ON A PARTIALLY DESCRIBED INVERSE QUADRATIC EIGENVALUE PROBLEM

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Abstract. The inverse eigenvalue problem of constructing square matrices M, C and K of size n for the quadratic pencil $Q(\lambda) \equiv \lambda^2 M + \lambda C + K$ so that $Q(\lambda)$ has a prescribed subset of eigenvalues and eigenvectors is considered. This paper offers a constructive proof showing that, given any $k \leq n$ distinct eigenvalues and linearly independent eigenvectors, the problem is solvable even under the restriction that M, C and K are all real and symmetric, and that M and K are positive definite and semi-definite, respectively. The construction also allows additional optimization conditions to be built into the solution so as to better refine the approximate pencil. The eigenstructure of the resulting $Q(\lambda)$ is completely analyzed.

Key words. quadratic eigenvalue problem, inverse eigenvalue problem, partially prescribed spectrum, partial eigenstructure assignment

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1. Introduction.

In a recent treatise, Tisseur and Meerbergen [9] surveyed the many applications, mathematical properties, and a variety of numerical techniques for the so called *quadratic eigenvalue problem* (QEP). The problem concerns, given $n \times n$ complex matrices M, C and K, finding scalars λ and nonzero vectors $\mathbf{x}$ such that

$$Q(\lambda)\mathbf{x} = 0, \tag{1.1}$$

where

$$Q(\lambda) := Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K \tag{1.2}$$

is called a *quadratic pencil* defined by M, C and K. The scalars λ and the corresponding vectors $\mathbf{x}$ are called, respectively, eigenvalues and eigenvectors of the pencil. The QEP has received much attention because its formation has repeatedly arisen from many different areas of disciplines, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, and signal processing. It is known that the QEP has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M is nonsingular. Quadratic eigenvalue problems arising in practice often entail some additional conditions on the matrices. For example, if M, C and K represent the mass, damping and stiffness matrices, respectively, in a mass-spring system, then it is required that all matrices are real-valued and symmetric, and that M and K should be positive definite and semi-definite, respectively. It is this class of constraints on the matrix coefficients in (1.2) that underlines our main contribution in this paper.

In mathematical modelling, we generally assume that there is a correspondence between the endogenous variables, that is, the internal parameters, and the exogenous variables, that is, the external behavior. The process of analyzing and deriving the spectral information and, hence, inducing the dynamical behavior of a system from *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a *direct* problem. The *inverse* problem, in contrast, is to validate, determine, or estimate the parameters of the system according to its observed or expected behavior. The concern in the direct problem is

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to express the behavior in terms of the parameters whereas in the inverse problem the concern is to express the parameters in term of the behavior. The inverse problem is just as important as the direct problem in applications. There has been a lot of interest in the inverse eigenvalue problem, including the notable pole assignment problem. Some general reviews and extensive bibliographies in this regard can be found, for example, in the first author's recent articles [3] and [4]. This paper concerns the inverse problem of the QEP.

The term *inverse quadratic eigenvalue problem* (IQEP) adopted in the literature usually is for general matrix coefficients. In this paper we shall use it distinctively to stress the additional structure imposed upon the matrix coefficients: Determine real, symmetric matrix coefficients M, C and K with M positive definite and K positive semi-definite so that the resulting QEP has a prescribed set of eigenvalues. We note that in several recent works, including those by Chu and Datta [2], Nichols and Kautsky [8], as well as Datta, Elhay, Ram and Sarkissian [5], studies are undertaken toward a feedback design problem for a second-order control system. That consideration eventually leads to either a full or a partial eigenstructure assignment problem for the QEP. The proportional and derivative state feedback controller designated in these studies is capable of assigning specific eigenvalues and making the resulting system insensitive to perturbations. Nonetheless, these results cannot meet the basic requirement that the quadratic pencil be symmetric.

In a large or complicated physical system, it is often impossible to obtain the entire spectral information. Furthermore, quantities related to high frequency terms generally are susceptible to measurement errors due to the finite bandwidth of measuring devices. Spectral information, therefore, should not be used at its full extent. For these reasons, it might be more sensible to consider a *partially described IQEP* (PD-IQEP) where only a *portion* of eigenvalues and eigenvectors is prescribed. More precisely, by a PD-IQEP we refer to the following inverse problem:

(PD-IQEP) Let $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ $(k \le n)$ be a given pair of matrices where

$$\Lambda = \operatorname{diag}\{\lambda_1^{[2]}, \cdots, \lambda_\ell^{[2]}, \lambda_{2\ell+1}, \cdots, \lambda_k\} \tag{1.3}$$

with

$$\lambda_j^{[2]} = \begin{bmatrix} \alpha_j, & \beta_j \\ -\beta_j, & \alpha_j \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_j \neq 0, \quad \text{for} \quad j = 1, \cdots, \ell, \tag{1.4}$$

and

$$X = [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \cdots, \mathbf{x}_{\ell R}, \mathbf{x}_{\ell I}, \mathbf{x}_{2\ell+1}, \cdots, \mathbf{x}_k]. \tag{1.5}$$

Find symmetric matrices M, C and K with M and K positive definite and semi-definite, respectively, so that the equation

$$MX\Lambda^2 + CX\Lambda + KX = 0, \tag{1.6}$$

is satisfied.

We remark that if the above problem is solvable, then the true eigenvalues and eigenvectors are readily identifiable. Indeed, let

$$R = \operatorname{diag}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1, & 1 \\ i, & -i \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}}\begin{bmatrix} 1, & 1 \\ i, & -i \end{bmatrix}, I_{k-2\ell}\right\}, \tag{1.7}$$

with $i = \sqrt{-1}$. We could induce the true eigenvalues and eigenvectors from (Λ, X) by defining $\tilde{\Lambda} = R^H \Lambda R$ and $\tilde{X} = XR$, respectively. Note that

$$\begin{aligned} \tilde{\Lambda} &= \operatorname{diag}\left\{\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_{2\ell-1}, \tilde{\lambda}_{2\ell}, \tilde{\lambda}_{2\ell+1}, \cdots, \tilde{\lambda}_k\right\}, \\ \tilde{X} &= [\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \cdots, \tilde{\mathbf{x}}_{2\ell-1}, \tilde{\mathbf{x}}_{2\ell}, \tilde{\mathbf{x}}_{2\ell+1}, \cdots, \tilde{\mathbf{x}}_k], \end{aligned} \tag{1.8}$$

where $\tilde{\mathbf{x}}_{2j-1} = \mathbf{x}_{jR} + i\mathbf{x}_{jI}$, $\tilde{\mathbf{x}}_{2j} = \mathbf{x}_{jR} - i\mathbf{x}_{jI}$, $\tilde{\lambda}_{2j-1} = \alpha_j + i\beta_j$, $\tilde{\lambda}_{2j} = \alpha_j - i\beta_j$, for $j = 1, \cdots, \ell$, and $\tilde{\mathbf{x}}_j = \mathbf{x}_j$, $\tilde{\lambda}_j = \lambda_j$, for $j = 2\ell+1, \cdots, k$. For convenience, we shall denote henceforth the set of diagonal elements of $\tilde{\Lambda}$, which is precisely the spectrum of Λ, by $\sigma(\Lambda)$. We shall also call the pair of matrices (Λ, X), with somewhat harmless ambiguity, an *eigenpair* of the quadratic pencil $Q(\lambda)$. Our study in this paper stems from the speculation that the notion of the PD-IQEP has the potential of leading to an important modification tool for model updating, model tuning, and model correction [1, 7, 10], when compared with an analytical model. We will discuss this specific application in a separate paper.

It seems appropriate to attribute the first technique for solving the inverse problem of QEP to a short exposition in the book [6, p. 173]. Unfortunately, the method derived from that discussion is not even capable of producing symmetric C and K. On the other hand, a natural question before formulating a PD-IQEP is to ask how many pairs of eigenvalues and eigenvectors can be prescribed so that the PD-IQEP remains solvable. Our contribution in this paper is fourfold: First, we give a recipe for solving the PD-IQEP numerically. Secondly, we specify necessary and sufficient conditions under which the PD-IQEP is solvable. Thirdly, we completely characterize the eigenstructure of the reconstructed quadratic pencil. Finally, we propose a way to refine the construction process so that the best approximation subject to some additional optimal conditions can be established.

2. Solving PD-IQEP.

In this section we present a general theory elucidating how the PD-IQEP could be solved with the prescribed spectral information (Λ, X). Our proof is constructive. As a by-product, numerical algorithms can also be developed thence. Examples of numerical schemes and applications will be discussed in Section 4. We shall assume henceforth, in the formulation of an PD-IQEP, that the given spectral information (Λ, X) is always in the form of (1.3) and (1.5).

Starting with the given pair of matrices (Λ, X), consider the null space $\mathcal{N}(\Omega)$ of the augmented matrix

$$\Omega := \begin{bmatrix} X^\top, & \Lambda^\top X^\top \end{bmatrix} \in \mathbb{R}^{k \times 2n}.$$

Denote the dimension of $\mathcal{N}(\Omega)$ by m. If X has linearly independently columns (as we will assume later), then $m = 2n - k$. Note that $m \geq n$, since we have assumed $k \leq n$ (for the reason to be seen later) in the formulation of the PD-IQEP. Let the columns of the matrix

$$\begin{bmatrix} U^\top \\ V^\top \end{bmatrix} \in \mathbb{R}^{2n \times m}$$

with $U^\top, V^\top \in \mathbb{R}^{n \times m}$ denote *any* basis of the subspace $\mathcal{N}(\Omega)$. The equation

$$\begin{bmatrix} X^\top, & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} U^\top \\ V^\top \end{bmatrix} = 0 \tag{2.1}$$

holds. Define the quadratic pencil $Q(\lambda)$ by the matrix coefficients

$$M = V^\top V, \tag{2.2}$$

$$C = V^\top U + U^\top V, \tag{2.3}$$

$$K = U^\top U. \tag{2.4}$$

We claim that the above definitions are sufficient for constructing a solution to the PD-IQEP. The theory will be established in several steps.

THEOREM 2.1. *Given any pair of matrices (Λ, X) in the form of (1.3) and (1.5), let U and V be an arbitrary solution to the equation (2.1). Then (Λ, X) is an eigenpair of the quadratic* *pencil $Q(\lambda)$ with matrix coefficients M, C and K defined according to (2.2), (2.3) and (2.4), respectively.*

Proof. Upon substitution, we see that

$$\begin{aligned} MX\Lambda^2 + CX\Lambda + KX &= V^\top VX\Lambda^2 + \left(V^\top U + U^\top V\right) X\Lambda + \left(U^\top U\right) X \\ &= V^\top \left(VX\Lambda + UX\right)\Lambda + U^\top \left(VX\Lambda + UX\right) = 0. \end{aligned}$$

The last equality is due to the properties of U and V in (2.1). ◻

By this construction, all matrix coefficients in $Q(\lambda)$ are obviously real and symmetric. Note also that both matrices M and K are positive semi-definite. However, it is not clear whether $Q(\lambda)$ is a trivial quadratic pencil. Toward that end, we claim that the assumption that X has full column rank is sufficient and necessary for the regularity of $Q(\lambda)$.

THEOREM 2.2. *The leading matrix coefficient $M = V^\top V$ is nonsingular, provided that X has full column rank. In this case, the resulting quadratic pencil $Q(\lambda)$ is regular, that is,* $\det(Q(\lambda))$ *is not identically zero.*

Proof. Suppose that $V^\top \in \mathbb{R}^{n\times m}$ is not of full row rank. There exists an orthogonal matrix $G \in \mathbb{R}^{m\times m}$ such that

$$V^\top G = \left[\begin{array}{cc} V_1^\top, & 0_{n\times m_2} \end{array}\right],$$

where $V_1^\top \in \mathbb{R}^{n\times m_1}$ and $0_{n\times m_2}$ denotes the zero matrix of size $n \times m_2$. Note that $m_1 < n$ and $m_2 = m - m_1$. Postmultiply the same G to $U^\top$ and partition the product into

$$U^\top G = \left[\begin{array}{cc} U_1^\top, & U_2^\top \end{array}\right],$$

where $U_1^\top \in \mathbb{R}^{n\times m_1}$ and $U_2^\top \in \mathbb{R}^{n\times m_2}$. Note that $m_2 > m - n$. On the other hand, we see from (2.1) that

$$X^\top U_2^\top = 0,$$

whereas the column of $U_2^\top$ are necessarily linearly independent by construction. It follows that

$$n - k \geq m_2 > m - n,$$

which contradicts with the fact that $m = 2n - k$. Thus, the matrix $V^\top$ must be of full row rank and then $M = V^\top V$ is nonsingular. ◻

THEOREM 2.3. *Suppose in a given pair of matrices* (Λ, X) *that all eigenvalues in* Λ *are distinct and that X is not of full column rank. Then the quadratic pencil $Q(\lambda)$ defined by (2.2), (2.3) and (2.4) is singular.*

Proof. It is easy to check that the equation (2.1) remains true if X and Λ are replaced by $\tilde{X}$ and $\tilde{\Lambda}$, respectively. Let μ be an arbitrary complex number not in $\sigma(\Lambda)$. Observe that

$$\left[\begin{array}{cc} \tilde{X}^\top, & \tilde{\Lambda}^\top \tilde{X}^\top \end{array}\right] \left[\begin{array}{cc} I, & -\mu I \\ 0, & I \end{array}\right] \left[\begin{array}{cc} I, & \mu I \\ 0, & I \end{array}\right] \left[\begin{array}{c} U^\top \\ V^\top \end{array}\right] = 0.$$

It follows that

$$\left[\begin{array}{cc} \tilde{X}^\top, & (\widetilde{\Lambda}^\top - \mu I)\widetilde{X}^\top \end{array}\right] \left[\begin{array}{c} \mu V^\top + U^\top \\ V^\top \end{array}\right] = 0.$$

By assumption, $\tilde{X}$ is not of full column rank. We may therefore assume that for some $2 \leq q \leq k$,

$$\tilde{\mathbf{x}}_q = \sum_{j=1}^{q-1} r_j \tilde{\mathbf{x}}_j,$$

where not all r_j, $j = 1, \cdots, q-1$, are zero. Define

$$\Gamma := \begin{bmatrix} 1 & & & r_{1,q} & & 0 \\ & \ddots & & \vdots & & \\ & & \ddots & r_{q-1,q} & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix} \in \mathbb{C}^{k\times k},$$

with $r_{j,q} = -\frac{\lambda_q - \mu}{\lambda_j - \mu} r_j$, $j = 1, \cdots, q-1$. Clearly,

$$\Gamma^\top \left[\begin{array}{cc} \tilde{X}^\top, & (\tilde{\Lambda}^\top - \mu I)\tilde{X}^\top \end{array} \right] \left[\begin{array}{c} \mu V^\top + U^\top \\ V^\top \end{array} \right] = 0. \tag{2.5}$$

Notice that, by construction, the q-th row of $\Gamma^\top(\tilde{\Lambda}^\top - \mu I)\tilde{X}^\top$ is zero. Let $y(\mu)^\top$ denote the q-th row of $\Gamma^\top \tilde{X}^\top$, which cannot be identically zero because the spectrum of Λ are distinct. We thus see from (2.5) that

$$y(\mu)^\top \left(\mu V^\top + U^\top\right) = 0.$$

It follows that $y(\mu)^\top Q(\mu) = 0$. Since $\mu \in \mathbb{C}$ is arbitrary, $Q(\lambda)$ must be singular. □

We conclude this section with one important remark. The rank condition $k = n$ plays a pivotal role in PD-IQEP. It is the critical value for the regularity of the quadratic pencil $Q(\lambda)$ defined by the matrix coefficients (2.2), (2.3) and (2.4). In fact, it is clear now that corresponding to any given $\Lambda \in \mathbb{R}^{k\times k}$, $X \in \mathbb{R}^{n\times k}$ in the form of (1.3) and (1.5), the quadratic pencil $Q(\lambda)$ can always be factorized into the product

$$Q(\lambda) = \left(\lambda V^\top + U^\top\right)(\lambda V + U). \tag{2.6}$$

If $k > n$, then $\operatorname{rank}(\lambda V + U) \le 2n - k < n$ and hence $\det(Q(\lambda)) \equiv 0$ for all λ. It is for this reason that we always assume that $k \le n$ in the formulation of an PD-IQEP.

3. Eigenstructure of $Q(\lambda)$. We have shown in the preceding section how to define the matrix coefficients so that the corresponding quadratic pencil possesses a prescribed set of k eigenvalues and eigenvectors. The PD-IQEP thereby is solved via construction. An interesting question to ask is how the unspecified eigenpair in the constructed pencil should look like. In this section we examine the remaining eigenstructure of the quadratic pencil $Q(\lambda)$ created from our scheme.

THEOREM 3.1. *Let $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ in the form of (1.3) and (1.5) denote the prescribed eigenpair and $Q(\lambda)$ be the quadratic pencil defined by coefficients (2.2), (2.3) and (2.4). Assume that X has full column rank k.*

1. *If $k = n$, then $Q(\lambda)$ has double eigenvalue $\tilde{\lambda}_j$ for each $\tilde{\lambda}_j \in \sigma(\Lambda)$;*
2. *If $k < n$, then $Q(\lambda)$ has double eigenvalue $\tilde{\lambda}_j$ for each $\tilde{\lambda}_j \in \sigma(\Lambda)$. The remaining $2(n-k)$ eigenvalues of $Q(\lambda)$ are all complex conjugate with nonzero imaginary parts. In addition, if the matrices U and V in (2.1) are chosen from an orthogonal basis of the null space of Ω, then the remaining $2(n-k)$ eigenvalues are only $\pm i$ with corresponding eigenvectors $\mathbf{z} \pm i\mathbf{z}$ where $X^\top \mathbf{z} = 0$.*

Proof. The case $k = n$ is easy. The matrices $U^\top$ and $V^\top$ involved in (2.1) forming the null space of Ω are square matrices of size n. We also know from Theorem 2.2 that $V^\top$ is nonsingular. Observe that

$$V^{-1}U = -X\Lambda X^{-1}. \tag{3.1}$$

Using the factorization (2.6), we see that

$$\det(Q(\lambda)) = (\det(\lambda V + U))^2 .$$

It is clear that $Q(\lambda)$ has double eigenvalue $\tilde{\lambda}_j$ at every $\tilde{\lambda}_j \in \sigma(\Lambda)$.

We now consider the case when $k < n$. Since $X^\top \in \mathbb{R}^{k\times n}$ is of full row rank, there exists an orthogonal matrix $P_1 \in \mathbb{R}^{n\times n}$ such that

$$X^\top P_1^\top = \begin{bmatrix} X_{11}^\top, & 0_{n\times(n-k)} \end{bmatrix}, \tag{3.2}$$

where $X_{11}^\top \in \mathbb{R}^{k\times k}$ is nonsingular. There also exists an orthogonal matrix $Q_1 \in \mathbb{R}^{m\times m}$ such that

$$P_1 V^\top Q_1 = \begin{bmatrix} V_{11}^\top, & 0_{k\times(n-k)}, & 0_{k\times(m-n)} \\ V_{21}^\top, & \mathcal{A}, & 0_{(n-k)\times(m-n)} \end{bmatrix} \in \mathbb{R}^{n\times m}, \tag{3.3}$$

with appropriate sizes for the other three submatrices. In particular, note that both $V_{11}^\top \in \mathbb{R}^{k\times k}$ and $\mathcal{A} \in \mathbb{R}^{(n-k)\times(n-k)}$ are nonsingular matrices, because $V^\top$ is of full row rank by Theorem 2.2. From the fact that

$$\begin{bmatrix} X^\top, & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} P_1^\top, & 0 \\ 0, & P_1^\top \end{bmatrix} \begin{bmatrix} P_1, & 0 \\ 0, & P_1 \end{bmatrix} \begin{bmatrix} U^\top \\ V^\top \end{bmatrix} Q_1 = 0, \tag{3.4}$$

we conclude that the structure of $P_1 U^\top Q_1$ must be of the form

$$P_1 U^\top Q_1 = \begin{bmatrix} U_{11}^\top, & 0_{k\times(n-k)}, & 0_{k\times(m-n)} \\ U_{21}^\top, & \Delta, & \mathcal{B} \end{bmatrix} \in \mathbb{R}^{n\times m}, \tag{3.5}$$

where $\mathcal{B} \in \mathbb{R}^{(n-k)\times(n-k)}$ is nonsingular. Because that $\begin{bmatrix} U^\top \\ V^\top \end{bmatrix}$ is of full column rank, together with the fact that both $\mathcal{A}$ and $\mathcal{B}$ in (3.3) and (3.5) are nonsingular, it follows that $\begin{bmatrix} U_{11}^\top \\ V_{11}^\top \end{bmatrix}$ must be of full column rank. Note that $U_{11}^\top$ is nonsingular if and only if Λ has no zero eigenvalue. Using $V_{11}^\top$ as a pivot matrix to eliminate $V_{21}^\top$ in (3.3), we may claim that there exits a nonsingular matrix P_2 such that

$$\tilde{U}^\top := P_2 P_1 U^\top Q_1 = \begin{bmatrix} U_{11}^\top, & 0, & 0 \\ \tilde{U}_{21}^\top, & \Delta, & \mathcal{B} \end{bmatrix},$$
$$\tilde{V}^\top := P_2 P_1 V^\top Q_1 = \begin{bmatrix} V_{11}^\top, & 0, & 0 \\ 0, & \mathcal{A}, & 0 \end{bmatrix}.$$

Compute the three matrices

$$\tilde{M} := \tilde{V}^\top \tilde{V} = \begin{bmatrix} V_{11}^\top V_{11}, & 0 \\ 0, & \mathcal{A}\mathcal{A}^\top \end{bmatrix},$$
$$\tilde{C} := \tilde{U}^\top \tilde{V} + \tilde{V}^\top \tilde{U} = \begin{bmatrix} U_{11}^\top V_{11} + V_{11}^\top U_{11}, & V_{11}^\top \tilde{U}_{21} \\ \tilde{U}_{21}^\top V_{11}, & \mathcal{A}\Delta^\top + \Delta\mathcal{A}^\top \end{bmatrix},$$
$$\tilde{K} := \tilde{U}^\top \tilde{U} = \begin{bmatrix} U_{11}^\top U_{11}, & U_{11}^\top \tilde{U}_{21} \\ \tilde{U}_{21}^\top U_{11}, & \tilde{U}_{21}^\top \tilde{U}_{21} + \mathcal{B}\mathcal{B}^\top + \Delta\Delta^\top \end{bmatrix},$$

and define the quadratic pencil $\tilde{Q}(\lambda) := \lambda^2 \tilde{M} + \lambda \tilde{C} + \tilde{K}$. By construction, it is clear that $\tilde{Q}(\lambda) = (P_2 P_1)\, Q(\lambda)(P_2 P_1)^\top$. This congruence relation ensures that $\tilde{Q}(\lambda)$ preserves the same eigenvalue information as $Q(\lambda)$. Define

$$Q_{11}(\lambda) := \lambda^2(V_{11}^\top V_{11}) + \lambda(V_{11}^\top U_{11} + U_{11}^\top V_{11}) + U_{11}^\top U_{11}, \tag{3.6}$$

$$P_3 := \begin{bmatrix} I, & 0 \\ -\tilde{U}_{21}^\top(\lambda V_{11} + U_{11})Q_{11}^{-1}(\lambda), & I \end{bmatrix}. \tag{3.7}$$

It is further seen that $\tilde{Q}(\lambda)$ can be factorized as

$$P_3 \begin{bmatrix} Q_{11}(\lambda), & 0 \\ 0, & (\lambda\mathcal{A} + \Delta)(\lambda\mathcal{A}^\top + \Delta^\top) + \mathcal{B}\mathcal{B}^\top \end{bmatrix} P_3^\top. \tag{3.8}$$

We thus have effectively decompose the quadratic pencil $\tilde{Q}(\lambda)$ into two subpencils.

By construction, we see from (3.2), (3.4) and (3.6) that the quadratic subpencil $Q_{11}(\lambda)$ in (3.6) solves exactly the PD-IQEP with spectral data (Λ, X_{11}). For this problem, we have already proved in the first part that $Q_{11}(\lambda)$ must have double eigenvalue λ_j for each $\lambda_j \in \tilde{\Lambda}$. It only remains to check the eigenvalues for the subpencil $(\mu\mathcal{A} + \Delta)(\mu\mathcal{A}^\top + \Delta^\top) + \mathcal{B}\mathcal{B}^\top$. Recall that the matrix $\mathcal{B}$ in (3.5) is nonsingular. The matrix $(\mu\mathcal{A} + \Delta)(\mu\mathcal{A}^\top + \Delta^\top) + \mathcal{B}\mathcal{B}^\top$ is positive definite for every $\mu \in \mathbb{R}$. In particular, its determinant cannot be zero for any real μ. Therefore, the remaining eigenvalues of $Q(\lambda)$ must be all complex conjugate with nonzero imaginary parts.

If, in addition, the columns of $\begin{bmatrix} U^\top \\ V^\top \end{bmatrix}$ in (3.4) are orthogonal to begin with, then both $\mathcal{A}$ and $\mathcal{B}$ are $(n-k) \times (n-k)$ orthogonal matrices and the submatrix Δ in (3.5) must be a zero matrix. By (3.8), the remaining eigenvalues of $Q(\lambda)$ can only be $\pm i$. Observer further that there exists a nonsingular $W \in \mathbb{R}^{k \times k}$ such that

$$\begin{bmatrix} I, & 0 \\ 0, & W \end{bmatrix} \begin{bmatrix} U, & V \\ X^\top, & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} U^\top, & X \\ V^\top, & X\Lambda \end{bmatrix} \begin{bmatrix} I, & 0 \\ 0, & W^\top \end{bmatrix} = \begin{bmatrix} I_{2n-k}, & 0 \\ 0, & I_k \end{bmatrix}. \tag{3.9}$$

It follows that

$$\begin{aligned} &U^\top U + XW^\top WX^\top = I_n, \\ &U^\top V + XW^\top W\Lambda^\top X^\top = 0, \\ &V^\top U + X\Lambda W^\top WX^\top = 0, \\ &V^\top V + X\Lambda W^\top W\Lambda^\top X^\top = I_n. \end{aligned} \tag{3.10}$$

For any $\mathbf{z}$ satisfying $X^\top\mathbf{z} = 0$, we see from the above equations that

$$\begin{aligned} U^\top U\mathbf{z} &= \mathbf{z}, \\ V^\top V\mathbf{z} &= \mathbf{z}, \\ U^\top V\mathbf{z} + V^\top U\mathbf{z} &= 0. \end{aligned}$$

This show that $Q(\pm i)(\mathbf{z} \pm i\mathbf{z}) = 0$. □

Theorem 3.1 is significant in several fronts. First, if $k = n$, then *all* eigenvalues of $Q(\lambda)$ are completely counted. Secondly, if $k < n$ and if the basis of null space $\mathcal{N}(\Omega)$ are selected to be mutually orthogonal (as we normally would do by using, say, MATLAB), then again all eigenvalues of $Q(\lambda)$ are completed determined. In other words, we are *not* allowed to supplement any additional $n - k$ eigenpairs to simplify this PD-IQEP. The solution of our method for PD-IQEP is the most natural way for $k(< n)$ prescribed pairs of eigenvalues and eigenvectors. In Section 4, we shall study how non-orthogonal basis of $\mathcal{N}(\Omega)$ can help to improve the PD-IQEP approximation.

We can further calculate the geometric multiplicity of the double roots characterized in Theorem 3.1.

THEOREM 3.2. *Let (Λ, X) in the form of (1.3) and (1.5) denote the prescribed eigenpair of the quadratic pencil $Q(\lambda)$ defined before. Assume that Λ has distinct spectrum and X has full column rank. Then*

1. *Each real-valued $\tilde{\lambda}_j \in \sigma(\Lambda)$ has an elementary divisor of degree 2, that is, the dimension of the null space $\mathcal{N}(Q(\tilde{\lambda}_j))$ is 1.*
2. *The dimension of $\mathcal{N}(Q(\tilde{\lambda}_j))$ of a complex-valued eigenvalue $\tilde{\lambda}_j \in \sigma(\Lambda)$ is generically 1. That is, pairs of matrices (Λ, X) of which a complex-valued eigenvalue has a linear elementary divisor forms a measure zero set.*

Proof. Real-valued eigenvalues correspond to those $\tilde{\lambda}_j \in \sigma(\Lambda)$ with $j = 2\ell + 1, \ldots, k$. We have already seen in Theorem 2.1 that $Q(\tilde{\lambda}_j)\mathbf{x}_j = 0$, where $\mathbf{x}_j$ is the j-th column of X. Suppose that the $\mathcal{N}(Q(\tilde{\lambda}_j))$ has dimension greater than 1. From (2.6), it must be that

$$\operatorname{rank}\left(\tilde{\lambda}_j V^\top + U^\top\right) \leq n - 2. \tag{3.11}$$

Rewrite (2.1) as

$$\begin{bmatrix} X^\top, & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} I, & -\tilde{\lambda}_j I \\ 0, & I \end{bmatrix} \begin{bmatrix} I, & \tilde{\lambda}_j I \\ 0, & I \end{bmatrix} \begin{bmatrix} U^\top \\ V^\top \end{bmatrix} = 0, \tag{3.12}$$

which is equivalent to

$$\begin{bmatrix} X^\top, & (\Lambda^\top - \tilde{\lambda}_j I) X^\top \end{bmatrix} \begin{bmatrix} \tilde{\lambda}_j V^\top + U^\top \\ V^\top \end{bmatrix} = 0. \tag{3.13}$$

Note that, since Λ has distinct spectrum and $X^\top$ has full row-rank,

$$\operatorname{rank}\left((\Lambda^\top - \tilde{\lambda}_j I) X^\top\right) = k - 1,$$

or equivalently,

$$\dim\left(\mathcal{N}\left((\Lambda^\top - \tilde{\lambda}_j I) X^\top\right)\right) = n - k + 1. \tag{3.14}$$

On the other hand, there exists an orthogonal $G_j \in \mathbb{R}^{m \times m}$ such that

$$\begin{bmatrix} \tilde{\lambda}_j V^\top + U^\top \\ V^\top \end{bmatrix} G_j = \begin{bmatrix} U_{j1}^\top, & 0 \\ V_{j1}^\top, & V_{j2}^\top \end{bmatrix}, \tag{3.15}$$

where, due to (3.11), $V_{j2}^\top$ has at least $m - (n-2) = n - k + 2$ linearly independent columns. We then see from (3.13) that

$$(\Lambda^\top - \tilde{\lambda}_j I) X^\top V_{j2}^\top = 0,$$

implying that $\dim\left(\mathcal{N}\left((\Lambda^\top - \tilde{\lambda}_j I) X^\top\right)\right) \geq n - k + 2$. This contradicts with (3.14).

To examine the complex-valued case, notice that (1.6) can be rewritten as

$$M(XR)(R^H \Lambda^2 R) + C(XR)(R^H \Lambda R) + KXR = 0,$$

where R is defined in (1.7). In particular, for $1 \leq j \leq 2\ell$, we have

$$Q(\tilde{\lambda}_j)\tilde{\mathbf{x}}_j = 0.$$

We first consider the case $k = n$. Two observations are due at the moment. First, the matrix V in the basis $\begin{bmatrix} U^\top \\ V^\top \end{bmatrix}$ for the null space $\mathcal{N}\left(\left[X^\top, \Lambda^\top X^\top\right]\right)$ can be an arbitrary nonsingular matrix. Secondly, if there exists another vector $\mathbf{z} \in \mathbb{C}^n$ independent of $\tilde{\mathbf{x}}_j$ such that $Q(\tilde{\lambda}_j)\mathbf{z} = 0$, we claim that for this kind of eigenvalue the matrix $(V^\top V)^{-1}$ must satisfy some kind of algebraic varieties in $\mathbb{R}^{n\times n}$. Putting these two facts together, we conclude that any complex-valued eigenvalue having a linear elementary divisor must come from a set of measure zero.

To see the claim concerning the algebraic varieties for $(V^\top V)^{-1}$, we use (3.1) to rewrite $\tilde{\lambda}_j V + U$ as

$$\tilde{\lambda}_j V + U = VXR(\tilde{\lambda}_j I - \tilde{\Lambda})R^H X^{-1},$$

and thus factorize $Q(\tilde{\lambda}_j)$ as

$$\begin{aligned} Q(\tilde{\lambda}_j) &= (\tilde{\lambda}_j V^\top + U^\top)(\tilde{\lambda}_j V + U) \\ &= X^{-\top}\bar{R}(\tilde{\lambda}_j I - \tilde{\Lambda})R^\top X^\top V^\top VXR(\tilde{\lambda}_j I - \tilde{\Lambda})R^H X^{-1}. \end{aligned} \tag{3.16}$$

If $Q(\tilde{\lambda}_j)\mathbf{z} = 0$, from (3.16) we have

$$R^\top X^\top V^\top VXR(\tilde{\lambda}_j I - \tilde{\Lambda})R^H X^{-1}\mathbf{z} = \tau \mathbf{e}_j, \tag{3.17}$$

where $\mathbf{e}_j$ is the jth standard unit vector and τ is some scalar. Rewrite (3.17) as

$$(\tilde{\lambda}_j I - \tilde{\Lambda})R^H X^{-1}\mathbf{z} = \tau R^H X^{-1}(V^\top V)^{-1}X^{-\top}\bar{R}\mathbf{e}_j.$$

Hence, a necessary condition for the existence of $\mathbf{z}$ is that $(V^\top V)^{-1}$ must satisfy the algebraic equation

$$\mathbf{e}_j^\top R^H X^{-1}(V^\top V)^{-1}X^{-\top}\bar{R}\mathbf{e}_j = 0. \tag{3.18}$$

We note in passing that the condition (3.18) for $(V^\top V)^{-1}$ is also sufficient since the above argument can be reversed to show the existence of a vector $\mathbf{z}$ in the null space of $Q(\tilde{\lambda}_j)$.

For the case $k < n$, a similar argument holds. Indeed, using to the decompositions (3.6) and (3.8) given in Theorem 3.1, a sufficient and necessary condition for the existence of $\mathbf{z}$ is exactly the same as (3.18) where X and V are replaced by X_{11} and V_{11}, respectively. In either case, outside the algebraic variety, the elementary divisor of a generically prescribed complex eigenvalues therefore is of degree 2. □

To further demonstrate the subtlety of the second statement in Theorem 3.2, we make an interesting observation as follows.

Corollary 3.3. *Suppose in the given* (Λ, X) *that* X *has full column rank and that* Λ *has distinct spectrum. Assume further that* $\{\pm i\} \subset \sigma(\Lambda)$. *Construct the quadratic pencil* $Q(\lambda)$ *by taking an orthogonal basis* $[U, V]^\top$ *for the null space* $\mathcal{N}(\Omega)$. *Then the dimension of* $\mathcal{N}(Q(\pm i))$ *is 2. In other words, in this non-generic case, both eigenvalues* $\pm i$ *have two linear elementary divisors.*

Proof. From (3.9), we have $W(X^\top X + \Lambda^\top X^\top X\Lambda)W^\top = I_n$. It follows that

$$W^\top W = (X^\top X + \Lambda^\top X^\top X\Lambda)^{-1}.$$

The last equation in (3.10) gives rise to

$$(V^\top V)^{-1} = (I - X\Lambda W^\top W\Lambda^\top X^\top)^{-1}.$$

Upon substitution, it holds that

$$\begin{aligned}
X^{-1}(V^\top V)^{-1}X^{-\top} &= X^{-1}(I - X\Lambda W^\top W\Lambda^\top X^\top)^{-1}X^{-\top} \\
&= X^{-1}(I - X\Lambda(X^\top X + \Lambda^\top X^\top X\Lambda)^{-1}\Lambda^\top X^\top)^{-1}X^{-\top} \\
&= X^{-1}(I - X\Lambda X^{-1}(I + X^{-\top}\Lambda^\top X^\top X\Lambda X^{-1})^{-1}X^\top\Lambda^\top X^\top)^{-1}X^{-\top} \\
&= X^{-1}(I + X\Lambda X^{-1}X^{-\top}\Lambda^\top X^\top)X^{-\top} \\
&= X^{-1}X^{-\top} + \Lambda X^{-1}X^{-\top}\Lambda^\top, \qquad (3.19)
\end{aligned}$$

where the forth equality is, after some algebraic manipulation, due to the Sherman-Morrison-Woodburg formula. Substituting (3.19) into (3.18) and assuming that j is the index that defines $\tilde{\lambda}_j = \pm i$, we find that

$$\begin{aligned}
\mathbf{e}_j^\top R^H X^{-1}(V^\top V)^{-1}X^{-\top}\bar{R}\mathbf{e}_j &= \mathbf{e}_j^\top(R^H X^{-1}X^{-\top}\bar{R} + R^H\Lambda R R^H X^{-1}X^{-\top}\bar{R}R^\top\Lambda^\top\bar{R})\mathbf{e}_j \\
&= \mathbf{e}_j^\top(\tilde{X}^{-1}\tilde{X}^{-\top} + \tilde{\Lambda}\tilde{X}^{-1}\tilde{X}^{-\top}\tilde{\Lambda}^\top)\mathbf{e}_j = 0.
\end{aligned}$$

The sufficient condition is met and, therefore, $\dim(\mathcal{N}(Q(\pm i))) = 2$. ☐

4. Numerical Experiment. In this section we intend to highlight two main points by numerical examples. The first example demonstrates the eigenstructure of a solution to a typical PD-IQEP. From the discussion in the preceding sections, we already have a pretty good idea about how the eigenstructure should look like. We now demonstrate numerically how the selection of U and V might affect the geometric multiplicity of the double eigenvalue. The second example has important meaning in applications. We demonstrate how some additional optimization constraints can be incorporated into the construction of a solution to PD-IQEP. These additional constraints are imposed by some logistic reasons with the hope to better approximate a real physical system. In this example, we also experiment with the effect of feeding various amount of information on eigenvalues and eigenvectors to the construction. In particular, we compare the discrepancy between a given (analytic) quadratic pencil and the resulting PD-IQEP approximation by varying the values of k and the optimal constraints. All calculation are done by using MATLAB in its full precision. For the ease of running text, however, we shall report all numerals in 5 digits only.

Example 4.1. Consider the PD-IQEP where the partial eigenstructure $(\Lambda,\ X) \in \mathbb{R}^{5\times 5} \times \mathbb{R}^{5\times 5}$ is randomly generated. Assume

$$X = \begin{bmatrix} -0.4132 & 5.2801 & 2.9437 & -6.6098 & -9.6715 \\ -4.3518 & 3.2758 & -5.1656 & 9.1024 & -9.1357 \\ -0.1336 & -4.0588 & 2.5321 & 3.3049 & -4.4715 \\ -5.1414 & 4.4003 & -2.2721 & 5.2872 & 6.9659 \\ 8.6146 & -4.0112 & -6.9380 & 1.4345 & -4.4708 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -0.2168 & -4.3159 & 0 & 0 & 0 \\ 4.3159 & -0.2168 & 0 & 0 & 0 \\ 0 & 0 & 2.0675 & -0.9597 & 0 \\ 0 & 0 & 0.9597 & 2.0675 & 0 \\ 0 & 0 & 0 & 0 & -0.3064 \end{bmatrix}.$$

Choose a basis $\begin{bmatrix} U_1^\top \\ V_1^\top \end{bmatrix}$ for the null space $\mathcal{N}\left(\left[X^\top, \Lambda^\top X^\top\right]\right)$, say,

$$U_1^\top = \begin{bmatrix} 0.26861 & 0.56448 & -0.08687 & 0.39491 & -0.24252 \\ 0.32690 & -0.24385 & 0.00804 & -0.32844 & 0.42471 \\ -0.33739 & 0.27725 & -0.15949 & -0.05883 & 0.58406 \\ -0.13374 & 0.43824 & 0.09638 & 0.28605 & 0.46936 \\ -0.42433 & 0.17867 & 0.69977 & -0.12829 & -0.16140 \end{bmatrix},$$

$$V_1^\top = \begin{bmatrix} 0.51817 & 0.09467 & 0.20341 & -0.04075 & 0.32693 \\ 0.25575 & 0.38674 & -0.09339 & -0.32830 & -0.22850 \\ 0.31749 & -0.02297 & 0.63841 & 0.01156 & 0.05987 \\ -0.02434 & -0.40196 & 0.09987 & 0.65755 & 0.09646 \\ 0.27184 & 0.02061 & -0.01859 & 0.30413 & -0.03669 \end{bmatrix}$$

and construct

$$Q_1(\lambda) = \lambda^2(V_1^\top V_1) + \lambda(V_1^\top U_1 + U_1^\top V_1) + (U_1^\top U_1).$$

This quadratic pencil has double eigenvalue $\tilde{\lambda}_j$ for each $\tilde{\lambda}_j \in \sigma(\Lambda)$, according to our theory. Furthermore, we compute the singular values of each $Q(\tilde{\lambda}_j)$ and find that

$$\begin{aligned} \mathrm{svd}(Q_1(-0.21683 \pm 4.3159i)) &= \{17.394, 15.039, 4.3974, 2.6136, 1.2483 \times 10^{-15}\}, \\ \mathrm{svd}(Q_1(2.0675 \pm 0.95974i)) &= \{5.9380, 4.9789, 1.1788, 0.45926, 4.6449 \times 10^{-16}\}, \\ \mathrm{svd}(Q_1(-0.30635)) &= \{1.0937, 1.0346, 0.89436, 0.18528, 3.8467 \times 10^{-17}\}, \end{aligned}$$

implying that the dimension of the null space $Q(\tilde{\lambda}_j)$ is precisely 1 for each $\tilde{\lambda}_j \in \sigma(\Lambda)$.

However, suppose we choose a special basis for $\mathcal{N}\left(\left[X^\top, \Lambda^\top X^\top\right]\right)$ by

$$\begin{bmatrix} U_2^\top \\ V_2^\top \end{bmatrix} = \begin{bmatrix} U_1^\top V_1^{-\top} X^{-1} \\ X^{-1} \end{bmatrix}$$

and construct

$$Q_2(\lambda) = \lambda^2(V_2^\top V_2) + \lambda(V_2^\top U_2 + U_2^\top V_2) + (U_2^\top U_2).$$

We find that

$$\begin{aligned} \mathrm{svd}(Q_2(-0.21683 \pm 4.3159i)) &= \{15.517,\ 0.12145,\ 0.07626,\ 3.4880 \times 10^{-15},\ 7.9629 \times 10^{-16}\}, \\ \mathrm{svd}(Q_2(2.0675 \pm 0.95974i)) &= \{21.064,\ 0.16325,\ 0.02540,\ 3.2321 \times 10^{-15},\ 5.2233 \times 10^{-16}\}, \\ \mathrm{svd}(Q_2(-0.30635)) &= \{20.995,\ 0.19733,\ 0.08264,\ 0.02977,\ 1.6927 \times 10^{-15}\}. \end{aligned}$$

In this case, each of the four the complex-valued eigenvalues of $\sigma(\Lambda)$ has linear elementary divisors.

Example 4.2. We can further exploit the freedom in the selection of basis for the null space $\mathcal{N}(\Omega)$. In this example we first demonstrate a few ways to select the basis under some special circumstances. We then illustrate the effect of available eigeninformation on the construction.

To fix the idea, we first generate randomly a 10×10 symmetric quadratic pencil $\hat{Q}(\lambda) = \lambda^2\hat{M} + \lambda\hat{C} + \hat{K}$, where $\hat{M}$ and $\hat{K}$ are also positive definite, as an analytic model. We then compare the effect of k on its PD-IQEP approximations for $k = 1, \ldots, 10$. To save the space, we shall not report the data of these test matrices $\hat{M}$, $\hat{C}$ and $\hat{K}$ in this paper, but will make them available upon request. We merely report that the spectrum of $\hat{Q}(\lambda)$ turns out to be the following 10 pairs of complex-conjugate values,

$$\{-0.27589 \pm 1.8585i,\ -0.19201 \pm 1.5026i,\ -0.15147 \pm 1.0972i,\ -0.11832 \pm 0.54054i,\\ -0.07890 \pm 1.3399i,\ -0.07785 \pm 0.76383i,\ -0.07716 \pm 0.86045i,\ -0.07254 \pm 1.1576i,\\ -0.06276 \pm 0.97722i,\ -0.05868 \pm 0.18925i\}.$$

These eigenvalues are not arranged in any specific order. Without loss of generality, we shall *pretend* that the first 5 pairs in the above list are the partially described eigenvalues and wish to reconstruct the quadratic pencil. For $\ell = 1, \ldots, 5$ (and hence $k = 2\ell$), denote these eigenvalues as $\alpha_\ell \pm \beta_\ell$ Also, define partial eigenpairs $(\Lambda_{2\ell},\ X_{2\ell})$ of $\hat{Q}(\lambda)$ according to (1.3) and (1.5), that is,

$$\Lambda_{2l} = \operatorname{diag}\left\{ \begin{bmatrix} \alpha_1, & \beta_1 \\ -\beta_1, & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_\ell, & \beta_\ell \\ -\beta_\ell, & \alpha_\ell \end{bmatrix} \right\}, \tag{4.1}$$

$$X_{2\ell} = [x_{1R}, x_{1I}, \ldots, x_{\ell R}, x_{\ell I}], \tag{4.2}$$

where $x_{\ell R} \pm i x_{\ell I}$ is the eigenvector of $\hat{Q}(\lambda)$ corresponding to $\alpha_\ell \pm i\beta_\ell$.

Let $\begin{bmatrix} U_\ell^\top \\ V_\ell^\top \end{bmatrix} \in \mathbb{R}^{2n\times(2n-2\ell)}$ be an orthogonal basis for $\mathcal{N}(\begin{bmatrix} X_{2\ell}^\top, & \Lambda_{2\ell}^\top X_{2\ell}^\top \end{bmatrix})$. We now introduce three ways to select a *new* basis for $\mathcal{N}(\begin{bmatrix} X_{2\ell}^\top, & \Lambda_{2\ell}^\top X_{2\ell}^\top \end{bmatrix})$, each of which is done for a different optimization purpose. The physical meaning of these optimal constraints will be explained at the end of this section.

Case 1. Suppose $\hat{K} = L_{\hat{K}} L_{\hat{K}}^\top$ and $\hat{M} = L_{\hat{M}} L_{\hat{M}}^\top$ are the Cholesky factorizations of $\hat{K}$ and $\hat{M}$ in the model pencil, respectively. Find a matrix $G_{\ell 1}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$ by solving the sequence of least-square problems

$$\min \left\| \begin{bmatrix} U_\ell^\top \\ V_\ell^\top \end{bmatrix} G_{\ell 1}^\top(:, j) - \begin{bmatrix} L_{\hat{K}}, & 0_{n-2\ell} \\ 0_{n-2\ell}, & L_{\hat{M}} \end{bmatrix} (:, j) \right\|_2, \tag{4.3}$$

for each of its columns $G_{\ell 1}^\top(:, j)$, $j = 1, \ldots, 2n - 2\ell$. For convenience, we have adopted here the MATLAB notation $(:, j)$ to denote the jth column of a matrix.

The solution of (4.3) is intended to, not only solve the PD-IQEP, but also best approximate the original $\hat{K}$ and $\hat{M}$ in the sense that the quantity

$$\|U_\ell^\top G_{\ell 1}^\top G_{\ell 1} U_\ell - \hat{K}\|_F + \|V_\ell^\top G_{\ell 1}^\top G_{\ell 1} V_\ell - \hat{M}\|_F. \tag{4.4}$$

is minimized among all possible $G_{\ell 1}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$. Once such a matrix $G_{\ell 1}^\top$ is found, we compute the coefficient matrices according to our recipe, that is,

$$\begin{aligned} M_{\ell 1} &= V_\ell^\top G_{\ell 1}^\top G_{\ell 1} V_\ell, \quad K_{\ell 1} = U_\ell^\top G_{\ell 1}^\top G_{\ell 1} U_\ell, \\ C_{\ell 1} &= U_\ell^\top G_{\ell 1}^\top G_{\ell 1} V_\ell + V_\ell^\top G_{\ell 1}^\top G_{\ell 1} U_\ell, \end{aligned} \tag{4.5}$$

and define the quadratic pencil

$$Q_{\ell 1}(\lambda) = \lambda^2 M_{\ell 1} + \lambda C_{\ell 1} + K_{\ell 1}, \tag{4.6}$$

according to $\ell = 1, \ldots, 5$.

Case 2. We first transform $V_\ell^\top$ to $[V_{\ell 0}^\top, 0]$ by an orthogonal transformation. Then we find a matrix $G_{\ell 2}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$ in the form

$$G_{\ell 2}^\top = \begin{bmatrix} E_{\ell 2}^\top, & 0 \\ 0, & F_{\ell 2}^\top \end{bmatrix}, \tag{4.7}$$

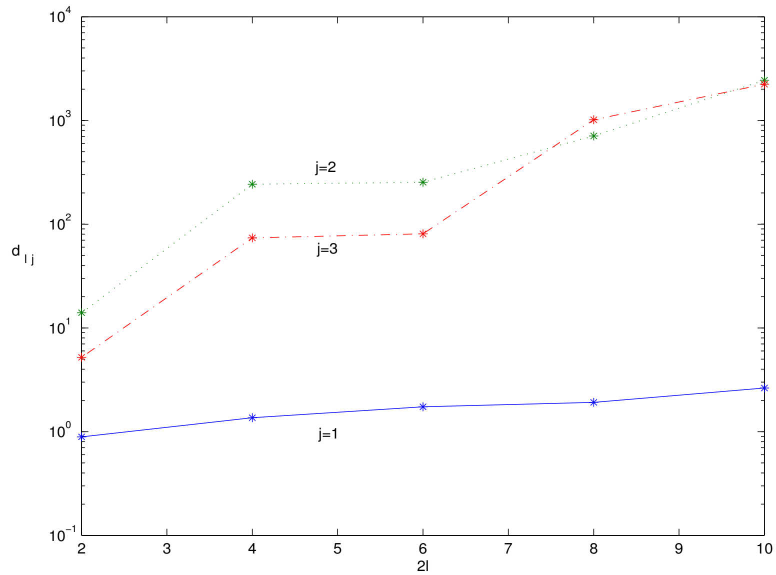

FIG. 4.1. *Errors of PD-IQEP approximations.*

where $E_{\ell 2}^\top = V_{\ell 0}^{-\top} L_{\hat{M}}$ and $F_{\ell 2}^\top$ is an arbitrary $(n-2\ell)\times(n-2\ell)$ orthogonal matrix.

Case 3. We transform $U_\ell^\top$ to $[U_{\ell 0}^\top, 0]$ by an orthogonal transformation. Then we find a matrix $G_{\ell 3}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$ in the form

$$G_{\ell 3}^\top = \begin{bmatrix} E_{\ell 3}^\top, & 0 \\ 0, & F_{\ell 3}^\top \end{bmatrix}, \tag{4.8}$$

where $E_{\ell 3}^\top = U_{\ell 0}^{-\top} L_{\hat{K}}$ and $F_{\ell 3}^\top$ is an arbitrary $(n-2\ell)\times(n-2\ell)$ orthogonal matrix.

The purpose of finding $G_{\ell 2}^\top$ and $G_{\ell 3}^\top$ in the form of (4.7) and (4.8) is to, not only solve the PD-IQEP, but also best approximate the original $\hat{M}$ and $\hat{K}$, respectively, in the sense that

$$\|V_\ell^\top G_{\ell 2}^\top G_{\ell 2} V_\ell - \hat{M}\|_F \tag{4.9}$$

and

$$\|U_\ell^\top G_{\ell 3}^\top G_{\ell 3} U_\ell - \hat{K}\|_F \tag{4.10}$$

are minimized by $G_{\ell 2}^\top$ and $G_{\ell 3}^\top$, respectively. Once these matrices are found, we define quadratic pencils $Q_{\ell 2}(\lambda)$ and $Q_{\ell 3}(\lambda)$ in exactly the same way as we define $Q_{\ell 1}(\lambda)$.

It would be interesting to see how the reconstructed quadratic pencils for the PD-IQEP, with the above-mentioned optimization in mind, approximate the original pencil. Toward that end, we define the measurement

$$d_{\ell j} = \|M_{\ell j} - \hat{M}\|_F + \|C_{\ell j} - \hat{C}\|_F + \|K_{\ell j} - \hat{K}\|_F, \tag{4.11}$$

for $j = 1, 2, 3$ and $\ell = 1, \ldots, 5$.

In Figure 4 we plot the *error* $d_{\ell j}$ between $\hat{Q}(\lambda)$ and $Q_{\ell j}(\lambda)$ for the various cases. Not surprisingly, we notice that the quadratic pencil $Q_{\ell 1}(\lambda)$ constructed from $G_{\ell 1}^\top$ is superior to the other two. What might be interesting to note is that in Case 1 the amount of eigeninformation available to the PD-QIEP does not seem to make any significance difference in the measurement of $d_{\ell 1}$. That is, all $d_{\ell 1}$ seems to be of the same order regardless of the value of ℓ. We think a reason for this happening is because $G_{\ell 1}^{\top}$ has somewhat more freedom to choose so that $M_{\ell 1}$ and $K_{\ell 1}$ better approximate $\hat{M}$ and $\hat{K}$, respectively.

In real application for vibrating systems, the stiffness matrix $\hat{K}$ and the mass matrix $\hat{M}$ of a mathematical model can usually be obtained by finite element or finite difference method. It is the damping matrix $\hat{C}$ in such a system that is generally not known. If some partial eigenstructure can be measured by experiment, then the construction proposed in Case 1 might be a good way to recover the original system by best approximating the stiffness matrix and the mass matrix in the sense of minimizing (4.4).

5. Conclusions. In a large or complicated system, often it is the case that only partial eigeninformation is available. To understand how a physical system modelled by a quadratic pencil should be modified based on partially available eigeninformation, it appears necessary to first understand how the PD-IQEP should be solved. This paper establishes some general theory toward that end. In particular, we find that the PD-IQEP is solvable, provided that the number of given eigenpairs is less or equal to the size of matrices and that the given vectors are linearly independent. A simple recipe for constructing such a matrix is described, which can serve as the basis for numerical computation. We also find that the unspecified eigenstructure of the reconstructed quadratic pencil is in fact quite limited in the sense discussed in Section 3. This observation should shed some light on how the long standing question of how much a quadratic pencil could be updated/modified/tuned if some of its eigenvalues and eigenvectors are to be keep invariant. We also demonstrate three different ways for the construction that not only satisfy the spectral constraints but also best approximate the original analytical model in some least squares sense.

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