Lest squares method

In general we defined the least squares error for any point to be given by

\[ E_i^2 = (y_i - (Ax_i + B))^2 \]

with the total squared error defined by

\[ E_{\text{tot}} = E^2 = \sum_i E_i^2 \]

To minimize the error we set the two derivatives equal to zero, i.e., we compute

\[ \frac{\partial E_{\text{tot}}(A, B)}{\partial A} = 0 \quad \text{and} \quad \frac{\partial E_{\text{tot}}(A, B)}{\partial B} = 0 \]

To check that the error is minimal we calculate and check

\[ D = \frac{\partial^2 E_{\text{tot}}}{\partial A^2} \frac{\partial^2 E_{\text{tot}}}{\partial B^2} - \left( \frac{\partial^2 E_{\text{tot}}}{\partial A \partial B} \right)^2 > 0 \quad \text{with} \quad \frac{\partial^2 E_{\text{tot}}}{\partial A^2} > 0. \]
For the example we did in class today we had the points (1,4), (2,5), and (3,8), so

\[ E_1^2 = (4 - (A + B))^2 = (4 - A - B)^2 \]
\[ E_2^2 = (5 - (2A + B))^2 = (5 - 2A - B)^2 \]
\[ E_3^2 = (8 - (3A + B))^2 = (8 - 3A - B)^2 \]

with

\[ E_{\text{tot}} = (4 - A - B)^2 + (5 - 2A - B)^2 + (8 - 3A - B)^2 \]

giving

\[ \frac{\partial E_{\text{tot}}}{\partial A} = -2(4 - A - B) - 2(5 - 2A - B) - 2(8 - 3A - B) \]
\[ = -8 + 2A + 2B - 20 + 8A + 4B - 48 + 18A + 6B \]
\[ = -76 + 28A + 12B \]
\[ \frac{\partial E_{\text{tot}}}{\partial B} = -2(4 - A - B) - 2(5 - 2A - B) - 2(8 - 3A - B) \]
\[ = -8 + 2A + 2B - 10 + 4A + 2B - 16 + 6A + 2B \]
\[ = -34 + 12A + 6B \]
\[ \frac{\partial^2 E_{\text{tot}}}{\partial A^2} = 28, \quad \frac{\partial^2 E_{\text{tot}}}{\partial B^2} = 14, \quad \frac{\partial^2 E_{\text{tot}}}{\partial A \partial B} = 12 \]

So setting the derivatives equal to zero gives

\[ \frac{\partial E_{\text{tot}}}{\partial A} = -76 + 28A + 12B = 0 \quad \Leftrightarrow \quad -38 + 14A + 6B = 0 \quad \text{or} \quad B = (38 - 14A) / 6 \]

\[ \frac{\partial E_{\text{tot}}}{\partial B} = -34 + 12A + 6B = 0 \quad \Leftrightarrow \quad -17 + 6A + 3B = -17 + 6A + 3(38 - 14A) / 6 = 0 \quad \Leftrightarrow \quad A = 2 \]

\[ B = (38 - 14A) / 6 = (38 - 14 \cdot 2) / 6 = \frac{5}{3}. \]

Now

\[ D = \frac{\partial^2 E_{\text{tot}}}{\partial A^2} \cdot \frac{\partial^2 E_{\text{tot}}}{\partial B^2} - \left( \frac{\partial^2 E_{\text{tot}}}{\partial A \partial B} \right)^2 = 28 \cdot 14 - 12^2 > 0 \]

and

\[ \frac{\partial^2 E_{\text{tot}}}{\partial A^2} = 28. \]