Chapter 3

Parameterized Inverse Eigenvalue Problems

- Overview.
- General Results.
- Additive Inverse Eigenvalue Problems.
- Multiplicative Inverse Eigenvalue Problems.
Overview

- The structural constraint is regulated by a set of parameters.
- Most discussion concentrates on linear dependence of the problem on the parameters.
Generic Form

- **Given**
  - A *family* of matrices $A(c) \in \mathcal{M}$ with parameters $c \in \mathbb{F}^m$,
  - A set of scalars $\Omega \subset \mathbb{F}$,

- **Find**
  - Values of parameter $c$ such that $\sigma(A(c)) \subset \Omega$.
    - $\mathcal{M} =$ One particular class of submatrices in $\mathbb{F}^{n\times n}$.
    - $\mathbb{F} =$ One particular field of scalars.

- **Remark:**
  - Degree of free parameters $m$ needs not be the same as the size $n$ of the matrix.
  - Commonly used $\Omega$:
    - $\Omega = \{\lambda^*_k\}_{k=1}^n$.
    - $\Omega =$ left-half complex plan.
  - Depending upon how $A(c)$ is defined, the PIEP can appear in very different form.
Variations

- Linear dependence on parameters (LiPIEP):

  \[ A(c) = A_0 + \sum_{i=1}^{m} c_i A_i. \]

  \( A_i \in \mathcal{R}(n), \, F = \mathbb{R} \).
  \( A_i \in \mathcal{S}(n), \, F = \mathbb{R} \).

- (AIEP) \( A(c) = A(X) = A_0 + X, \, X \in \mathcal{N} \).

  \( \mathcal{N} = \) Some special class of submatrices.
  \( X \) can be expressed in terms of linear combinations of basis \( \{A_i\} \) of \( \mathcal{N} \).

- (MIEP) \( A(c) = A(X) = XA_0, \, X \in \mathcal{N} \).

  \( XA_0 \) can still be expressed as a linear combination of some \( A_i, \, i = 1, \ldots, m \).
  If \( X = \text{diag}\{c_1, \ldots, c_n\} \), write \( A_0 = [a_1^T, \ldots, a_n^T]^T \) in rows. Then

  \[ XA_0 = \sum_{i=1}^{n} c_i e_i a_i^T A_i. \]

- (Generalized Pole Assignment Problem)

  \( A(c) = A(K_1, \ldots, K_q) = A_0 + \sum_{i=1}^{q} B_i K_i C_i. \)
General Results

- Lot of attention has been paid to the theory and numerical method of the LiPIEP.
  - Finding a solution over real field is more complicated and difficult than over complex field.
- Whatever is known about LiPIEP applies to AIEP and MIEP.
- Pole assignment problem itself stands alone as an important application for decades.
  - Has been extensively studied already.
  - Many theoretical results and numerical techniques are available.
  - Approaches include skills from linear system theory, combinatorics, complex analysis to algebraic geometry.
  - Will not be discussed in this note.
Existence Theory for Linear PIEP

• Most discussions concentrate on the LiPIEP.

\[ A(c) = A_0 + \sum_{i=1}^{m} c_i A_i. \]

• Complex solvability is generally expected by solving polynomial systems.

• Presence of multiple eigenvalues in real case makes a big difference.
Complex Solvability

- Given $n$ complex numbers $\{\lambda_k^*\}_{k=1}^n$,
  - For almost all $A_i \in \mathbb{C}^{n \times n}$, there exists $c \in \mathbb{C}^n$ such that $A(c) = A_0 + \sum_{k=1}^n c_k A_k$ has eigenvalues $\{\lambda_k^*\}_{k=1}^n$.
  - There are at most $n!$ distinct solutions.
Real Solvability \((n = m)\)

- Notation and Definitions:

\[ A_k := \begin{bmatrix} a_{ij}^{(k)} \end{bmatrix}, \quad k = 0, 1, \ldots, n, \]

\[ E := \begin{bmatrix} a_{ii}^{(k)} \end{bmatrix}, \quad i, k = 1, \ldots, n, \]

\[ S := \sum_{i=1}^{m} |A_k|, \]

\[ \pi(M) := \|M - \text{diag}(M)\|_{\infty}, \]

\[ d(\lambda) := \min_{i \neq j} |\lambda_i - \lambda_j| \]

- Normalize the diagonals of \(A_j\):

\[ \diamond \text{Assume } E^{-1} = [\ell_{ij}] \text{ exists and } \tilde{c} := Ec. \]

\[ \diamond \text{Rewrite } \]

\[ A(c) = A_0 + \sum_{k=1}^{n} c_k A_k = A_0 + \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \ell_{k,j} \tilde{c}_j \right) A_k \]

\[ = A_0 + \sum_{j=1}^{n} \tilde{c}_j \left( \sum_{k=1}^{n} \ell_{k,j} A_k \right) \cdot \tilde{A}_j \]

\[ \text{diag}(\tilde{A}_j) = e_j, \quad j = 1, \ldots n. \]
• [34] Sufficient condition:

◊ Given
  ▶ $n$ real numbers $\lambda^* = \{\lambda_k^*\}_{k=1}^n$, and
  ▶ $n + 1$ real $n \times n$ matrices $A_i$, $i = 0, 1, \ldots, n$,

◊ Assume
  ▶ $\text{diag}(A_k) = e_k$, $k = 1, \ldots, n$,
  ▶ $\pi(S) < 1$,
  ▶ The gap $d(\lambda^*)$ is sufficiently large, i.e.,

$$
d(\lambda^*) \geq 4\frac{\pi(S)\|\text{diag}(\lambda^*) - \text{diag}(A_0)\|_\infty + \pi(A_0)}{1 - \pi(S)}.
$$

◊ Then the LiPIEP (with $m = n$) has a real solution $c \in \mathbb{R}^n$.

◊ Idea of proof:
  ▶ Prove that Gerschgorin circles of $A(c)$ are disjoint.
  ▶ Use Brouser fixed-point theorem to find a fixed point for the map $T(c) = \lambda^* + c - \lambda(A(c))$.

• Open Question: What can be said if $m > n$?
Multiple Eigenvalue

- Consider the LiPIEP associated with
  - Matrices $A_i \in \mathbb{R}^{n \times n}$, $i = 0, 1, \ldots, m$, and
  - $k$ real eigenvalue $\{\lambda_1^*, \ldots, \lambda_k^*\}$,
    - $\lambda_i^*$ has multiplicity $r_i \geq 0$.
    - $r_1 + \ldots + r_k = n$.
- Let $r = \max\{r_1, \ldots, r_k\}$ = maximal multiplicity.
- [310, 332] The LiPIEP is unsolvable almost everywhere if $n - m + r(r - 1) > 1$.
  - If $n = m$, then the LiPIEP is unsolvable almost everywhere if and only if $r > 1$. 
Sensitivity Analysis

- The solution to an IEP is generally not unique.
- The IEP is generally ill-posed.
  - Even if a solution depends continuously upon the problem data, the numerical solution could differ by a great deal with small perturbation.
Forward Problem for General $A(c)$

• Assume
  ◇ $A(c) \in \mathbb{C}^{n \times n}$ is analytic in $c \in \mathbb{C}^m$ over a neighborhood of 0.
  ◇ $\lambda_0$ is a simple eigenvalue of $A(0)$.
  ◇ $x_0$ and $y_0$ are the right and left unit eigenvector, respectively, of $A(0)$ corresponding to $\lambda_0$.

• Then
  ◇ There exists an analytic function $\lambda(c)$ in a neighborhood $N$ of $0 \in \mathbb{C}^m$ such that
    ◦ $\lambda(c)$ is a simple eigenvalue of $A(c)$.
    ◦ $\lambda(0) = \lambda_0$.
  ◇ There exist analytic functions $x(c)$ and $y(c)$ in $N$ such that
    ◦ $x(c)$ is a right eigenvector corresponding to $\lambda(c)$.
    ◦ $y(c)$ is a left eigenvector corresponding to $\lambda(c)$.
    ◦ $x(0) = x_0$, $y(0) = y_0$.

• Furthermore,
  \[
  \left( \frac{\partial \lambda(c)}{\partial c_i} \right)_{c=0} = y_0^T \left( \frac{\partial A(c)}{\partial c} \right)_{c=0} x_0.
  \]
Inverse Problem for Linear Symmetric $A(c)$

- Assume all matrices are symmetric and the LiPIEP
  
  $A(c) = A_0 + \sum_{i=1}^{n} c_i A_i$

  is solvable.

- Assume $A(c) = Q(c)\text{diag}\{\lambda_k^*\}_{k=1}^{n} Q(c)^T$ and define
  
  $J(c) = \begin{bmatrix} q_i(c)^T A_j q_i(c) \end{bmatrix}, \quad i, j = 1, \ldots, n,$

  $b = \begin{bmatrix} q_1(c)^T A_0 q_1(c), \ldots, q_n^T A_0 q_n(c) \end{bmatrix}^T.$

- [360] If

  \[ \delta = \|\lambda^* - \tilde{\lambda}\|_\infty + \sum_{i=0}^{n} \|A_i - \tilde{A}_i\|_2 \]

  is sufficiently small, then

  - The PIEP associated with $\tilde{A}_i, \ i = 0, \ldots, n$ and
    \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\} is solvable.

  - There is a solution $\tilde{c}$ near to $c,$

  \[ \frac{\|c - \tilde{c}\|_\infty}{\|c\|_\infty} \leq \kappa_\infty(J(c)) \left( \frac{\|\lambda^* - \tilde{\lambda}\|_\infty + \|A_0 - \tilde{A}_0\|_2}{\|\lambda^* - b\|_\infty} + \sum_{i=1}^{n} \frac{\|A_i - \tilde{A}_i\|_2}{\|J(c)\|_\infty} \right) + O(\delta^2). \]
Numerical Methods

- Direct methods.
  - Lanczos method.
- Iterative methods.
  - Newton’s method.
  - Orthogonal reduction method.
- Continuous Methods.
  - Homotopy method.
  - Projected gradient method.
  - ASVD flow method.
Direct Method

- Solution can be found in finite number of steps.
- Formulation exists for IEP with Jacobi structure.
- Will be discussed in Chapter 4.
Iterative Methods

- Newton’s method.
  - Applicable to real symmetric LiPIEP.
  - Fast, but only local convergence.
  - Multiple eigenvalue case needs to be handled more carefully.

- Orthogonal reduction method.
  - Employs QR-like decomposition.
  - Can handle multiple eigenvalues easily.
Newton’s Method (for real symmetric LiPIEP)

- Assume:
  - All matrices in
    \[ A(c) = A_0 + \sum_{i=1}^{n} c_i A_i \]
    are real and symmetric.
  - All eigenvalues \( \lambda_1^*, \ldots, \lambda_n^* \) are distinct.

- Consider:
  - The affine subspace
    \[ \mathcal{A} := \{ A(c) | c \in R^n \} \]
  - The isospectral surface
    \[ \mathcal{M}_e(\Lambda) := \{ Q\Lambda Q^T | Q \in O(n) \} \]
    where
    \[ \Lambda := diag\{ \lambda_1^*, \ldots, \lambda_n^* \} \]
  - Any tangent vector \( T(X) \) to \( \mathcal{M}_e(\Lambda) \) at a point \( X \in \mathcal{M}_e(\Lambda) \) must be of the form
    \[ T(X) = XK - KX \]
    for some skew-symmetric matrix \( K \in R^{n \times n} \).
A Classical Newton Method

- A function $f : \mathbb{R} \rightarrow \mathbb{R}$.

- The scheme:
  \[ x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)}) \]

- The intercept:
  - The new iterate $x^{(\nu+1)} = \text{The } x\text{-intercept of the tangent line of the graph of } f \text{ from } (x^{(\nu)}, f(x^{(\nu)}))$.

- The lifting:
  - $(x^{(\nu+1)}, f(x^{(\nu+1)})) = \text{The natural "lift" of the intercept along the y-axis to the graph of } f \text{ from which the next tangent line will begin.}$
An Analogy of the Newton Method

- Think of:
  - The surface $\mathcal{M}_e(\Lambda)$ as playing the role of the graph of $f$.
  - The affine subspace $\mathcal{A}$ as playing the role of the $x$-axis.

- Given $X^{(\nu)} \in \mathcal{M}_e(\Lambda)$,
  - There exist a $Q^{(\nu)} \in \mathcal{O}(n)$ such that
    \[ Q^{(\nu)T} X^{(\nu)} Q^{(\nu)} = \Lambda. \]
  - The matrix $X^{(\nu)} + X^{(\nu)} K - K X^{(\nu)}$ with any skew-symmetric matrix $K$ represents a tangent vector to $\mathcal{M}_e(\Lambda)$ emanating from $X^{(\nu)}$.

- Seek an $\mathcal{A}$-intercept $A(c^{(\nu+1)})$ of such a vector with the affine subspace $\mathcal{A}$.

- Lift up the point $A(c^{(\nu+1)}) \in \mathcal{A}$ to a point $X^{(\nu+1)} \in \mathcal{M}_e(\Lambda)$. 
Find the Intercept

- Find a skew-symmetric matrix $K^{(\nu)}$ and a vector $c^{(\nu+1)}$
  such that

$$X^{(\nu)} + X^{(\nu)}K^{(\nu)} - K^{(\nu)}X^{(\nu)} = A(c^{(\nu+1)}).$$

- Equivalently, find $\tilde{K}^{(\nu)}$ such that

$$\Lambda + \Lambda\tilde{K}^{(\nu)} - \tilde{K}^{(\nu)}\Lambda = Q^{(\nu)^T} A(c^{(\nu+1)}) Q^{(\nu)}.$$

  $\diamond \tilde{K}^{(\nu)} := Q^{(\nu)^T} K^{(\nu)} Q^{(\nu)}$ is skew-symmetric.

- Can find $c^{(\nu)}$ and $K^{(\nu)}$ separately.
- Diagonal elements in the system $\Rightarrow$

$$J^{(\nu)} c^{(\nu+1)} = \lambda^* - b^{(\nu)}.$$

- Known quantities:

$$J^{(\nu)}_{ij} := q^{(\nu)^T}_i A_j q^{(\nu)}_i, \text{ for } i, j = 1, \ldots, n$$

$$\lambda^* := (\lambda_1^*, \ldots, \lambda_n^*)^T$$

$$b^{(\nu)}_i := q^{(\nu)^T}_i A_0 q^{(\nu)}_i, \text{ for } i = 1, \ldots, n$$

$$q^{(\nu)}_i = \text{ the } i\text{-th column of the matrix } Q^{(\nu)}.$$  

- The vector $c^{(\nu+1)}$ can be solved.

- Off-diagonal elements in the system together with $c^{(\nu+1)}$ $\Rightarrow \tilde{K}^{(\nu)}$ (and, hence, $K^{(\nu)}$):

$$\tilde{K}^{(\nu)}_{ij} = \frac{q^{(\nu)^T}_i A(c^{(\nu+1)}) q^{(\nu)}_j}{\lambda^*_i - \lambda^*_j}, \text{ for } 1 \leq i < j \leq n.$$
Find the Lift-up

- No obvious coordinate axis to follow.
- Solving the IEP $\equiv$ Finding $\mathcal{M}_e(\Lambda) \cap \mathcal{A}$.
- Suppose all the iterations are taking place near a point of intersection. Then
  
  $$X^{(\nu+1)} \approx A(c^{(\nu+1)}).$$

- Also should have
  
  $$A(c^{(\nu+1)}) \approx e^{-K^{(\nu)}} X^{(\nu)} e^{K^{(\nu)}}.$$

- Replace $e^{K^{(\nu)}}$ by the Cayley transform:
  
  $$R := (I + \frac{K^{(\nu)}}{2})(I - \frac{K^{(\nu)}}{2})^{-1} \approx e^{K^{(\nu)}}.$$

- Define
  
  $$X^{(\nu+1)} := R^T X^{(\nu)} R \in \mathcal{M}_e(\Lambda).$$

- The next iteration is ready to begin.
Remarks

• Note that
\[ X^{(\nu+1)} \approx R^T e^{K^{(\nu)}} A^{(\nu+1)} e^{-K^{(\nu)}} R \approx A^{(\nu+1)} \]
represents a lifting of the matrix \( A^{(\nu+1)} \) from the affine subspace \( \mathcal{A} \) to the surface \( \mathcal{M}_e(\Lambda) \).

• The above offers a geometrical interpretation of Method III developed by Friedland et al [145].

• Quadratic convergence even for multiple eigenvalues case.
Continuous Methods

- Homotopy method.
  - Homotopy theory for some AIEP’s can be established.
    - **Open Question:** Describe a homotopy for general PIEP.
  - Provides both an existence proof and a numerical method.
  - See discussion in AIEP.
- Projection gradient method.
  - General, least squares setting.
  - Can be generalized to SIEP with any linear structure.
  - The method enjoys the globally descent property, but slow.
- ASVD flow method.
  - Provides stable coordinate transformations for non-symmetric matrices.
  - Will be discussed in SIEP for stochastic structure.
Projected Gradient Method (for SIEP)

• The idea works for general symmetric $A(c)$ so long as the projection $P(X)$ of a matrix $X$ to $\mathcal{A}$ can be calculated.

• The idea applies to SIEP and is described in that setting.

• Idea:
  - $X \in \mathcal{M}_e(\Lambda)$ satisfies the spectral constraint.
  - $P(X) \in \mathcal{V}$ has the desirable structure in $\mathcal{V}$.
  - Minimize the undesirable part $\|X - P(X)\|$.

• Working with the parameter $Q$ is easier:

  \[
  \text{Minimize } F(Q) := \frac{1}{2} \langle Q^T \Lambda Q - P(Q^T \Lambda Q), Q^T \Lambda Q - P(Q^T \Lambda Q) \rangle
  \]

  \[
  \text{Subject to } Q^T Q = I
  \]

  - $\langle A, B \rangle = \text{trace}(AB^T)$ is the Frobenius inner product.
Feasible Set \( O(n) \) & Gradient of \( F \)

- The set \( O(n) \) is a regular surface.

- The tangent space of \( O(n) \) at any orthogonal matrix \( Q \) is given by
  \[
  T_QO(n) = QK(n)
  \]
  where
  \[
  K(n) = \{ \text{All skew-symmetric matrices} \}.
  \]

- The normal space of \( O(n) \) at any orthogonal matrix \( Q \) is given by
  \[
  N_QO(n) = QS(n).
  \]

- The Fréchet Derivative of \( F \) at a general matrix \( A \) acting on \( B \):
  \[
  F'(A)B = 2\langle \Lambda A(A^T\Lambda A - P(A^T\Lambda A)), B \rangle.
  \]

- The gradient of \( F \) at a general matrix \( A \):
  \[
  \nabla F(A) = 2\Lambda A(A^T\Lambda A - P(A^T\Lambda A)).
  \]
The Projected Gradient

- A splitting of $R^{n \times n}$:
  $$R^{n \times n} = T_QO(n) + N_QO(n)$$
  $$= QK(n) + QS(n).$$

- A unique orthogonal splitting of $X \in R^{n \times n}$:
  $$X = Q \left\{ \frac{1}{2}(Q^TX - X^TQ) \right\} + Q \left\{ \frac{1}{2}(Q^TX + X^TQ) \right\}.$$ 

- The projection of $\nabla F(Q)$ into the tangent space:
  $$g(Q) = Q \left\{ \frac{1}{2}(Q^T\nabla F(Q) - \nabla F(Q)^TQ) \right\}$$
  $$= Q[P(Q^T\Lambda Q), Q^T\Lambda Q].$$
An Isospectral Descent Flow

- A descent flow on the manifold $O(n)$:
  \[ \frac{dQ}{dt} = Q[Q^T \Lambda Q, P(Q^T \Lambda Q)]. \]

- A descent flow on the manifold $M(\Lambda)$:
  \[ \frac{dX}{dt} = \frac{dQ^T}{dt} \Lambda Q + Q^T \Lambda \frac{dQ}{dt} \]
  \[ = [X, [X, P(X)]]_{k(X)}. \]

- The entire concept can be obtained by utilizing the Riemannian geometry on the Lie group $O(n)$.
Additive Inverse Eigenvalue Problems

• Subvariations.
• Solvability Issues.
• Sensitivity Issues.
• Numerical Methods.
• Applications.
Subvariations

• Generic form:
  ◊ Given
    ▶ A fixed matrix $A$ and a class of matrices $\mathcal{N}$ in $F^{n \times n}$,
    ▶ A subset $\Omega \subset F$,
  ◊ Find
    ▶ $X \in \mathcal{N}$ such that $\sigma(A + X) \subset \Omega$.

• Some special cases:
  ◊ (AIEP1) $A$ is real, $X$ is real diagonal, and $F$ is real.
  ◊ (AIEP2) $A$ is real symmetric, $X$ is real diagonal, and $F$ is real.
  ◊ (AIEP3) $A$ is complex general, $X$ is complex diagonal, and $F$ is complex.
  ◊ Open Question: $A = 0$, $X$ has prescribed entries at specific location.
Solvability Issues

• If $X = \text{diag}(c_1, \ldots, c_n)$, then consider

$$A + X = A + \sum_{i=1}^{n} c_i e_i e_i^T.$$

This is a special PIEP.

• Complex solvability [2, 63, 138]:

  ◇ For any specified $\{\lambda_{k}^\ast\}_{k=1}^{n} \in \mathbb{C}$, the AIEP3 is solvable.
  ◇ There are at most $n!$ solutions.
  ◇ For almost all $\{\lambda_{k}^\ast\}_{k=1}^{n}$, there are exactly $n!$ solutions.
• Real solvability

◇ Some sufficient conditions:
  ▶ If $d(\lambda^*) > 4\pi(A)$, then AIEP1 is solvable [101].
  ▶ If $d(\lambda^*) > 2\sqrt{3}(\pi(A \circ A))^{1/2}$, then AIEP2 is solvable [170].

◇ Some necessary conditions:
  ▶ If AIEP1 is solvable, then
    \[
    \sum_{i \neq j} (\lambda_i^* - \lambda_j^*)^2 \geq 2n \sum_{i \neq j} a_{ij}a_{ji}.
    \]

• Unsolvability:

◇ Both AIEP1 and AIEP2 are unsolvable almost everywhere if an multiple eigenvalue is present [332].
Sensitivity Issues (for AIEP2)

- Suppose that the AIEP2 is solvable.
- Assume
  - $A(X) := A + X = Q(X)^T \Lambda Q(X)$,
  - Define
    \[ J(X) := [q_{ji}^2(X)], \]
    \[ b(X) := [q_1(X)^T Aq_1(X), \ldots, q_n(X)^T Aq_n(X)]^T. \]
  - $J(X)$ is nonsingular,
  - The perturbation
    \[ \delta = \|\lambda^* - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2 \]
    is small enough.
- Then
  - The AIEP2 associated with $\tilde{A}$ and $\tilde{\lambda}$ is solvable.
  - There is a solution $\tilde{X}$ near to $X$, i.e.,
    \[ \frac{\|X - \tilde{X}\|_\infty}{\|X\|_\infty} \leq \kappa_\infty(J(X)) \left( \frac{\|\lambda^* - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2}{\|\lambda^* - b\|_\infty} \right) + O(\delta^2). \]
Numerical Methods

- Most methods for symmetric or Hermitian problem depend heavily on the fact that the eigenvalues are real and can be totally ordered.
  - Can consider each eigenvalue $\lambda_i$ as piecewise differential function $\lambda_i(X)$.
  - Newton’s iteration for AIEP2 is easy to formulate.
- For general matrices where eigenvalues are complex, tracking each eigenvalue requires some kind of matching mechanism.
  - Homotopy method naturally track each individual eigenvalue curves as are predetermined by initial values.
  - Homotopy method for AIEP3 gives rise to both an existence proof and a numerical method for finding all solutions.
Newton’s Method (for AIEP2)

- At the \( \nu \)-th iterate, assume \( Z^{(\nu)} \in \mathcal{M}_e(\Lambda) \),
  \[
  Z^{(\nu)} = Q^{(\nu)T} \Lambda Q^{(\nu)},
  \]
  \[
  A(X^{(\nu)}) := A + X^{(\nu)},
  \]
  \[
  J^{(\nu)} := \begin{bmatrix} q_{ji}^{(\nu)} \\ \end{bmatrix},
  \]
  \[
  b^{(\nu)} := \begin{bmatrix} q_1^{(\nu)T} A q_1^{(\nu)} , \ldots , q_n^{(\nu)T} A q_n^{(\nu)} \end{bmatrix}^T.
  \]
- Solve \( J^{(\nu)} X^{(\nu+1)} = \lambda^* - b^{(\nu)} \) for \( X^{(\nu+1)} \).
- Define skew-symmetric matrix
  \[
  K^{(\nu)} := Q^{(\nu)} \begin{bmatrix} q_i^{(\nu)T} \frac{A(X^{(\nu+1)}) q_j^{(\nu)}}{\lambda_i^* - \lambda_j^*} \end{bmatrix} Q^{(\nu)T}.
  \]
- Update the lift,
  \[
  R^{(\nu)} := \left( I + \frac{K^{(\nu)}}{2} \right) \left( I - \frac{K^{(\nu)}}{2} \right)^{-1},
  \]
  \[
  Z^{(\nu+1)} := R^{(\nu)T} X^{(\nu)} R^{(\nu)},
  \]
  \[
  Q^{(\nu+1)} := R^{(\nu)T} Q^{(\nu)}.
  \]
Homotopy Method (for AIEP3)
Multiplicative Inverse Eigenvalue Problems

- Subvariations.
- Solvability Issues.
- Sensitivity Issues.
- Numerical Methods.
- Applications.
Subvariations

- Generic form:
  - Given
    - A fixed matrix $A$ and a class of matrices $\mathcal{N}$ in $\mathbb{F}^{n \times n}$,
    - A subset $\Omega \subset \mathbb{F}$,
  - Find
    - $X \in \mathcal{N}$ such that $\sigma(XA) \subset \Omega$.

- Some special cases:
  - (MIEP1) $A$ is real, $X$ is real diagonal, and $\mathbb{F}$ is real.
  - (MIEP2) $A$ is real, symmetric, and positive definite, $X$ is nonnegative diagonal, and $\mathbb{F}$ is real.
  - (MIEP3) $A$ is complex general, $X$ is complex diagonal, and $\mathbb{F}$ is complex.
  - (MIEP4) $A$ is complex Hermitian, $\mathbb{F}$ is real, want $\sigma(X^{-1}AX^{-1}) = \{\lambda_k^*\}^{n}_{k=1}$ [120].
  - (MIEP5) Preconditioning applications.
Solvability Issues

• If $X = \text{diag}(c_1, \ldots, c_n)$ and $A = [a_1^T, \ldots, a_n^T]^T$, then write

$$XA = \sum_{i=1}^{n} c_i e_i a_i^T.$$  

This is a special PIEP.

• Complex solvability [137]:
  
  ◇ Assume that
  
  ▶ All principal minors of $A$ are distinct from zero,
  
  ◇ Then
  
  ▶ For any specified $\{\lambda_k^*\}_{k=1}^{n} \in \mathbb{C}$, the MIEP3 is solvable.
  
  ▶ There are at most $n!$ solutions.
• Real solvability:

  ◦ Some sufficient conditions:

    ▶ Suppose

      • The diagonals of $A$ are normalized to 1, i.e.,
        $a_{ii} = 1$.
      • $\pi(A) < 1$.
      • $d(\lambda) \geq \frac{4\pi(A)\|\lambda^*\|_{\infty}}{1-\pi(A)}$.

    ▶ Then the MIEP1 is solvable.

  ◦ Some necessary conditions:

    ▶ If MIEP1 is solvable, then

      $$
      \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \lambda_i^*,
      \det(A) \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} \lambda_i^*.
      $$
Optimal Conditioning by Diagonal Matrices

- Friedland’s result on MIEP1 suggests that a general complex matrix $A$ can be perfectly conditioned.
  - Lacks an efficient algorithm to implement the result.
- **Open question:** Want to know the optimal preconditioner of a given sparsity pattern \cite{165}.
  - Suppose
    - $A$ is symmetric and positive definite.
    - $A$ has property $A$, i.e., $A$ can be symmetrically permuted into
      \[
      \begin{bmatrix}
      D_1 & B \\
      B^T & D_2
      \end{bmatrix}
      \]
      where $D_1$ and $D_2$ are diagonal.
    - $D = \text{diag}(A)$.
  - Then \cite{134}
    \[
    \kappa(D^{-1/2}AD^{-1/2}) = \min_{\hat{D} > 0, \hat{D} = \text{diagonal}} \kappa(\hat{D}A\hat{D}).
    \]
Sensitivity Issues (for MIEP2)

- Suppose that the MIEP2 is solvable.
- MIEP2 is equivalent to the symmetrized problem:
  \[ X^{-1/2} A(X) X^{1/2} = X^{1/2} A X^{1/2}. \]
- Define
  \[ X^{1/2} A X^{1/2} = U(X)^T \Lambda U(X), \]
  \[ W(X) := [u^2_{ji}(X)]. \]
- Assume
  - \( W(X) \) is nonsingular,
  - The perturbation
    \[ \delta = \| \lambda^* - \tilde{\lambda} \|_\infty + \| A - \tilde{A} \|_2 \]
    is small enough.
- Then
  - The MIEP2 associated with \( \tilde{A} \) and \( \tilde{\lambda} \) is solvable.
  - There is a solution \( \tilde{X} \) near to \( X \), i.e.,
    \[ \frac{\| X - \tilde{X} \|_\infty}{\| X \|_\infty} \leq \frac{\lambda^*_i}{\lambda_i^*} \| W(X)^{-1} \|_\infty \left( \frac{\| \lambda^* - \tilde{\lambda} \|_\infty}{\| \lambda^* \|_\infty} + \| A - \tilde{A} \|_2 \right) + O(\delta^2). \]
Numerical Methods

- For preconditioning purpose, no need to solve the MIEP precisely.
  - There are many techniques for picking up a preconditioner.
  - Will not be discussed in this note.
- The MIEP is a linear, but not symmetric PIEP even if $A$ is symmetric.
  - The numerical methods for symmetric PIEP need to be modified.
  - If $A$ is a Jacobi matrix, the problem can be solved by direct methods. Will be discussed in Chapter 4.
Reformulate MIEP1 as Nonlinear Equations

- Formulate MIEP1 as solving \( f(X) = 0 \) for some nonlinear function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \).

- Different ways to formulate \( f(X) \):
  - \( f_i(X) := \det(XA - \lambda_i^*I) \).
  - \( f_i(X) := \lambda_i(XA) - \lambda_i^* \).
  - \( f_i(X) := \alpha_n(XA - \lambda_i^*I) \), where \( \alpha_n(M) = \) the smallest singular values of \( M \).

- Assume that \( \lambda_i \neq 0 \) and, therefore, \( Y = X^{-1} \) exists.
  - \( g_i(Y) := \det(A - \lambda_i^*Y) \).
  - \( g_i(Y) := \lambda_i(A, Y) - \lambda_i^* \).
Newton’s Method (for MIEP2)

• Reformulate the MIEP2 as solving equations

\[ \lambda_i(A, Y) - \lambda_i^* = 0, \quad i = 1, \ldots, n \]

to maintain symmetry.

• At the \( \nu \)-th stage [213],

  ◦ Solve the generalized eigenvalue problem

  \[
  \left( A - \lambda(\nu)Y^{(\nu)} \right) x^{(\nu)} = 0.
  \]

  ◦ Normalize \( x_i^{(\nu)} \) so that \( x_i^{(\nu)T} Y^{(\nu)} x_i^{(\nu)} = 1 \).

  ◦ Denote \( Q^{(\nu)} = [x_1^{(\nu)}, \ldots, x_n^{(\nu)}] = [q_{ij}^{(\nu)}] \).

  ◦ Define (the Jacobian matrix of \( \lambda(A, Y) \))

    \[
    J(Y^{(\nu)}) := [-\lambda_{ij}^{(\nu)} q_{ji}^{(\nu)}].
    \]

  ◦ Solve \( J(Y^{(\nu)}) d^{(\nu)} = \lambda^* - \lambda^{(\nu)} \).

  ◦ Update \( Y^{(\nu+1)} := Y^{(\nu)} + \text{diag}(d^{(\nu)}) \).
Numerical Experience

- **Open Question:** Given the standard Jacobi matrix $A$ with nonzero row entries $[-1, 2, -1]$, what is the set of all reachable spectra of $XA$ via a nonnegative diagonal matrix $X$?

- **Open Question:** In structure design, often we are only interested in a few low order natural frequencies. Indeed, for large structures, it is impractical to calculate all of the frequencies and modes. How should one solve the problem if only a few low order frequencies are given?

- The above Newton method is only a locally convergent method. It appears that in the case of divergence, the Jacobian matrix $J$ becomes highly ill-conditioned and nearly singular.

- To effectively develop an algorithm for controlling the vibration of a string with a specified set of natural frequencies, for example, we need to have another mechanism that can somehow provide a good initial guess before the Newton’s method can be employed.