1. (25 pts) Differentiate the following functions using the appropriate rules (Don’t simplify)

A (5 pts): \( f(x) = 4x^5 + 2x^3 - 7x + \frac{1}{x} - 3 \)

**Answer:** \( f'(x) = 20x^4 + 6x^2 - 7 - \frac{1}{x^2} \)

B (5 pts): \( f(x) = \sqrt{4 - 5x} = (4 - 5x)^{1/2} \)

**Answer:** \( f'(x) = -\frac{5}{2}(4 - 5x)^{-1/2} \)

C (5 pts): \( f(x) = (1 + 3x^2 - 2x^3)^4 \)

**Answer:** \( f'(x) = 4(1 + 3x^2 - 2x^3)^3(6x - 6x^2) \)

D (5 pts): \( f(x) = (13x - x^2)(22 - 4x^2) \)

**Answer:** \( f'(x) = (13 - 2x)(22 - 4x^2) + (13x - x^2)(-8x) \)

E (5 pts): \( f(x) = \frac{x^{3/2} - 2x + 4}{x^3 - 2\sqrt{x} - 1} \)

**Answer:** \( f'(x) = \frac{(x^3 - 2\sqrt{x} - 1)(3/2x^{1/2} - 2) - (x^{3/2} - 2x + 4)(3x^{1/2} - x^{-1/2})}{(x^3 - 2\sqrt{x} - 1)^2} \)

2. (30 pts) An object travels \( s(t) \) km in \( t \) hours, where \( s(t) = t^{3/8} - 2t^2 + 6t + 15 \)

**Answer:** Before answering any of the questions we need to find the velocity and acceleration.

\[ v(t) = s'(t) = \frac{3t^2}{8} - 4t + 6 \]
\[ a(t) = v'(t) = s''(t) = \frac{6t}{8} - 4 \]

A (5 pts): What is the velocity at \( t = 6 \) hours?

**Answer:** \( v(6) = 3 \cdot 6^2/8 - 4 \cdot 6 + 6 = -4.5 \) km/hour

B (5 pts): What is the acceleration after \( t = 1 \) hour?

**Answer:** \( a(1) = 6 \cdot 1/8 - 4 = -4.5 = -3.25 \) km/hour$^2$

C (5 pts): How far can the object travel in \( t = 2 \) hours?

**Answer:** \( s(2) = 2^{3/8} - 2^2 + 6 \cdot 2 + 15 = 20 \) km

D (5 pts): When is the velocity zero?

**Answer:** \( v(t) = 0 \iff 3t^2/8 - 4t + 6 = 0 \iff t = \{1.8057, 8.8610\} \)
E (5 pts): What is the maximum height (find both relative and absolute maxima). Assume \( t \leq 12 \)

**Answer:** The maximum height occurs when \( s'(t) = v(t) = 0 \), i.e. at \( t = 1.8057 \) or \( t = 8.8610 \).

To find what point is the maximum, we check concavity, i.e. compute \( s''(t) = v'(t) = a(t) \) and evaluate \( a(t) \) at the two points.

\[ a(1.8057) = -2.6458 < 0 \] this point is concave down and therefore a maximum.

\[ a(8.8610) = 2.2658 > 0 \] this point is concave up and therefore a minimum.

The function value \( s(1.8057) = 20.049 \). This should be compared with the endpoint \( t = 12 \). We get \( s(12) = 15 \).

Conclusion: The local max at \( t = 1.8057 \) is also the global max.

F (5 pts): Write an equation for the tangent line at \( t = 2 \) hours.

**Answer:** To get this equation we use the point-slope equation \( y - y_0 = m(t - t_0) \) going through the point \( (t_0, y_0) \) with slope \( y'(t_0) = m \). For this problem \( t_0 = 2 \) and \( y_0 = s(2) = 2^3/8 - 2 \cdot 2^2 + 6 \cdot 2 + 15 = 20 \). The slope at \( t = 2 \) can be found from the velocity \( v(2) = -0.5 \). Inserting these values give

\[ y - 20 = -0.5(t - 2) \iff y = -0.5t + 21 \]

3. (10 pts) For the function \( f(x) = \frac{1}{3x} \)

A (5 pts): Differentiate the function using the power rule

**Answer:** \( f(x) = \frac{1}{3}(x)^{-1} \) giving \( f'(x) = -\frac{1}{3}(x)^{-2} = -\frac{1}{3x^2} \)

B (5 pts): Differentiate the function using the limit definition, find

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
 f'(x) = \frac{1}{3} \lim_{x \to 0} \frac{1}{x+h} - \frac{1}{x} \\
 f'(x) = \frac{1}{3} \lim_{x \to 0} \frac{x}{x(x+h)} - \frac{x+h}{x(x+h)} \\
 f'(x) = \frac{1}{3} \lim_{x \to 0} \frac{1}{h} \frac{1}{x(x+h)} \\
 f'(x) = \frac{1}{3} \lim_{x \to 0} \frac{1}{h} \frac{-h}{x(x+h)} \\
 f'(x) = \frac{1}{3} \lim_{x \to 0} \frac{-1}{x(x+h)} = -\frac{1}{3x^2}
\]

4. (20 pts) Find the limit for the functions
A (5 pts): \( f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} \)

**Answer:** Factoring \( x^2 - x - 2 = (x - 2)(x + 1) \) allows us to rewrite

\[
f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3
\]

B (5 pts): \( f(x) = \lim_{x \to \infty} \frac{x^2 - 4}{x^2 + 4} \)

**Answer:** Noting that the denominator has a higher power, i.e. it goes to infinity faster we get

\[
f(x) = \lim_{x \to \infty} \frac{x^2 - 4}{x^2 + 4} = 0
\]

C (5 pts): \( f(x) = \lim_{x \to \infty} \frac{x^2 - 2x + 3}{4x^2 + 1} \)

**Answer:** Noting that the numerator and denominator has the same power, dividing by highest power gives

\[
f(x) = \lim_{x \to \infty} \frac{x^2/x^2 - 2x/x^2 + 3/x^2}{4x^2/x^2 + 4/x^2} = \lim_{x \to \infty} \frac{1 - 2/x + 3/x^2}{4 + 4/x^2} = \frac{1}{4}
\]

D (5 pts): \( f(x) = \lim_{x \to \infty} \frac{x^2 + x - 2}{5x - 1} \)

**Answer:** Noting that the numerator has the highest power, we know that this goes to infinity faster, i.e.

\[
f(x) = \lim_{x \to \infty} \frac{x^2 + x - 2}{5x - 1} \to \infty
\]

5. (35 pts). Curve sketching for the function \( f(x) = x^5 - 5x^4 \)

A (5 pts): Calculate \( f'(x) \) and \( f''(x) \)

**Answer:**

\[
f'(x) = 5x^4 - 20x^3 \quad f''(x) = 20x^3 - 60x^2
\]

B (5 pts): Find the relative mins and maxes (solve \( f'(x) = 0 \)) and sketch points and associated function values on a graph

**Answer:** Solving \( f'(x) = 0 \) gives

\[
f'(x) = 5x^4 - 20x^3 = x^3(5x - 20) = 0 \iff x = 0 \text{ or } x = 4
\]

C (5 pts): Evaluate the function \( f(x) \) at the extremum points

**Answer:** We found two extremum points in the previous question, evaluating the function values give

\[
f(0) = 0 \text{ and } f(4) = 4^5 - 5 \cdot 4^4 = -256
\]
D (5 pts): Study concavity at points where $f'(x) = 0$ and determine if they are minima or maxima

**Answer:** To study concavity we need to evaluate $f''(x)$ at these points

$$f''(0) = 0 \text{ and } f''(4) = 320$$

Given that $f''(0) = 0$ we cannot make any conclusions, but at $x = 4$, the second order derivative $f'' > 0$, i.e. it is a maximum.

E (5 pts): Find inflection points (solve $f''(x) = 0$)

**Answer:** Inflection points are found where $f'' = 0$, for this problem we need to solve

$$f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3)$$

Inflection points are $x = 0$ and $x = 3$, the first point is not a true inflection point since $f' = 0$ at this point.

F (5 pts): Study concavity for the complete function and sketch concavity and function values at extremum points and inflection points

**Answer:** To study concavity, we need to examine the function for the intervals:

(a) $x < 0$ no points of interest were found for smaller values of x. Picking $x = -1$ we get $f''(-1) = 20(-1)^3 - 60(-1)^2 = -80$ (concave down).
(b) $0 \leq x < 3$. In this interval, we pick $x = 2$. For this value $f''(2) = 20(2)^3 - 60(2)^2 = -80$ (concave down).
(c) $3 \leq x < 4$. In this interval, we pick $x = 3.5$. For this value $f''(3.5) = 20(3.5)^3 - 60(3.5)^2 = 320$ (concave up).
(d) $4 \leq x < \infty$. In this interval, we pick $x = 5$. For this value $f''(5) = 20(5)^3 - 60(5)^2 = 1000$ (concave up).

G (5 pts): Examine endpoints and complete the sketch

**Answer:** The end-points include $\lim_{x \to \pm\infty} f(x)$. For $x \to -\infty$, the function $f(x) \to -\infty$ and for $x \to -\infty$, the function $f(x) \to -\infty$ and for $x \to \infty$, the function $f(x) \to \infty$. 
Sketch: