R}^{n\times n}$ be an arbitrary symmetric matrix and let B be a* *solution to the equation (3.12). Then the self-adjoint quadratic pencil with its coefficients* (M, C, K) *defined by*

$$M = X^{-\top} A X^{-1}, \tag{3.15}$$

$$C = X^{-\top} B X^{-1}, \tag{3.16}$$

$$K = -X^{-\top} \Lambda^{\top} (B + \Lambda^{\top} A) X^{-1}, \tag{3.17}$$

has the prescribed pair (X, Λ) *as part of its eigenstructure.*

Proof. The proof is already contained in our construction above, except that we need to remove the reference to the intermediate parameters Ψ, S and T. The relationship (3.8) implies that $M = X^{-\top} S \Psi$ for some $\Psi \in \mathbb{R}^{2n \times n}$. We also know from (3.9) that $A = S \Psi X$. Together, we can express M as $M = X^{-\top} A X^{-1}$. Similar arguments can be applied to C and K. □

It is worth mentioning that if ***A **is selected to be symmetric and positive definite, then so is the leading coefficient** M. Indeed, the above construction parameterizes all possible solutions.

Corollary 3.2. *The solutions* (M, C, K) *to the quadratic inverse eigenvalue problem with eigenstructure* (X, Λ) *as described in Theorem 3.1 form a subspace of dimensionality* $\frac{n(n+3)}{2}$ *in the product space* $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$.

Corollary 3.3. *With* (M, C, K) *being defined in Theorem 3.1, the corresponding quadratic pencil can be factorized as*

$$\begin{aligned} \lambda^2 M + \lambda C + K &= X^{-\top} \left(\lambda I_n - \Lambda^{\top} \right) \left(B + (\lambda I_n + \Lambda^{\top}) A \right) X^{-1} \\ &= X^{-\top} \left(B + A(\lambda I_n + \Lambda) \right) \left(\lambda I_n - \Lambda \right) X^{-1}. \end{aligned} \tag{3.18}$$

Based on Corollary 3.3, the remaining eigenvalues therefore are determined by those of the linear pencil $\lambda A + B + A\Lambda$. Since the entire A and part of B are free, there is room to impose additional eigeninformation to the pencil. In [19], for instance, it was argued that additional n eigenvalues could be specified. In our context, we ask how many more *eigenpairs* can be prescribed. We shall study the general case in Section 3.3.

3.2. The case $k < n$. If less than n eigenpairs (X, Λ) are given, we can solve the QIEP by embedding this given eigeninformation in a larger set of n eigenparis, giving leeway of more free parameters. In particular, we expand $X \in \mathbb{R}^{n\times k}$ to

$$\widehat{X} := [X, \widetilde{X}] \in \mathbb{R}^{n\times n}, \tag{3.19}$$

where $\widetilde{X} \in \mathbb{R}^{n\times(n-k)}$ is arbitrary while making $\hat{X}$ nonsingular. A caution should be taken when counting the degrees of freedom — the columns in $\widetilde{X}$ should be considered as being *normalized* because, otherwise, any normalization factor would have been included in the arbitrariness of A. With this normalization in mind, this expansion of eigenvectors involves additional $(n-1)(n-k)$ degrees of freedom. We then expand $\Lambda \in \mathbb{R}^{k\times k}$ to

$$\widehat{\Lambda} := \operatorname{diag}\{\Lambda, \widetilde{\Lambda}\}, \tag{3.20}$$

where $\widehat{\Lambda}$ is a diagonal matrix with distinct eigenvalues. This expansion of eigenvalues gives rise to another $n-k$ degrees of freedom. With $(\widehat{X}, \widehat{\Lambda})$ playing the role of (X, Λ) in Theorem 3.1, we can now construct the coefficient matrices M, C and K according to the formulas (3.15), (3.16) and (3.17), respectively, whereas A is taken as an arbitrary symmetric matrix in $\mathbb{R}^{n\times n}$ and B, depending on $\widehat{\Lambda}$ through the relationship (3.12), maintains n degrees of freedom. Note that the parametrization of (M, C, K) in this embedding approach is *nonlinear* in A, B, $\widetilde{X}$ and $\widetilde{\Lambda}$. We summarize this construction as follows:

THEOREM 3.4. *The quadratic inverse eigenvalue problem with k, $k < n$, prescribed eigenpairs is always solvable. For almost all prescribed eigenstructure (X, Λ), the solutions form a subspace of dimensionality $\frac{n(n+3)}{2} + n(n-k)$.*

Proof. We only need to justify the dimensionality. It is clear that the solutions (M, C, K) to the homogeneous system (3.1) with k prescribed eigenpairs form a subspace of dimensionality at least $\frac{3n(n+1)}{2} - nk$, but in the above we have just found a parametric representation of (M, C, K) which involves precisely $\frac{n(n+1)}{2}$ free parameters in A, n in B, $n-k$ in $\widetilde{\Lambda}$, and $(n-1)(n-k)$ in $\widetilde{X}$, giving a total of $\frac{3n(n+1)}{2} - nk$ free parameters. $\square$

3.3. The case $k > n$. This case is rather involved and, to our knowledge, has never been discussed in the literature. At a first glance, we know from the relationships (3.8), (3.9), (3.10) and (3.11) that

$$A = S\Psi X = X^\top M X, \tag{3.21}$$

$$B = T\Psi X = X^\top C X, \tag{3.22}$$

$$F = U\Psi X = X^\top K X, \tag{3.23}$$

remain symmetric even in the case $k > n$. However, we cannot obtain a parametric representation of M, C and K from A and B directly because $X \in \mathbb{R}^{n\times k}$ is no longer an injection transformation. The challenge is to retrieve M, C and $K \in \mathbb{R}^{n\times n}$ from the "seemingly" over-specified A and B.

Rewrite the eigenvectors as

$$X = [Z_1, Z_2],$$

where $Z_1 \in \mathbb{R}^{n\times n}$ and $Z_2 \in \mathbb{R}^{n\times(k-n)}$. Then we see that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix} = \begin{bmatrix} Z_1^\top M Z_1 & Z_1^\top M Z_2 \\ Z_2^\top M Z_1 & Z_2^\top M Z_2 \end{bmatrix}, \tag{3.24}$$

where A_{ij}, $i, j = 1, 2$, are blocks with appropriate sizes. This relationship suggests that we may choose a symmetric matrix $A_{11} \in \mathbb{R}^{n\times n}$ arbitrarily and define

$$M = Z_1^{-\top} A_{11} Z_1^{-1}. \tag{3.25}$$

This selection give rises to $\frac{n(n+1)}{2}$ degrees for freedom. Once $M \in \mathbb{R}^{n\times n}$ is determined, the matrix $A \in \mathbb{R}^{k\times k}$ is completely specified. There is no additional freedom in the choice of A. With $A \in \mathbb{R}^{k\times k}$ given, we need to determine the matrix $B \in \mathbb{R}^{k\times k}$ so that the necessary condition (3.12) is satisfied.

Note that

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^\top & B_{22} \end{bmatrix} = \begin{bmatrix} Z_1^\top C Z_1 & Z_1^\top C Z_2 \\ Z_2^\top C Z_1 & Z_2^\top C Z_2 \end{bmatrix}. \tag{3.26}$$

Consider the $(1,1)$ block of B first. With A_{11} given, using an argument similar to that made in Section 3.1 (see (3.14)), we see that the submatrix B_{11} is completely determined up to n free parameters. These would have determined a symmetric matrix

$$C = Z_1^{-\top} B_{11} Z_1^{-1} \tag{3.27}$$

and, hence, the matrix B up to n free parameters. However, the very same C should also equate the two sides of equation (3.12) at the $(1,2)$ and $(2,2)$ blocks, respectively. These blocks involve more than n equations to be satisfied. We have no choice but to go back to modify the selection of A_{11} and sacrifice some of the freedom. In other words, the n free parameters in B_{11} and the matrix A_{11} must be further restricted so that the remaining part of B also satisfies (3.12). Toward that end, we re-examine the relationship between A and B block by block. If we define

$$\Xi := Z_1^{-1} Z_2, \tag{3.28}$$

then it follows that

$$A = \begin{bmatrix} A_{11} & A_{11}\Xi \\ \Xi^\top A_{11} & \Xi^\top A_{11} \Xi \end{bmatrix},$$

$$B = \begin{bmatrix} B_{11} & B_{11}\Xi \\ \Xi^\top B_{11} & \Xi^\top B_{11} \Xi \end{bmatrix}.$$

If we partition the given eigenvalues as

$$\Lambda = \operatorname{diag}\{\Upsilon_1, \Upsilon_2\}$$

where $\Upsilon_1 \in \mathbb{R}^{n\times n}$ and $\Upsilon_2 \in \mathbb{R}^{(k-n)\times(k-n)}$, then the critical condition (3.12) to be satisfied can be expressed as three equations,

$$\Upsilon_1^\top B_{11} - B_{11}\Upsilon_1 = A_{11}\Upsilon_1^2 - \Upsilon_1^{2\top} A_{11}, \tag{3.29}$$

$$\Upsilon_1^\top B_{11}\Xi - B_{11}\Xi\Upsilon_2 = A_{11}\Xi\Upsilon_2^2 - \Upsilon_1^{2\top} A_{11}\Xi, \tag{3.30}$$

$$\Upsilon_2^\top \Xi^\top B_{11}\Xi - \Xi^\top B_{11}\Xi\Upsilon_1 = \Xi^\top A_{11}\Xi\Upsilon_2^2 - \Upsilon_2^{2\top}\Xi^\top A_{11}\Xi. \tag{3.31}$$

Post-multiplying (3.29) by Ξ and subtracting (3.30), we obtain that

$$A_{11}\Upsilon_1^2\Xi + B_{11}\Upsilon_1\Xi = A_{11}\Xi\Upsilon_2^2 + B_{11}\Xi\Upsilon_2.$$

It follows that

$$\begin{aligned}\Xi^\top(A_{11}\Xi\Upsilon_2^2 + B_{11}\Xi\Upsilon_2) &= \Xi^\top(A_{11}\Upsilon_1^2\Xi + B_{11}\Upsilon_1\Xi)\\ &= (\Xi^\top A_{11}\Upsilon_1^2 + \Xi^\top B_{11}\Upsilon_1)\Xi = (\Upsilon_2^\top\Xi^\top B_{11} + \Upsilon_2^{2\top}\Xi^\top A_{11})\Xi,\end{aligned}$$

which is precisely (3.31). In the above, the last equality follows from taking the transpose of equation (3.30). What we have just proved is that if we can solve equations (3.29) and (3.30), then (3.31) is automatically solved. We have indicated earlier that any given A_{11} will determine B_{11} through (3.29) up to n free parameters. Thus, it only remains to choose the n free parameters in B_{11} and the $n \times n$ symmetric matrix A_{11} to satisfy the $n(k-n)$ linear equations imposed by (3.30). Totally, only

$$\frac{n(n+1)}{2} + n - n(k-n) = \frac{3n(n+1)}{2} - nk$$

degrees of freedom remain.

Together with results proved in the preceding sections, we have now established the following results.

THEOREM 3.5. *Given any* $1 \le k < \frac{3(n+1)}{2}$*, the quadratic inverse eigenvalues with* k *prescribed eigenpairs is always solvable. For almost all prescribed eigenstructure* (X, Λ)*, the solutions form a subsapce of dimensionality precisely* $\frac{3n(n+1)}{2} - nk$*. The maximal allowable number of prescribed eigenpairs is given by (3.3).*

Example 2. Suppose that the prescribed eigenstructure is given by

$$X = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

and

$$\Lambda = \operatorname{diag}\{1, 2, 3, 5, 8\}.$$

This is a case where $n = 3$ and $k = k_{max} = 5$. By our theory, the solution has three degrees of freedom. ***Using formulas (3.15)-(3.17) we find that the general solution to the QIEP can be represented as

$$M = \begin{bmatrix} s & -s+4u & u \\ -s+4u & w & -\frac{7}{10}s + \frac{14}{5}u \\ u & -\frac{7}{10}s + \frac{14}{5}u & -\frac{3}{10}u + \frac{7}{10}s \end{bmatrix},$$

$$C = \begin{bmatrix} -9s+10u & 3s-12u & -4u \\ 3s-12u & -\frac{27}{5}s + \frac{108}{5}u - 7w & \frac{7}{2}s - 14u \\ -4u & \frac{7}{2}s - 14u & \frac{34}{5}u - \frac{77}{10}s \end{bmatrix},$$

$$K = \begin{bmatrix} 8s-10u & -2s+8u & 3u \\ -2s+8u & \frac{54}{5}s - \frac{216}{5}u + 10w & -\frac{21}{5}s + \frac{84}{5}u \\ 3u & -\frac{21}{5}s + \frac{84}{5}u & -\frac{177}{10}u + \frac{84}{5}s \end{bmatrix}.$$

It can be computed that

$$\det(M) = -\frac{1}{100}\,(7s - 10u)\left(272u^2 - 136su - 10wu - 10ws + 17s^2\right).$$

Obviously, the parameters s, u and w can be chosen so that M is positive definite. We also find that the sixth eigenvalue is given by

$$\lambda_6 = -2\frac{52u^2 + 37s^2 - 161su + 40st - 35tu}{17s^2 - 136su - 10wu + 272u^2 - 10sw}$$

with its corresponding eigenvector given by

$$\mathbf{x}_6 = \left[\frac{2}{5}\frac{9s - 36u + 5w}{7s - 10u}, 1, \frac{2}{5}\frac{9s - 36u + 5w}{7s - 10u}\right]^\top .$$

It is clear that $\mathbf{x}_6$ cannot be arbitrarily assigned and, hence, no spill-over *cannot* be maintained.

4. A case study of no spill-over. We have argued in the preceding section that in the process of updating a quadratic pencil the phenomenon of spill-over is inevitable in general. Thus it is fitting to ask under what stringent conditions we can maintain no spill-over. To answer this question requires laborious and careful analysis which is accomplished in the paper [10]. It might be informative if we demonstrate in this section by a numerical example on how specific it has to be in order to maintain the no spill-over.

Given the original self-adjoint quadratic pencil $Q(\lambda) = \lambda^2 M_0 + \lambda C_0 + K_0$, let its eigenvector and eigenvalue matrices be expressed in real-value form as we have described before. Partition the eigenstructure as $[X_1, X_2] \in \mathbb{R}^{n\times 2n}$ and $\mathrm{diag}\{\Lambda_1, \Lambda_2\} \in \mathbb{R}^{2n\times 2n}$, respectively, where the portion $(X_1, \Lambda_1) \in \mathbb{R}^{n\times k} \times \mathbb{R}^{k\times k}$ is to be updated by newly measured eigenpair (Y, Σ). Recall that the updating with no spill-over means to find symmetric matrices ΔM, ΔC and ΔK such that the equations,

$$(M_0 + \Delta M)X_2\Lambda_2^2 + (C_0 + \Delta C)X_2\Lambda_2 + (K_0 + \Delta K)X_2 = 0, \tag{4.1}$$

$$(M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y = 0, \tag{4.2}$$

are satisfied simultaneously. Considering this updating problem as a QIEP with prescribed eigenvectors $[Y, X_2]$ and eigenvalues $\mathrm{diag}\{\Sigma, \Lambda_2\}$, we are facing a homogeneous system with $2n^2$ equations in $\frac{3n(n+1)}{2}$ unknowns. If $n > 3$, the system is over-determined. In order to have a nontrivial solution, (Y, Σ) must satisfy some consistency conditions. This is in contrast to the undamped case studied in Section 2.

We start with an explanation of the subtlety involved in the "genericness" of the eigenpair (X_2, Λ_2). Observe that if (4.1) holds, then the incremental pencil,

$$\Delta Q(\lambda) := \lambda^2 \Delta M + \lambda \Delta C + \Delta K, \tag{4.3}$$

necessarily has the $\hat{k} = 2n - k$ eigenpairs (X_2, Λ_2) as part of its eigenstructure. If $\hat{k} < \frac{3(n+1)}{2}$, that is, if $k > \frac{n-3}{2}$, then our theory asserts that there are nontrivial solutions $(\Delta M, \Delta C, \Delta K)$ which, *for almost all* (X_2, Λ_2), form a subspace of dimension $\frac{n(2k-n+3)}{2}$. The pencil $(\Delta M, \Delta C, \Delta K)$ can be characterized by the procedures described in Section 3.3. If $\hat{k} \geq \frac{3(n+1)}{2}$, that is, if $k \leq \frac{n-3}{2}$, then our theory implies that the solution space to the QIEP with "generic" eigenpairs (X_2, Λ_2) should be made of the trivial solution only. However, since we have already assumed that (M_0, C_0, K_0) has (X_2, Λ_2) as part of its eigenstructure, we have to conclude that (X_2, Λ_2) is *not* generic in the sense that the *seemingly* over-determined algebraic system

$$\Delta M X_2 \Lambda_2^2 + \Delta C X_2 \Lambda_2 + \Delta K X_2 = 0, \tag{4.4}$$

is not overly determined at all and, in fact, has nontrivial solutions. The following example serves to shed some insight into this situation.

Example 3. Consider the case $n = 5$ and $k = 1$ with

$$M_0 = \begin{bmatrix} 3.3308 & 1.9508 & 2.0792 & 1.0873 & 2.3424 \\ 1.9508 & 1.6595 & 1.3898 & 0.6036 & 1.5318 \\ 2.0792 & 1.3898 & 1.7062 & 0.8195 & 1.5197 \\ 1.0873 & 0.6036 & 0.8195 & 0.5217 & 0.7819 \\ 2.3424 & 1.5318 & 1.5197 & 0.7819 & 1.7472 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1.0454 & 0.8031 & 1.1669 & 1.0143 & 0.7795 \\ 0.8031 & 1.3832 & 0.6174 & 1.3404 & 0.8307 \\ 1.1669 & 0.6174 & 1.6762 & 0.6650 & 1.0423 \\ 1.0143 & 1.3404 & 0.6650 & 0.9317 & 1.2889 \\ 0.7795 & 0.8307 & 1.0423 & 1.2889 & 0.5037 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 2.6981 & 2.2257 & 1.5499 & 1.6738 & 1.5832 \\ 2.2257 & 2.2472 & 1.4826 & 1.6162 & 1.3072 \\ 1.5499 & 1.4826 & 1.2197 & 1.0846 & 1.1743 \\ 1.6738 & 1.6162 & 1.0846 & 1.5889 & 0.8304 \\ 1.5832 & 1.3072 & 1.1743 & 0.8304 & 1.4532 \end{bmatrix}.$$

The pencil (M_0, C_0, K_0) has eigenvectors

$$\underbrace{\begin{bmatrix} -0.0101 \\ 0.0395 \\ -0.0537 \\ 0.0805 \\ -0.0073 \end{bmatrix}}_{X_1}, \underbrace{\begin{bmatrix} 0.1148 & -0.0410 & -0.5947 & -0.2308 & -0.3380 \\ 0.0625 & -0.0001 & -0.2369 & -0.0038 & 0.4726 \\ 0.0243 & -0.0536 & 0.6170 & 0.1379 & -1.0000 \\ 0.0927 & -0.0737 & 0.4292 & 0.2892 & 0.2785 \\ -0.2859 & 0.1619 & 0.5020 & 0.1943 & 0.5821 \end{bmatrix}}_{Z_1}, \underbrace{\begin{bmatrix} 0.4771 & -0.0893 & 0.4573 & -0.2609 \\ 0.5463 & 0.0699 & -0.6302 & 0.3698 \\ -0.3110 & -0.3052 & -0.4527 & 0.1497 \\ -0.5810 & -0.4190 & 0.4052 & -0.2067 \\ -0.1483 & 0.6559 & 0.2904 & -0.1039 \end{bmatrix}}_{Z_2}$$

and eigenvalues

$$\underbrace{-12.4263}_{\Lambda_1}, \underbrace{\begin{bmatrix} 2.4975 & 1.5414 \\ -1.5414 & 2.4975 \end{bmatrix} \begin{bmatrix} -1.0079 & 0.6851 \\ -0.6851 & -1.0079 \end{bmatrix}, -0.0603,}_{\Upsilon_1} \underbrace{\begin{bmatrix} 0.3444 & 0.9859 \\ -0.9859 & 0.3444 \end{bmatrix} \begin{bmatrix} -0.1218 & 0.7665 \\ -0.7665 & -0.1218 \end{bmatrix}}_{\Upsilon_2},$$

which we have partitioned into $[X_1, Z_1, Z_2]$ and $[\Lambda_1, \Upsilon_1, \Upsilon_2]$ with $X_1 \in \mathbb{R}^{5\times 1}$, $Z_1 \in \mathbb{R}^{5\times 5}$, $Z_2 \in \mathbb{R}^{5\times 4}$, $\Lambda_1 \in \mathbb{R}$, $\Upsilon_1 \in \mathbb{R}^{5\times 5}$, and $\Upsilon_2 \in \mathbb{R}^{4\times 4}$, respectively. According to the theory developed earlier, the general solution $(\Delta M, \Delta C, \Delta K)$ to (4.4) can be expressed in the form

$$\begin{aligned} \Delta M &= Z_1^{-\top} A_{11} Z_1^{-1}, \\ \Delta C &= Z_1^{-\top} B_{11} Z_1^{-1}, \\ \Delta K &= -Z_1^{-\top} (A_{11} \Upsilon_1^2 + B_{11} \Upsilon_1) Z_1^{-1}, \end{aligned}$$

where the symmetric matrices A_{11} and B_{11} must satisfy equations (3.29) and (3.30) simultaneously. Using our data above, we find that

$$A_{11} = \alpha A^{[1]} + \beta A^{[2]}, \tag{4.5}$$

$$B_{11} = \alpha B^{[1]} + \beta B^{[2]}, \tag{4.6}$$

with arbitrary $\alpha, \beta \in \mathbb{R}$, where

$$A^{[1]} = \begin{bmatrix} 0.0010 & -0.0015 & 0.0005 & -0.0044 & -0.0297 \\ -0.0015 & 0.0025 & -0.0008 & 0.0070 & 0.0472 \\ 0.0005 & -0.0008 & 0.0006 & -0.0022 & -0.0161 \\ -0.0044 & 0.0070 & -0.0022 & 0.0201 & 0.1351 \\ -0.0297 & 0.0472 & -0.0161 & 0.1351 & 0.9106 \end{bmatrix},$$

$$A^{[2]} = \begin{bmatrix} -0.0026 & 0.0020 & -0.0047 & -0.0023 & 0.0050 \\ 0.0020 & -0.0021 & 0.0068 & 0.0031 & 0.0016 \\ -0.0047 & 0.0068 & -0.1916 & -0.0800 & 0.1530 \\ -0.0023 & 0.0031 & -0.0800 & -0.0295 & 0.0994 \\ 0.0050 & 0.0016 & 0.1530 & 0.0994 & 0.0379 \end{bmatrix},$$

$$B^{[1]} = \begin{bmatrix} -0.0096 & 0.0100 & -0.0051 & 0.0170 & 0.1452 \\ 0.0100 & -0.0075 & 0.0052 & -0.0031 & -0.0693 \\ -0.0051 & 0.0052 & -0.0020 & 0.0091 & 0.0754 \\ 0.0170 & -0.0031 & 0.0091 & 0.0439 & 0.1554 \\ 0.1452 & -0.0693 & 0.0754 & 0.1554 & 0.1111 \end{bmatrix},$$

$$B^{[2]} = \begin{bmatrix} 0.0159 & -0.0113 & 0.0159 & 0.0116 & -0.0097 \\ -0.0113 & 0.0075 & -0.0007 & -0.0057 & -0.0115 \\ 0.0159 & -0.0007 & -0.3738 & -0.1494 & 0.2315 \\ 0.0116 & -0.0057 & -0.1494 & -0.0719 & 0.0013 \\ -0.0097 & -0.0115 & 0.2315 & 0.0013 & -0.7634 \end{bmatrix},$$

serve as a basis. In other words, the solutions to (4.4), including the original pencil (M_0, C_0, K_0) which corresponds to $\alpha_0 = 0.5820$ and $\beta_0 = -1.8046$, form a two-dimensional subspace.

It is now clear that in order to update the eigenpair (X_1, Λ_1) of the pencil (M_0, C_0, K_0) to newly measured (Y, Σ) while maintaining *no* spill-over to (X_2, Λ_2), the newly measured eigenpair (Y, Σ) must satisfy the algebraic system

$$(\alpha A^{[1]}+\beta A^{[2]})Z_1^{-1}Y\Sigma^2+(\alpha B^{[1]}+\beta B^{[2]})Z_1^{-1}Y\Sigma = \Big((\alpha A^{[1]}+\beta A^{[2]})\Upsilon_1^2+(\alpha B^{[1]}+\beta B^{[2]})\Upsilon_1\Big)\,Z_1^{-1}Y,$$

for some $\alpha, \beta \in \mathbb{R}$. At its glance, this is a system of five polynomials in eight unknowns whose real solutions form an algebraic variety of dimension three. Two degrees of this freedom come from the choice of α and β and the third degree of freedom comes from the scaling of the eigenvector. More specifically, the system by construction already has one real eigenvalue and four pairs of complex conjugate eigenvalues. So the remaining eigenvalue Σ and the associated eigenvector Y must be real and is completely determined by α and β. In fact, in the case when $\beta \neq 0$, then Σ is determined by

Fig. 4.1. *Admissible values of Σ as a function of ρ*

the ratio $\rho = \frac{\alpha}{\beta}$ through the characteristic polynomial,

$$\det\Big(\lambda^2(\rho A^{[1]} + A^{[2]}) + \lambda(\rho B^{[1]} + B^{[2]}) - ((\rho A^{[1]} + A^{[2]}))\Upsilon_1^2 + (\rho B^{[1]} + B^{[2]})\Upsilon_1)\Big) = 0.$$

In Figure 4.1 we sketch the admissible values of Σ as a function ρ. Note that the quadratic pencil becomes singular when ρ is an eigenvalue of the linear pencil $\rho A^{[1]} + A^{[2]}$. In Figure 4.1, this happens at approximately $\rho \approx -0.2388$. The other $-\infty$ to ∞ jump depicted in the lower drawing of Figure 4.1 indicates $\Sigma = 0$ at approximately $\rho \approx 0.9050$. It is seen empirically that Σ can be *arbitrary* real numbers. However, in order to keep the eigenstructure (X_2, Λ_2) in the updated model, the admissible eigenvectors Y corresponding to Σ form at most a two-dimensional manifold in $\mathbb{R}^5$.

5. Conclusion. One common procedure to mend the discrepancy between a mathematical model and the corresponding real-world system is to modify the model parameters in such a way so as to achieve a good correspondence between the analytic solution and the real data. In this paper we have considered one such model updating of self-adjoint quadratic pencils using a few measured natural frequencies and mode shapes. The model updating problem is cast as a quadratic inverse eigenvalue problem with prescribed eigenpairs.

We have given a constructive proof on that the QIEP with no damping can be solved with any number of arbitrarily assigned eigenpairs and that the QIEP with damping can be solved with up to maximal allowable k_{max} arbitrarily assigned eigenpairs. Consequently, updating with no spill-over is entirely possible for undamped quadratic pencils, whereas spill-over for damped quadratic pencils generally is unpreventable. Examples are given to demonstrate both the phenomenon of spill-over and the conditions under which no spill-over might be maintained. For the later, more work is still needed to characterize the conditions analytically.

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