1. (20 pts): Given the difference equation $y_{n+1} = 0.5y_n + 5$.

   A (5 pts): Find $y_1, y_2, y_3$ given $y_0 = 5$.
   Answer: Use the iterative formula $y_{n+1} = ay_n + b$.
   
   $y_0 = 5$,  
   $y_1 = 0.5 \cdot y_0 + 5 = 0.5 \cdot 5 + 5 = 7.5$,  
   $y_2 = 0.5 \cdot y_1 + 5 = 0.5 \cdot 7.5 + 5 = 8.75$,  
   $y_3 = 0.5 \cdot y_2 + 5 = 0.5 \cdot 8.75 + 5 = 9.375$.

   B (5 pts): Calculate the limit $b/(1 - a)$.
   Answer: The answer can be calculated directly,
   
   \[
   \frac{b}{1 - a} = \frac{5}{1 - 0.5} = \frac{5}{0.5} = 10.
   \]

   C (5 pts): Solve the equation and find $y_{20}$.
   Answer: Use the solution formula
   
   \[
   y_{20} = \frac{b}{1 - a} + \left( y_0 - \frac{b}{1 - a} \right) a^n = 10 + (5 - 10)(0.5)^{20} = 9.9999952
   \]
   correct until 6'th decimal.

   D (5 pts): Sketch the graph.
   Answer: First, investigate properties, then make the sketch. The slope $a < 0$, which implies that the graph is monotonic. Furthermore $|a| < 1$, which implies that the graph is bounded, it is bounded by $y = \frac{b}{1 - a} = 10$. Sketch is included below.

2. (10 pts): A cell in a dish is surrounded by a fluid with a concentration of 4 mg/l of solute. Initially the concentration in the cell is 2 mg/l. The solute passes through the cell membrane at a rate that increases the concentration in the cell by 5% of the difference between the outside concentration and the inside concentration.
A (5 pts): Find the difference equation for $y_n$.

Answer: First use the iterative formula $y_{n+1} = \text{previous amount } + \text{fractional change } + \text{amount added or removed}$.

$$y_{n+1} = y_n + (4 - y_n)0.05 + 0.95y_n + 0.2.$$  

Note in this problem nothing is added or removed.

B (5 pts): Sketch the graph.

Answer: First, investigate properties, then make the sketch. The slope $a < 0$, which implies that the graph is monotonic. Furthermore $|a| < 1$, which implies that the graph is bounded, it is bounded by $y = \frac{b}{1-a} = \frac{0.2}{1-0.95} = 4$. Sketch is included below.

3. (20 pts): Given the function $f(x) = x^3 - 3x$.

A (5 pts): At $x = 2$ find the equation for the tangent line.

Answer: The equation for the tangent line is given by $y = ax + b$, where $a$ is the slope of the function $f(x = 2)$ at $x = 2$, and $b$ is the $y$-intercept. Then function $f(x)$ is given, from this we find $f'(x) = 3x^2 - 3$ and thus $f(2) = 2^3 - 3 \cdot 2 = 2$ and $f'(2) = 3 \cdot 2^2 - 3 = 9$. Consequently $a = 9$ and using the point $f(2) = 2$, which intersects with the tangent line, i.e., $2 = 9 \cdot 2 + b$ giving $b = -16$. Inserting these values the equation becomes

$$y = 9x - 16.$$  

B (5 pts): For $2 \leq x \leq 3$ find the average rate of change for the function $f$.

Answer: The average rate of change is given by

$$a = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(3) - f(2)}{3 - 2} = \frac{(3^3 - 3 \cdot 3) - (2^3 - 3 \cdot 2)}{3 - 2} = 16.$$  

B (5 pts): Sketch the function and its tangent line, and the secant line for $2 \leq x \leq 3$.

Answer: Note the secant line intersects $f$ twice (at $x = 2$ and $x = 3$), while the tangent line only has one common point with $f$ at $x = 2$. Also note, both the tangent line and the secant line are straight lines. These lines will always be straight.
C (5 pts): Find the value of $x$ for which the tangent line is horizontal?

**Answer:** When the tangent line is horizontal $f' = 0$ (i.e., it is the points at which the function has maxima or minima). For this function $f'(x) = 3x^2 - 3 = 0$ implies $3x^2 = 3$ or $x = \pm 1$. See tangent lines on graph below.

4. (20 pts): For the function $f(x) = x^2 + 4$.

**A** (10 pts): Use the power rule to find the first and second order derivative of the function.

**Answer:** For $f(x) = x^2 + 4$ we get

$$f'(x) = 2x,$$

$$f''(x) = 2.$$

**B** (10 pts): Use definition of differentiability $f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to confirm that the derivative found in A is correct.

**Answer:** Using the definition given above we get

$$f' = \lim_{h \to 0} \frac{[(x+h)^2 + 4] - [x^2 + 4]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh + 4 - x^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh}{h}$$

$$= \lim_{h \to 0} (h + 2x) = 2x.$$
A (5 pts): \( \lim_{x \to 4} \frac{x^2 - 16}{x - 4} \).

**Answer:** Factoring the numerator gives
\[
\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \to 4} (x + 4) = 8.
\]

B (5 pts): \( \lim_{x \to \infty} \frac{3x^2 + 2x + 5}{4x - 7} \).

**Answer:** Dividing by the highest common factor \( x^2 \) gives
\[
\lim_{x \to \infty} \frac{3x^2 + 2x + 5}{4x - 7} = \lim_{x \to \infty} \frac{\frac{3x^2}{x^2} + \frac{2x}{x^2} + \frac{5}{x^2}}{\frac{4x}{x^2} - \frac{7}{x^2}} = \lim_{x \to \infty} \frac{3 + \frac{2}{x} + \frac{5}{x^2}}{4 \cdot \frac{1}{x^2} - \frac{7}{x^2}} = \infty
\]

C (5 pts): \( \lim_{x \to \infty} \frac{x^2 - 2x + 3}{2x^3 + 4x - 1} \).

**Answer:** Factoring the numerator gives
\[
\lim_{x \to \infty} \frac{x^2 - 2x + 3}{2x^3 + 4x - 1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^3} - \frac{2x}{x^3} + \frac{3}{x^3}}{\frac{2x^3}{x^3} + \frac{4x}{x^3} - \frac{1}{x^3}} = \lim_{x \to \infty} \frac{1 - \frac{2}{x} + \frac{3}{x^2}}{2x + \frac{4}{x^2} - \frac{1}{x^3}} = 0
\]

6. (10 pts): Use the power rule to find the first and second order derivative for the functions

A (5 pts): \( f(x) = \sqrt{3x^2 + 2x - 5} \).

**Answer:** Rewriting the expression gives \( f(x) = (3x^2 + 2x - 5)^{1/2} \), then apply the power rule to get
\[
f'(x) = \frac{1}{2}(3x^2 + 2x - 5)^{-1/2}(6x + 2) \]
\[
f''(x) = -\frac{1}{4}(3x^2 + 2x - 5)^{-3/2}(6x + 2)^2 + \frac{6}{2}(3x^2 + 2x - 5)^{-1/2}
\]

To find the second order derivative we used the product rule \((fg)' = f'g + fg'\).

B (5 pts): \( f(x) = \frac{4x - 2}{x + 3} \).
7. (20 pts): An object travels $s(t)$ feet in $t$ minutes following the line $s(t) = -0.5t^2 + 10t - 18$.

**Answer:** First for this function find the general equations for velocity and acceleration, use differentiation to find these.

\[
\begin{align*}
s(t) &= -0.5t^2 + 10t - 18 \\
v(t) &= s'(t) = -t + 10 \\
a(t) &= v'(t) = s''(t) = -1
\end{align*}
\]

A (5 pts): What is the velocity at $t = 7$ min.

**Answer:** Evaluating the velocity equation at $t = 7$ gives $v(7) = -7 + 10 = 3$.

B (5 pts): What is the acceleration at $t = 4$ min.

**Answer:** Evaluating the acceleration equation at $t = 4$ gives $a(4) = -1$. Note the acceleration $a(t) = -1$ for all values of $t$.

C (5 pts): When does the object hit the ground for an object released at $t = 3$ min?

**Answer:** When the object hits the ground $s(t) = 0$ so to answer this we need to solve this equation for $t$.

\[
\begin{align*}
s(t) &= 0 \\
t &= \frac{-10 \pm \sqrt{100 - 4 \cdot 0.5 \cdot 18}}{-1} \\
t &= \frac{-10 \pm 8}{-1} = 2, 9
\end{align*}
\]

Note we were told that the object was released at $t = 3$, thus only one solution $t = 9$ is valid.

D (5 pts): When does the velocity shift from positive to negative?

**Answer:** To find when the velocity shifts from positive to negative solve $v(t) = 0$ i.e., $-t + 10 = 0$ giving $t = 10$. For $t < 10$ the velocity $v(t) > 0$ and for $t > 10$ the velocity $v(t) < 0$. 