1 (55 pts) Differentiate the following functions:

A (5 pts)

\[ f(x) = e^{(3x^3 - 5x^2 + 3)} \]

**Answer:** Using the chain rule we get

\[ f'(x) = e^{(3x^3 - 5x^2 + 3)} (9x^2 - 10x) \]

B (5 pts)

\[ f(x) = e^{(\sin(2x) + \cos(3x))} \]

**Answer:** Using the chain rule we get

\[ f'(x) = e^{(\sin(2x) + \cos(3x))} (2 \cos(2x) - 3 \cos(3x)) \]

C (5 pts)

\[ f(x) = \frac{e^{(2x-4)} - x^2}{3x + e^{(2x+3)}} \]

**Answer:** Using the quotient rule

\[ f'(x) = \frac{(3x + e^{(2x+3)})(2e^{(2x-4)} - 2x) - (e^{(2x-4)} - x^2)(3 + 2e^{(2x+3)})}{(3x + e^{(2x+3)})^2} \]

D (5 pts)

\[ f(x) = \ln(x^3 - x^2 + 2) \]

**Answer:** Recall that for \( g(x) = \ln(x) \) then \( g'(x) = 1/x \). Combining this with the chain rule gives

\[ f'(x) = \frac{3x^2 - 2x}{x^3 - x^2 + 2} \]

E (5 pts)

\[ f(x) = \frac{5}{\ln(x+4)} \]

**Answer:** Using the quotient rule combined with the chain rule gives

\[ f'(x) = \frac{-5}{(\ln(x+4))^2} \frac{1}{x+4} \]
\[ f(x) = \ln \left( \frac{3x + 5}{x - 4} \right) \]

**Answer:** First rewrite \( f(x) \) using the logarithm rule \( \ln(a/b) = \ln(a) - \ln(b) \). Then differentiate.

\[ f(x) = \ln(3x + 5) - \ln(x - 4) \]

Giving

\[ f'(x) = \frac{3}{3x + 5} - \frac{1}{x - 4} \]

\[ f(x) = e^{\ln(3x)} \]

**Answer:** Again start by simplifying the function, then differentiate.

\[ f(x) = e^{\ln(3x)} = 3x \]

Giving

\[ f'(x) = 3 \]

\[ f(x) = \sin(x^3 + 2x + 4) \]

**Answer:** Using the chain rule gives

\[ f'(x) = \cos(x^3 + 2x + 4)(3x^2 + 2) \]

\[ f(x) = \frac{\sin(2x - 3)}{\cos(2x - 3)} \]

**Answer:** Using the quotient rule (and the chain rule) gives

\[ f'(x) = \frac{2\cos(2x - 3)\cos(2x - 3) + 2\sin(2x - 3)\sin(2x - 3)}{\cos^2(2x - 3)} \]

\[ f(x) = \sin(x - 2) \cos(x + 5) \]

**Answer:** Using the product rule gives

\[ f'(x) = \cos(x - 2)\cos(x + 5) - \sin(x - 2)\sin(x + 5) \]

\[ f(x) = \cos(\sin(4x)) \]

**Answer:** Using the chain rule gives

\[ f'(x) = -\sin(\sin(4x))\cos(4x) \]
2 (40 pts) A company makes cans for soda with a volume of $128\pi$ cm$^3$. To cut costs, the company wants to minimize the amount of material needed to make the can. In other words, the objective is to minimize the surface area of the can.

A (5 pts) Sketch the can.

**Answer:**

B (5 pts) Find the objective function.

**Answer:** The objective function gives the surface area, given by the sides, the top and the bottom. $SA = 2\pi r^2 + 2\pi rh$.

C (5 pts) Find the constraint function.

**Answer:** The constraint is that the can’s volume $V = 128\pi = \pi r^2 h$. Simplifying this function gives $h = 128/r^2$.

D (5 pts) Simplify the objective function.

**Answer:** Simplifying this function gives $h = 128/r^2$.

E (5 pts) Minimize the simplified objective function (solve $f' = 0$).

**Answer:** Inserting the constraint equation in the objective function gives $SA = 2\pi r^2 + 256\pi/r$. Minimizing involves computing $SA'$ and solving $SA' = 0$ gives

$$SA' = 4\pi r - \frac{256\pi}{r^2} = 0$$

$$r^3 = \frac{350}{4}$$

$$r = 4.$$ 

F (5 pts) Check that the point found is a minimum (use the second derivative rule).

**Answer:** To check that the point is a minimum, check $SA''$ and evaluate the derivative at $r = 4$. $SA'' = 4\pi + 512\pi/r^3 > 0$ for $r > 0$, i.e., the point is a minimum since the function is concave up.

G (5 pts) Find all dimensions (height and radius) of the can, and calculate how much material is needed for the can (calculate the surface area).

**Answer:** In E we found $r$, and using the constraint equation $h = 128/r^2 = 128/16 = 8$. 
(5 pts) Sketch the simplified objective function, on the sketch mark the minimum value.

Answer:
3 (30 pts): A new animal protection program that has been in effect for 25 years has led to an increase in the Appalachian mountain bear population. At the start of the program the projected bear enclosure had 10 bears, and during this period the population was \( P(t) = P_0 e^{0.05t} \). To judge the effectiveness of the program the world’s nature organization wanted to know:

A (5 pts): What is the total population at the beginning of the program and today?

**Answer:** At the beginning \( t = 0 \) the population \( P(0) = P_0 = 10 \). The population today can be found by evaluating \( P(t) \) at \( t = 25 \) using \( P_0 = 10 \)

\[
P(25) = 10e^{0.05 \cdot 25} = 34.9
\]

B (5 pts): Sketch \( P(t) \)

C (5 pts): What is the differential equation satisfied by this solution?

**Answer:** The general form for the differential equation is \( P''(t) = kP(t) \) for this problem \( k = 0.05 \) giving the equation \( P'(t) = 0.05P(t) \).

D (5 pts): Show that the solution \( P(t) \) given above satisfy the differential equation.

**Answer:** Differentiating \( P(t) = 10e^{0.05t} \) gives

\[
P'(t) = 10 \cdot 0.05e^{0.05t} = 0.05 \cdot 10e^{0.05t} = 0.05P(t)
\]

as predicted by the differential equation.

E (5 pts): Researchers were not happy with the progress and wanted to double the number of bears over the next 25 years. How much should they increase the growth rate \( k \) to satisfy this goal?

**Answer:** Since the new study starts after the first 25 years, we have that the initial number of bears was \( P(25) = 35 \) (rounding up to a whole number). Now the objective is to find a value for \( k \) that doubles the population in 25 years. This can be done by solving

\[
\begin{align*}
70 & = 35e^{k \cdot 25} \iff 2 = e^{k \cdot 25} \iff \\
\ln(2) & = k \cdot 25 \iff k = \ln(2)/25 = 0.0277
\end{align*}
\]
F (5 pts): Discussion: How could you increase $k$ (this is not a math question)?

**Answer:** One way would be to have female bears in closer proximity to the male bears increasing rate of re-production, another would be by providing additional food making it easier for the bears to survive during winter.