A Shifting Bloom Filter Framework for Set Queries

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ABSTRACT
Set queries are fundamental operations in computer systems and applications. This paper addresses the fundamental problem of designing a probabilistic data structure that can quickly process set queries using a small amount of memory. We propose a Shifting Bloom Filter (ShBF) framework for representing and querying sets. We demonstrate the effectiveness of ShBF using three types of popular set queries: membership, association, and multiplicity queries. The key novelty of ShBF is on encoding the auxiliary information of a set element in a location offset. In contrast, prior BF based set data structures allocate additional memory to store auxiliary information. We conducted experiments using real-world network traces, and results show that ShBF significantly advances the state-of-the-art on all three types of set queries.

1. INTRODUCTION
1.1 Motivations
Set queries, such as membership queries, association queries, and multiplicity queries, are fundamental operations in computer systems and applications. Membership queries check whether an element is a member of a given set. Network applications, such as IP lookup, packet classification, and regular expression matching, often involve membership queries. Association queries identify which set(s) among a pair of sets contain a given element. Network architectures such as distributed servers often use association queries. For example, when data is stored distributively on two servers and the popular content is replicated over both servers to achieve load balancing, for any incoming query, the gateway needs to identify the server(s) that contain the data corresponding to that query. Multiplicity queries check how many times an element appears in a multi-set. A multi-set allows elements to appear more than once. Network measurement applications, such as measuring flow sizes, often use multiplicity queries.

This paper addresses the fundamental problem of designing a probabilistic data structure that can quickly process set queries, such as the above-mentioned membership, association, and multiplicity queries, using a small amount of memory. Set query processing speed is critical for many systems and applications, especially for networking applications as packets need to be processed at wire speed. Memory consumption is also critical because small memory consumption may allow the data structure to be stored in SRAM, which is an order of magnitude faster than DRAM.

Widely used set data structures are the standard Bloom Filter (BF) [3] and the counting Bloom Filter (CBF) [11]. Let \( h_1(.) \), \( \ldots \), \( h_k(.) \) be \( k \) independent hash functions with uniformly distributed outputs. Given a set \( S \), BF constructs an array \( B \) of \( m \) bits, where each bit is initialized to 0, and for each element \( e \in S \), BF sets the \( k \) bits \( B[h_1(e)\%m], \ldots, B[h_k(e)\%m] \) to 1. To process a membership query of whether element \( e \) is in \( S \), BF returns true if all corresponding \( k \) bits are 1 (i.e., returns \( \land_{i=1}^{k} B[h_i(e)\%m] \)). BF has no false negatives (FNs), i.e., it never says that \( e \notin S \) when actually \( e \in S \). However, BF has false positives (FPs), i.e., it may say that \( e \in S \) when actually \( e \notin S \) with a certain probability. Note that BF does not support element deletion. CBF overcomes this shortcoming by replacing each bit in BF by a counter. Given a set of elements, CBF first constructs an array \( C \) of \( m \) counters, where each counter is initialized to 0. For each element \( e \) in \( S \), for each \( 1 \leq i \leq k \), CBF increments \( C[h_i(e)\%m] \) by 1. To process a membership query of whether element \( e \) is in set \( S \), CBF returns true if all corresponding \( k \) counters are at least 1 (i.e., returns \( \land_{i=1}^{k} (C[h_i(e)\%m] \geq 1) \)). To delete an element \( e \) from \( S \), for each \( 1 \leq i \leq k \), CBF decrements \( C[h_i(e)\%m] \) by 1.

1.2 Proposed Approach
In this paper, we propose a Shifting Bloom Filter (ShBF) framework for representing and querying sets. Let \( h_1(.) \), \( \ldots \), \( h_k(.) \) be \( k \) independent hash functions with uniformly distributed outputs. In the construction phase, ShBF first constructs an array \( B \) of \( m \) bits, where each bit is initialized to 0. We observe that in general a set data structure needs to store two types of information for each element \( e \): (1) existence in-
is another hash function with uniformly distributed outputs and \( \mathbb{W} \) is a function of machine word size \( w \). For the third case, i.e., \( e \in S_2 - S_1 \), the offset function \( o(e) = o_2(e) = a_1(e) + h_{k+2}(e)\% ((\mathbb{W} - 1)/2) + 1 \), where \( h_{k+2}(e) \) is yet another hash function with uniformly distributed outputs.

In the construction phase, for each element \( e \in S_1 \cup S_2 \), we set the \( k \) bits \( B[h_1(e)\% m + o(e)], \ldots, B[h_k(e)\% m + o(e)] \) to 1 using an appropriate value of \( o(e) \) as just described for the three cases. In the query phase, given an element \( e \in S_1 \cup S_2 \), for each \( 1 \leq i \leq k \), we read the \( i \) bits \( B[h_i(e)\% m], B[h_i(e)\% m + o(e)] \), and \( B[h_i(e)\% m + o_2(e)] \). If all the \( k \) bits \( B[h_1(e)\% m], \ldots, B[h_k(e)\% m] \) are 1, then \( e \) may belong to \( S_1 - S_2 \). If all the \( k \) bits \( B[h_1(e)\% m + o_1(e)], \ldots, B[h_k(e)\% m + o_1(e)] \) are 1, then \( e \) may belong to \( S_1 \cap S_2 \). If all the \( k \) bits \( B[h_1(e)\% m + o_2(e)], \ldots, B[h_k(e)\% m + o_2(e)] \) are 1, then \( e \) may belong to \( S_2 - S_1 \).

There are a few other possibilities that we will discuss later in Section 4.2, that ShBF takes into account when answering the association queries. In comparison, the standard BF based association query scheme, namely iBF, constructs a BF for each set. In terms of accuracy, iBF is prone to false positives whenever it declares an element \( e \in S_1 \cup S_2 \) in a query to be in \( S_1 \cap S_2 \), whereas ShBF achieves an FPR of zero. In terms of performance, ShBF is almost twice as fast as iBF because iBF needs 2k hash functions and 2k memory accesses per query, whereas ShBF needs only \( k + 2 \) hash functions and memory accesses per query.

### 1.2.3 Multiplicity Queries

For multiplicity queries, for each element \( e \) in a multi-set \( S \), the offset function \( o(e) = c(e) - 1 \) where \( c(e) \) is \( e \)'s counter (i.e., the number of occurrences of \( e \) in \( S \)). In the construction phase, for each element \( e \), we set the \( k \) bits \( B[h_1(e)\% m + c(e) - 1], \ldots, B[h_k(e)\% m + c(e) - 1] \) to 1. In the query phase, for an element \( e \), for each \( 1 \leq i \leq k \), we read the \( i \) bits \( B[h_i(e)\% m], B[h_i(e)\% m + c(e) - 1], \ldots, B[h_i(e)\% m + c - 1] \), where \( c \) is the maximum number of occurrences that an element can appear in \( S \). For these \( k \) bits, for each \( 1 \leq j \leq c \), if all the \( k \) bits \( B[h_i(e)\% m + j - 1], \ldots, B[h_i(e)\% m + j - 1] \) are 1, then we output \( j \) as one possible value of \( c(e) \). Due to false positives, we may output multiple possible values.

### 1.3 Novelty and Advantages over Prior Art

The key novelty of ShBF is on encoding the auxiliary information of a set element in its location by the use of offsets. In contrast, prior BF based set data structures allocate additional memory to store such auxiliary information.

To evaluate our ShBF framework in comparison with prior art, we conducted experiments using real-world network traces. Our results show that ShBF significantly advances the state-of-the-art on all three types of set queries: membership, association, and multiplicity. For membership queries, in comparison with the standard BF, ShBF has about the same FPR but is about 2 times faster; in comparison with 1MemBF [17], which represents the state-of-the-art in membership query BFs, ShBF has 10%~19% lower FPR and 1.27~1.44 times faster query speed. For association queries, in comparison with iBF, ShBF has 1.47 times higher probability of a clear answer, and has 1.4 times faster query speed. For multiplicity queries, in comparison with Spectral BF [8], which represents the state-of-the-art in multiplicity query BFs, ShBF has 1.45~1.62 times higher correctness rate and the query speeds are comparable.
2. RELATED WORK

We now review related work on the three types of set queries: membership, association, and multiplicity queries, which are mostly based on Bloom Filters. Elaborate surveys of the work on Bloom Filters can be found in [5, 14, 18, 19].

2.1 Membership Queries

Prior work on membership queries focuses on optimizing BF in terms of the number of hash operations and the number of memory accesses. Fan et al. proposed the Cuckoo filter and found that it is more efficient in terms of space and time compared to BF [10]. This improvement comes at the cost of non-negligible probability of failing when inserting an element. To reduce the number of hash computation, Kirsch et al. proposed to use two hash functions $h_1(.)$ and $h_2(.)$ to simulate $k$ hash functions $(h_1(.) + i * h_2(.)) \% m$, where $(1 \leq i \leq k)$; but the cost is increased FPR [13]. To reduce the number of memory accesses, Qiao et al. proposed to confine the output of the $k$ hash functions within certain number of machine words, which reduces the number of memory accesses during membership queries; but the cost again is increased FPR [17]. In contrast, ShBF reduces the number of hash operations and memory access by about half while keeping FPR about the same as BF.

2.2 Association Queries

Prior work on association queries focuses on identifying the set, among a group of pair-wise disjoint sets, to which an element belongs. A straightforward solution is iBF, which builds one BF for each set. To query an element, iBF generates a membership query for each set’s BF and finds out which set(s) the unknown element is in. This solution is used in the Summary-Cache Enhanced ICP protocol [11]. Other notable solutions include kBF [20], Bloomtree [22], Bloomier [6], Coded BF [16], Combinatorial BF [12], and SVBF [15]. When some sets have intersections, there will be consecutive 1s in the filters, and the false positive rate will increase and formulas will change. In this paper, we focus on the query of two sets with intersections.

2.3 Multiplicity Queries

BF cannot process multiplicity queries because it only tells whether an element is in a set. Spectral BF, which was proposed by Cohen and Matias, represents the state-of-the-art scheme for multiplicity queries [8]. There are three versions of Spectral BF. The first version proposes some modifications to CBF to record the multiplicities of elements. The second version increases only the counter with the minimum value when inserting an element. This version reduces FPR at the cost of not supporting updates. The third version minimizes space for counters with a secondary spectral BF and auxiliary tables, which makes querying and updating procedures time consuming and more complex. Aguilar-Saborit et al. proposed Dynamic Count Filters (DCF), which combines the ideas of spectral BF and CBF, for multiplicity queries [2]. DCF uses two filters: the first filter uses fixed size counters and the second filter dynamically adjusts counter sizes. The use of two filters degrades query performance. Another well-known scheme for multiplicity queries is the Count-Min (CM) Sketch [9]. A CM Sketch is actually a partitioned Counting Bloom filter. We will show that our scheme is much more-memory efficient than CM sketches.

3. MEMBERSHIP QUERIES

In this section, we first present the construction and query phases of ShBF for membership queries. Membership queries are the “traditional” use of a BF. We use ShBFM to denote the ShBF scheme for membership queries. Second, we describe the updating method of ShBFM. Third, we derive the FPR formulas of ShBFM. Fourth, we compare the performance of ShBFM with that of BF. Last, we present a generalization of ShBFM. Table 1 summarizes the symbols and abbreviations used in this paper.

<p>| Table 1: Symbols &amp; abbreviations used in the paper |</p>
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$m$</td>
<td>size of a Bloom Filter</td>
</tr>
<tr>
<td>$n$</td>
<td># of elements of a Bloom Filter</td>
</tr>
<tr>
<td>$k$</td>
<td># of hash functions of a Bloom Filter</td>
</tr>
<tr>
<td>$k_{opt}$</td>
<td>the optimal value of $k$</td>
</tr>
<tr>
<td>$S$</td>
<td>a set</td>
</tr>
<tr>
<td>$e$</td>
<td>one element of a set</td>
</tr>
<tr>
<td>$h_i(s)$</td>
<td>the $i$-th hash function</td>
</tr>
<tr>
<td>$FP$</td>
<td>false positive</td>
</tr>
<tr>
<td>$FPR$</td>
<td>false positive rate</td>
</tr>
<tr>
<td>$f$</td>
<td>the FP rate of a Bloom Filter</td>
</tr>
<tr>
<td>$p'$</td>
<td>the probability that one bit is still 0 after inserting all elements into BF</td>
</tr>
<tr>
<td>$BF$</td>
<td>standard Bloom Filter</td>
</tr>
<tr>
<td>$iBF$</td>
<td>individual BF: the solution that builds one BF for each set</td>
</tr>
<tr>
<td>$ShBF$</td>
<td>Shifting Bloom Filters</td>
</tr>
<tr>
<td>$ShBF_M$</td>
<td>Shifting Bloom Filters for membership qrs.</td>
</tr>
<tr>
<td>$ShBF_A$</td>
<td>Shifting Bloom Filters for association qrs.</td>
</tr>
<tr>
<td>$ShBF_x$</td>
<td>Shifting Bloom Filters for multiplicities qrs.</td>
</tr>
<tr>
<td>$Qps$</td>
<td>queries per second</td>
</tr>
<tr>
<td>multi-set</td>
<td>a generalization of the notion of a set in which members can appear more than once</td>
</tr>
<tr>
<td>$o(.)$</td>
<td>offset(.), referring to the offset value for a given input</td>
</tr>
<tr>
<td>$w$</td>
<td># of bits in a machine word</td>
</tr>
<tr>
<td>$\overline{w}$</td>
<td>the maximum value of $o(.)$ for membership query of a single set</td>
</tr>
<tr>
<td>$c$</td>
<td>the maximum number of times an element can occur in a multi-set</td>
</tr>
</tbody>
</table>

3.1 ShBFM – Construction Phase

The construction phase of ShBFM proceeds in three steps. Let $h_1(.), h_2(.), \ldots, h_{k+1}(.)$ be $\frac{k}{m} + 1$ independent hash functions with uniformly distributed outputs. First, we construct an array $B$ of $m$ bits, where each bit is initialized to 0. Second, to store the existence information of an element $e$ of set $S$, we calculate $\frac{k}{m}$ hash values $h_1(e) \% m, h_2(e) \% m, \ldots, h_{k+1}(e) \% m$. To leverage our ShBF framework, we also calculate the offset values for the element $e$ of set $S$ as the auxiliary information for each element, namely $o(e) = h_{k+1}(e) \% (\overline{w} + 1) + 1$. We will later discuss how to choose an appropriate value for $\overline{w}$. Third, we set the $\frac{k}{m}$ bits $B[h_1(e) \% m], \ldots, B[h_{k+1}(e) \% m]$ to 1 and the other $\frac{k}{m}$ bits $B[h_1(e) \% m + o(e)], \ldots, B[h_{k+1}(e) \% m + o(e)]$ to 1. Note that $o(e) \neq 0$ because if $o(e) = 0$, the two bits $B[h_1(e) \% m]$
and $B[h_i(e)\bmod m+o(e)]$ are the same bits for any value of $i$ in the range $1 \leq i \leq \frac{w}{2}$. For the construction phase, the maximum number of hash operations is $\frac{w}{2} + 1$. Figure 2 illustrates the construction phase of ShBF_M.

![Figure 2: Illustration of ShBF_M construction phase.](Image)

We now discuss how to choose a proper value for $w$ so that for any $1 \leq i \leq \frac{w}{2}$, we can access both bits $B[h_i(e)\bmod m]$ and $B[h_i(e)\bmod m+o(e)]$ in one memory access. Note that modern architecture like x86 platform CPU can access data starting at any byte, i.e., can access data aligned on any boundary, not just on word boundaries. Let $B[h_i(e)\bmod m]$ be the $j$-th bits of a byte where $1 \leq j \leq 8$. To access bit $B[h_i(e)\bmod m]$, we always need to read the $j - 1$ bits before it. To access both bits $B[h_i(e)\bmod m]$ and $B[h_i(e)\bmod m+o(e)]$ in one memory access, we need to access $j + 1 + w$ bits in one memory access. Thus, $j - 1 + w \leq w$, which means $w \leq w + 1 - j$. When $j = 8$, $w + 1 - j$ has the minimum value of $w - 7$. Thus, we choose $w \leq w - 7$ as it guarantees that we can read both bits $B[h_i(e)\bmod m]$ and $B[h_i(e)\bmod m + o(e)]$ in one memory access.

### 3.2 ShBF_M – Query Phase

Given a query $e$, we first read the two bits $B[h_i(e)\bmod m]$ and $B[h_i(e)\bmod m + o(e)]$ in one memory access. If both bits are 1, then we continue to read the next two bits $B[h_2(e)\bmod m]$ and $B[h_2(e)\bmod m + o(e)]$ in one memory access; otherwise we output that $e \notin S$ and the query process terminates. If for all $1 \leq i \leq \frac{w}{2}$, $B[h_i(e)\bmod m]$ and $B[h_i(e)\bmod m + o(e)]$ are 1, then we output $e \in S$. For the query phase, the maximum number of memory accesses is $\frac{w}{2}$.

### 3.3 ShBF_M – Updating

Just like BF handles updates by replacing each bit by a counter, we can extend ShBF_M to handle updates by replacing each bit by a counter. We use CShBF_M to denote this counting version of ShBF_M. Let $C$ denote the array of $m$ counters. To insert an element $e$, instead of setting $k$ bits to 1, we increment each of the corresponding $k$ counters by 1; that is, we increment both $C[h_i(e)\bmod m]$ and $C[h_i(e)\bmod m + o(e)]$ by 1 for all $1 \leq i \leq \frac{w}{2}$. To delete an element $e$ in $C$, we decrement both $C[h_i(e)\bmod m]$ and $C[h_i(e)\bmod m + o(e)]$ by 1 for all $1 \leq i \leq \frac{w}{2}$. In most applications, 4 bits for a counter are enough. Therefore, we can further reduce the number of memory accesses for updating CShBF_M. Similar to the analysis above, if we choose $w \leq 2^{\lceil z/2 \rceil}$ where $z$ is the number of bits for each counter, we can guarantee to access both $C[h_i(e)\bmod m]$ and $C[h_i(e)\bmod m + o(e)]$ in one memory access. Consequently, one update of CShBF_M needs only $k/2$ memory accesses.

Due to the replacement of bits by counters, array $C$ in CShBF_M uses much more memory than array $B$ in ShBF_M. To have the benefits of both fast query processing and small memory consumption, we can maintain both ShBF_M and CShBF_M, but store array $B$ in fast SRAM and array $C$ in DRAM. Note that SRAM is at least an order of magnitude faster than DRAM. Array $B$ in fast SRAM is for processing queries and array $C$ in slow DRAM is only for updating. After each update, we synchronize array $C$ with array $B$.

The synchronization is quite straightforward: when we insert an element, we insert it to both array $C$ and $B$; when we delete an element, we first delete it from $C$, if there is at least one of the $k$ counters becomes 0, we clear the corresponding bit in $B$ to 0.

### 3.4 ShBF_M – Analysis

We now calculate the FPR of ShBF_M, denoted as $f_{\text{ShBF_M}}$. Then, we calculate the minimum value of $w$ so that ShBF_M can achieve almost the same FPR as BF. Last, we calculate the optimum value of $k$ that minimizes $f_{\text{ShBF_M}}$.

#### 3.4.1 False Positive Rate

We calculate the false positive rate of ShBF_M in the following theorem.

**Theorem 1.** The FPR of ShBF_M for a set of $n$ elements is calculated as follows:

$$f_{\text{ShBF_M}} \approx (1 - p)^{\frac{k}{m}} \left(1 - \frac{1}{w} - p\right)$$

(1)

where $p = \frac{w}{\text{np}}$.

*Proof.* Let $p'$ represent the probability that one bit (suppose it is at position $i$) in the filter $B$ is still 0 after inserting information of all $n$ elements. For an arbitrary element $e$, if $h_i(e)\bmod m$ does not point to $i$ or $i - o(e)$, where $o(e) = h_i(e)\bmod (w - 1) + 1$, then the bit at position $i$ will still be 0, thus $p'$ is given by the following equation.

$$p' = \left(1 - \frac{2}{m} \right)^{\frac{kn}{2}} = \left(1 - \frac{2}{m} \right)^{\frac{kn}{2}}$$

(2)

When $m$ is large, we can use the identity $\sum_{x=0}^{m} \left(1 - \frac{1}{x}\right)^{-x} = e$, to get the following equation for $p'$.

$$p' = \left(1 - \frac{2}{m} \right)^{\frac{kn}{2}} = \left(1 - \frac{2}{m} \right)^{\frac{kn}{m}} \approx e^{-nk}$$

(3)

Let $X$ and $Y$ be the random variables for the event that the bit at position $h_i(\cdot)$ and the bit at position $h_i(\cdot) + h_i^+\bmod m(\cdot)$ is 1, respectively. Thus, $P\{X\} = 1 - p'$. Suppose we look at a hash pair $\langle h, h_+ + 1 \rangle$, we want to calculate $P\{XY\}$.

As $P\{XY\} = P\{X\} \times P\{Y\}$, next we calculate $P\{Y\}$. There are $w - 1$ bits on the left side of position $h_i$. The $1$s in these $w - 1$ bits could be due to the first hash function in a pair and/or due to the second hash function in the pair. In other words, event $Y$ happens because a hash pair $\langle h_i, h_+ + 1 \rangle$ sets the position $h_i$ to 1 during the construction phase. When event $X$ happens, there are two cases:

1. The event $X_1$ happens, i.e., the position $h_i$ is set to 1 by $h_+ + 1$, i.e., the left $w - 1$ bits cause $h_i$ to be 1, making $X$ and $Y$ independent. Thus, in this case $P\{Y\} = 1 - p'$. 

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2. The event $X_2$ happens, i.e., the position $h_i$ is set to 1 by $h_i$. In this case, $P \{X_1\} + P \{X_2\} = 1$, thus, $P \{Y \mid X, X_1\} = P \{Y \mid X, X_2\} = P \{X_1\} \times P \{X_2\} \times P \{X_1\} = P \{X_2\}$.

Next, we compute $P \{X_1\}$ and $P \{X_2\}$.

As there are $r-1$ bits on the left side of position $h_i$, there are $\binom{r}{1}$ combinations, i.e., $\binom{r}{1} = r-1$. Probability that any bit of the $\binom{r}{1}$ bits is 1 is $1 - \rho'$. When one bit in the $\binom{r}{1}$ bits is 1, probability that this bit sets the bit at location $h_i$ using the hash function $h_i + h_\bot + 1$ to 1 is $\frac{1}{r-1}$. Therefore, $P \{X_1\} = \binom{r}{1} \times (1 - \rho') \times \frac{1}{r-1} = 1 - \rho'$. Consequently, $P \{X_2\} = 1 - P \{X_1\} = \rho'$. Again there are two cases:

1. If the bit which $h_i(x)$ points to is set to 1 by the left 1s, $X$ and $Y$ are independent, and thus $P \{Y\} = \binom{r}{1} \times (1 - \rho') \times \frac{1}{r-1} = 1 - \rho'$.

2. If the bit which $h_i(x)$ points to is not set to 1 by the left 1s, then it must set one bit of the latter $\binom{r}{1}$ bits to 1. This case will cause one bit of the latter $\binom{r}{1}$ bits after position $h_i$ to be 1. In this case, there are following two situations for the second hashing $h_i + h_\bot + 1$:

   a) when the second hash points to this bit, the probability is $\frac{1}{r-1} \times 1$;

   b) otherwise, the probability is $(1 - \frac{1}{r-1}) \times (1 - \rho')$.

When the second case above happens, $P \{Y \mid X, X_2\}$ is given by the following equation.

$$P \{Y \mid X, X_2\} = \frac{(1 - \rho')(\binom{r}{1} - 1) + \frac{1}{r-1}}{\binom{r}{1} - 1} = \left(1 - \frac{\binom{r}{1} - 2}{\binom{r}{1} - 1}\rho'\right)$$

Integrating the two cases, we can compute $P \{Y \mid X\}$ as follows.

$$P \{Y \mid X\} = (1 - \rho')(1 - \rho') + \left(1 - (1 - \rho')\right) \left(1 - \frac{\binom{r}{1} - 2}{\binom{r}{1} - 1}\rho'\right)$$

(5)

The probability that all the first hashes point to bits that are 1 is $(1 - \rho')^{\frac{1}{r-1}}$. The probability that the second hash points to a bit that is 1 is the $\frac{1}{r-1}$-th power of Equation (5). Thus, the overall FPR of ShBFM is given by the following equation.

$$f_{\text{ShBFM}} = (1 - \rho')^{\frac{1}{r}} \left(1 - \rho'(1 - \rho') + \rho'(1 - \frac{\binom{r}{1} - 2}{\binom{r}{1} - 1}\rho')\right)^{\frac{1}{r}}$$

$$= (1 - \rho')^{\frac{1}{r}} \left(1 - \rho' + \frac{1}{\binom{r}{1} - 1}\rho^2\right)^{\frac{1}{r}}$$

(6)

Note that when $\binom{r}{1} \rightarrow \infty$, this formula becomes the formula of the FPR of BF.

Let us represent $e^{\frac{m}{nk}}$ by $p$. Thus, according to equation 3, $p' \approx p$. Consequently, we get:

$$f_{\text{ShBFM}} \approx (1 - p)^{\frac{1}{r}} \left(1 - p + \frac{1}{\binom{r}{1} - 1}p^2\right)^{\frac{1}{r}}$$

Note that the above calculation of FPRs is based on the original Bloom’s FPR formula [3]. In 2008, Bose et al. pointed out that Bloom’s formula [3] is slightly flawed and gave a new FPR formula [4]. Specifically, Bose et al. explained that the second independence assumption needed to derive $f_{\text{Bloom}}$ is too strong and does not hold in general, resulting in an underestimation of the FPR. In 2010, Christensen et al. further pointed out that Bose’s formula is also slightly flawed and gave another FPR formula [7]. Although Christensen’s formula is final, it cannot be used to compute the optimal value of $k$, which makes the FPR formula practically not much useful. Although Bloom’s formula underestimates the FPR, both studies pointed out that the error of Bloom’s formula is negligible. Therefore, our calculation of FPRs is still based on Bloom’s formula.

### 3.4.2 Optimizing System Parameters

#### Minimum Value of $\binom{r}{1}$

Recall that we proposed to use $\binom{r}{1} \leq w - 7$. According to this inequality, $\binom{r}{1} \leq 25$ for 32-bit architectures and $\binom{r}{1} \leq 57$ for 64-bit architectures. Next, we investigate the minimum value of $\binom{r}{1}$ for ShBF to achieve the same FPR with BF. We plot $f_{\text{ShBFM}}$ of ShBFM as a function of $\binom{r}{1}$ in Figures 3(a) and 3(b). Figure 3(a) plots $f_{\text{ShBFM}}$ vs. $\binom{r}{1}$ for $n = 10000$, $m = 100000$, and $k = 4, 8$, and 12. Figure 3(b) plots $f_{\text{ShBFM}}$ vs. $\binom{r}{1}$ for $n = 10000$, $k = 10$, and $m = 100000, 110000$, and 120000. The horizontal solid lines in these two figures plot the FPR of BF. From these two figures, we observe that when $\binom{r}{1} > 20$, the FPR of ShBFM becomes almost equal to the FPR of BF. Therefore, to achieve similar FPR as of BF, $\binom{r}{1}$ needs to be larger than 20. Thus, by using $\binom{r}{1} = 25$ for 32-bit and $\binom{r}{1} = 57$ for 64-bit architecture, ShBFM will achieve almost the same FPR as BF.

#### Optimum Value of $k$

Now we calculate the value of $k$ that minimizes the FPR calculated in Equation (1). The standard method to obtain the optimal value of $k$ is to differentiate Equation (1) with respect to $k$, e.g., $\frac{df_{\text{ShBFM}}}{dk} = 0$, and solve this equation for $k$. Unfortunately, this method does not yield a closed form solution for $k$. Thus, we use standard numerical methods to solve the equation $\frac{df_{\text{ShBFM}}}{dk} = 0$ to get the optimal value of $k$ for given values of $m$, $n$, and $\binom{r}{1}$. For $\binom{r}{1} = 57$, differentiating Equation (1) with respect to $k$ and solving for $k$ results in the following optimum value of $k$.

$$k_{opt} = 0.7009 \frac{m}{n}$$

Substituting the value of $k_{opt}$ from the equation above into Equation (1), the minimum value of $f_{\text{ShBFM}}$ is given by the following equation.

$$f_{\text{ShBFM}}^{\min} = 0.6204 \frac{m}{\binom{r}{1}}$$

(7)
3.5 Comparison of ShBFₘ FPR with BF FPR

Our theoretical comparison of ShBFₘ and BF shows that the FPR of ShBFₘ is almost the same as that of BF. Figure 4 plots FPRs of ShBFₘ and BF using Equations (1) and (8), respectively for \( m = 10000 \) and \( n = 4000, 6000, 8000, 10000, 12000 \). The dashed lines in the figure correspond to ShBFₘ whereas the solid lines correspond to BF. We observe from this figure that the sacrificed FPR of ShBFₘ in comparison with the FPR of BF is negligible, while the number of memory accesses and hash computations of ShBFₘ are half in comparison with BF.

![Figure 4: ShBFₘ FPR vs. BF FPR.](image)

Next, we formally arrive at this result. We calculate the minimum FPR of BF as we calculated for ShBFₘ in Equation (7) and show that the two FPRs are practically equal. For a membership query of an element \( u \) that does not belong to set \( S \), just like ShBFₘ, BF can also report true with a small probability, which is the FPR of BF and has been well studied in literature [3]. It is given by the following equation.

\[
f_{BF} = \left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^k \approx \left(1 - e^{-\frac{kn}{m}}\right)^k
\]  

For given values of \( m \) and \( n \), the value of \( k \) that minimizes \( f_{BF} \) is \( k = \frac{\ln n}{\ln 2} \). Substituting this value of \( k \) into Equation (8), the minimum value of \( f_{BF} \) is given by the following equation.

\[
f_{BF}^{\text{min}} = \left(\frac{1}{2}\right)^{\left(\frac{\ln n}{\ln 2}\right)} \approx 0.6185^{\frac{n}{m}}
\]

By comparing Equations (7) and (9), we observe that the FPRs of ShBFₘ and BF are almost the same. Thus, ShBFₘ achieves almost the same FPR as BF while reducing the number of hash computations and memory accesses by half.

3.6 Generalization of ShBFₘ

As mentioned earlier, ShBFₘ reduces \( k \) independent hash functions to \( k/2 + 1 \) independent hash functions. Consequently, it calculates \( k/2 \) locations independently and remaining \( k/2 \) locations are correlated through the equation \( h_i(e) + o_i(e) (1 \leq i \leq k/2) \). Carrying this construction strategy one step further, one could replace the first \( k/2 \) hash functions with \( k/4 \) independent hash functions and an offset \( o_2(e) \), i.e., \( h_j(e) + o_j(e) (1 \leq j \leq k/4) \). Continuing in this manner, one could eventually arrive at \( \log(k) + 1 \) hash functions. Unfortunately, it is not trivial to calculate the FPR for this case because \( \log(k) \) is seldom a integer. In this subsection, we simplify this \( \log \) method into a \( \text{linear} \) method by first using a group of \( \frac{1}{2} \) hash functions to calculate \( \frac{1}{2} \) hash locations and then applying shifting operation \( t \) times on these hash locations.

Consider a group of hash function comprising of \( t + 1 \) elements, i.e., \( h_1(x), h_2(x), \ldots, h_{t+1}(x) \). After completing the construction phase using this group of hash functions, the probability that any given bit is 0 is \( \frac{m - w}{m} + \frac{w - 1 - t}{m} + \cdots + \frac{w - t - 1}{m} \). To insert \( n \) elements, we need \( t + 1 \) such group insertion operations. After completing the insertion, the probability \( p' \) that one bit is still 0 is given by the following equation.

\[
p' = \left(1 - \frac{t+1}{m}\right) \approx e^{-\frac{kn}{m}}
\]  

Note that this probability formula is essentially \( k \) times product of \( e^{-\frac{n}{m}} \). Thus, we can treat our ShBFₘ as a \text{partitioned Bloom filter}, where the output of each hash function covers a distinct set of consecutive \( \frac{w}{m} \) bits. Setting \( w = m \) makes this scheme partitioned Bloom filter. The equations below calculate the FPR \( f \) for this scheme.

\[
f = (1-p')^t \times (f_{\text{group}})^k
\]  

\[
f_{\text{group}} = \frac{1}{t} \times \left(1 - p'\right)^{t} = \frac{1}{t} \times \left(1 - \frac{w - 1}{w - 1} \times p'\right)^{t}
\]

4. ASSOCIATION QUERIES

In this section, we first describe the construction and query phases of ShBF for association queries, which are also called membership test. We use ShBFₐ to denote the ShBF scheme for association queries. Second, we describe the updating methods of ShBFₐ. Third, we derive the FPR of ShBFₐ. Last, we analytically compare the performance of ShBFₐ with that of iBF.

4.1 ShBFₐ – Construction Phase

The construction phase of ShBFₐ proceeds in three steps. Let \( h_1(\cdot), \ldots, h_k(\cdot) \) be \( k \) independent hash functions with uniformly distributed outputs. Let \( S_1 \) and \( S_2 \) be the two given sets. First, ShBFₐ constructs a hash table \( T_1 \) for set \( S_1 \) and a hash table \( T_2 \) for set \( S_2 \). Second, it constructs an array \( B \).
of $m$ bits, where each bit is initialized to 0. Third, for each element $e \in S_1$, to store its existence information, ShBF$_A$ calculates $k$ hash functions $h_i(e) \% m$, $h_k(e) \% m$ and searches $e$ in $T_2$. If it does not find $e$ in $T_2$, to store its auxiliary information, it sets the offset $o(e) = 0$. However, if it does find $e$ in $T_2$, to store its auxiliary information, it calculates the offset $o(e)$ as $o(e) = o_2(e) + h_k(e) \% ((w - 1)/2) + 1$, where $h_k(e)$ is a hash function with uniformly distributed output and $w$ is a function of machine word size $w$, which we will discuss shortly. Fourth, it sets the $k$ bits $B[h_1(e) \% m + o(e)], \ldots, B[h_k(e) \% m + o(e)]$ to 1. Fifth, for each element $e \in S_2$, to store its existence information, ShBF$_A$ calculates the $k$ hash functions and searches it in $T_1$. If it finds $e$ in $T_1$, it does not do anything because its existence as its auxiliary information have already been stored in the array $B$. However, if it does not find $e$ in $T_1$, to store its auxiliary information, it calculates the offset $o(e)$ as $o(e) = o_2(e) + h_k(e) \% ((w - 1)/2) + 1$, where $h_k(e)$ is also a hash function with uniformly distributed output. Last, it sets the $k$ bits $B[h_1(e) \% m + o(e)], \ldots, B[h_k(e) \% m + o(e)]$ to 1. To ensure that ShBF$_A$ can read $B[h_i(e) \% m], B[h_i(e) \% m + o(e)],$ and $B[h_i(e) \% m + o(e)]$ in a single memory access when querying, we let $w \leq 7$ earlier at the end of Section 3.1. As the maximum value of $h_i(e) \% m + o(e)$ can be equal to $m + w - 2$, we append the $m$-bit array $B$ with $w - 2$ bits.

### 4.2 ShBF$_A$ – Query Phase

We assume that the incoming elements always belong to $S_1 \cup S_2$. ShBF$_A$ finally searches for the element $e$ belongs to in the load balance applications for convenience. To query an element $e \in S_1 \cup S_2$, ShBF$_A$ simply checks whether the element $e$ belongs to the following three steps. First, it computes $o_1(e), o_2(e)$, and the $k$ hash functions $h_i(e) \% m$ ($1 \leq i \leq k$). Second, for each $1 \leq i \leq k$, it reads the $3k$ bits $B[h_i(e) \% m], B[h_i(e) \% m + o(e)],$ and $B[h_i(e) \% m + o(e)]$. Third, for these $3k$ bits, if all the $k$ bits $B[h_1(e) \% m], \ldots, B[h_k(e) \% m]$ are 1, $e$ may belong to $S_1 \cup S_2$. In this case, ShBF$_A$ records (but does not yet declare) $e \in S_1 \cup S_2$. Similarly, if all the $k$ bits $B[h_1(e) \% m + o_1(e)], \ldots, B[h_k(e) \% m + o_1(e)]$ are 1, $e$ may belong to $S_1 \cup S_2$ and ShBF$_A$ records $e \in S_1 \cup S_2$. Finally, if all the $k$ bits $B[h_1(e) \% m + o_2(e)], \ldots, B[h_k(e) \% m + o_2(e)]$ are 1, $e$ may belong to $S_1 \cup S_2$ and ShBF$_A$ records $e \in S_1 \cup S_2$. Based on what ShBF$_A$ recorded after analyzing the $3k$ bits, there are following 7 outcomes. If ShBF$_A$ records that:

1. only $e \in S_1 \cup S_2$, it declares that $e$ belongs to $S_1 \cup S_2$.
2. only $e \in S_1 \cap S_2$, it declares that $e$ belongs to $S_1 \cap S_2$.
3. only $e \in S_2 \setminus S_1$, it declares that $e$ belongs to $S_2 \setminus S_1$.
4. both $e \in S_1 \setminus S_2$ and $e \in S_1 \cap S_2$, it declares that $e$ belongs to $S_1$ but is unsure whether or not it belongs to $S_2$.
5. both $e \in S_2 \setminus S_1$ and $e \in S_1 \cap S_2$, it declares that $e$ belongs to $S_2$ but is unsure whether or not it belongs to $S_1$.
6. both $e \in S_1 \setminus S_2$ and $e \in S_2 \setminus S_1$, it declares that $e$ belongs to $S_1 \setminus S_2 \cup S_2 \setminus S_1$.  

Note that for all these seven outcomes, the decisions of ShBF$_A$ do not suffer from false positives or false negatives. However, decisions 4 through 6 provide slightly incomplete information and the decision 7 does not provide any information because it is already given that $e$ belongs to $S_1 \cup S_2$. We will shortly show that the probability that decision of ShBF$_A$ is one of the decisions 4 through 7 is very small, which means that with very high probability, it gives a decision with clear meaning, and we call it a clear answer.

### 4.3 ShBF$_A$ – Updating

Just like BF handles updates by replacing each bit by a counter, we can also extend ShBF$_A$ to handle updates by replacing each bit by a counter. We use CShBF$_A$ to denote this counting version of ShBF$_A$. Let $C$ denote the array of $2^7$ counters. To insert an element $e$, after querying $T_1$ and $T_2$ and determining whether $o(e) = 0$, $o_1(e)$, or $o_2(e)$, instead of setting $k$ bits to 1, we increment each of the corresponding $k$ counters by 1; that is, we increment the $k$ counters $C[h_1(e) \% m + o_1(e)], \ldots, C[h_k(e) \% m + o_1(e)]$ by 1. To delete an element $e$, after querying $T_1$ and $T_2$ and determining whether $o(e) = 0$, $o_1(e)$, or $o_2(e)$, we decrement $C[h_1(e) \% m + o_1(e)]$ by 1 for all $1 \leq i \leq k$. To have the benefits of both fast query processing and small memory consumption, we maintain both ShBF$_A$ and CShBF$_A$, but store array $B$ in fast SRAM and array $C$ in slow DRAM. After each update, we synchronize array $C$ with array $B$.

### 4.4 ShBF$_A$ – Analysis

Recall from Section 4.2 that ShBF$_A$ may report seven different outcomes. Next, we calculate the probability of each outcome. Let $P_i$ denote the probability of the $i$th outcome. Before proceeding, we show that $h_i(\cdot) + o_i(\cdot)$ and $h_i(\cdot) + o(\cdot)$, when $i \neq j$, are independent of each other. For this we show that given two random variables $X$ and $Y$ and a number $z \in R^+$, where $R^+$ is the set of positive real numbers, if $X$ and $Y$ are independent, then $X + z$ and $Y + z$ are independent. As $X$ and $Y$ are independent, for any $x \in R$ and $y \in R$, we have

$$P(X \leq x, Y \leq y) = P(X \leq x) \ast P(Y \leq y) \quad (13)$$

Adding $z$ to both sides of all inequality signs in $P(X \leq x, Y \leq y)$, we get

$$P(X + z \leq x + z, Y + z \leq y + z) = P(X \leq x, Y \leq y)$$

$$= P(X \leq x) \ast P(Y \leq y) \quad (14)$$

$$= P(X + z \leq x + z) \ast P(Y = y + z)$$

Therefore, $X + z$ and $Y + z$ are independent.

Let $n'$ be the number of distinct elements in $S_1 \cup S_2$, and let $k$ be the number of hash functions. After inserting all $n'$ elements into ShBF$_A$, the probability $p'$ that any given bit is still 0 is given by the following equation.

$$p' = \left(1 - \frac{1}{m}\right)^{kn'} \quad (15)$$

This is similar to one minus the false positive probability of a standard BF. When $k = \ln 2 \frac{1}{o'}$, $p' \approx 0.5$. 

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Note that the probabilities for outcomes 1, 2, and 3 are the same. Similarly, the probabilities for outcomes 4, 5, and 6 are also the same. Following equations state the expressions for these probabilities.

\[
P_1 = P_2 = P_3 = (1 - 0.5^m)^2
\]

\[
P_4 = P_5 = P_6 = 0.5^m \times (1 - 0.5^m)
\]

\[
P_7 = (0.5^m)^2
\]

When the incoming element \( e \) actually belongs to one of the three sets: \( S_1 - S_2, S_1 \cap S_2, \) and \( S_2 - S_1 \), there is one combination each for \( S_1 - S_2 \) and \( S_2 - S_1 \) and two combinations for \( S_1 \cap S_2 \). Consequently, the total probability is \( P_1 + P_2 + P_3 \) which equals 1. This validates our derivation of the expressions in Equation 16. As an example, let \( k = \frac{m}{2} \ln 2 = 10 \). Thus, \( P_1 = P_2 = P_3 = (1 - 0.5^{10})^2 \approx 0.998 \), \( P_4 = P_5 = P_6 = 0.5^{10} \times (1 - 0.5^{10}) = 9.756 \times 10^{-4} \), and \( P_7 = (1 - 0.5^{10})^2 \approx 9.54 \times 10^{-7} \). This example shows that with probability of 0.998, ShBF \(_x\) gives a clear answer, and with probability of only 9.756 \( \times 10^{-4} \), ShBF \(_x\) gives an answer with incomplete information. The probability with which it gives an answer with no information is just 9.54 \( \times 10^{-7} \), which is negligibly small.

4.5 Comparison between ShBF \(_x\) with iBF

For association queries, a straightforward solution is to build one individual BF (iBF) for each set. Let \( n_1, n_2, \) and \( n_3 \) be the number of elements in \( S_1, S_2, \) and \( S_1 \cap S_2, \) respectively. For iBF, let \( n_1, n_2, n_3 \) be the size of the Bloom filter for \( S_1 \) and \( S_2, \) respectively. Table 2 presents a comparison between ShBF \(_x\) and iBF. We observe from the table that ShBF \(_x\) needs less memory, less hash computations, and less memory accesses, and has no false positives. For the iBF, as we use the traffic trace that hits the two sets with the same probability, iBF is optimal when the two BF's use identical values for the optimal system parameters and have the same number of hash functions. Specifically, for iBF, when \( n_1 + n_2 = (n_1 + n_3)k/l, \) the probability of answering a clear answer is \( \frac{k}{1 - 0.5^k}. \) For ShBF \(_x\), when \( m = (n_1 + n_2 - n_3)k/l, \) the probability of answering a clear answer is \( (1 - 0.5^k)^m. \)

5. MULTIPlicitY QUERIES

In this section, we first present the construction and query phases of ShBF for multiplicity queries. Multiplicity queries check how many times an element appears in a multi-set. We use ShBF \(_x\) to denote the ShBF scheme for multiplicity queries. Second, we describe the updating methods of ShBF \(_x\). Last, we derive the FPR and correctness rate of ShBF \(_x\).

5.1 ShBF \(_x\) – Construction Phase

The construction phase of ShBF \(_x\) proceeds in three steps. Let \( h_1(\cdot), \ldots, h_k(\cdot) \) be \( k \) independent hash functions with uniformly distributed outputs. First, we construct an array of \( B \) of \( m \) bits, where each bit is initialized to 0. Second, to store the existence information of an element \( e \) of multi-set \( S \), we calculate \( k \) hash values \( h_1(e)\%m, \ldots, h_k(e)\%m \). To calculate the auxiliary information of \( e \), which in this case is the count \( c(e) \) of element \( e \) in \( S \), we calculate offset \( o(e) \) as \( o(e) = c(e) - 1. \) Third, we set the \( k \) bits \( B[h_1(e)\%m + o(e)], \ldots, B[h_k(e)\%m + o(e)] \) to 1. To determine the value of \( c(e) \) for any element \( e \in S \), we store the count of each element in a hash table and use the simplest collision handling method called collision chain.

5.2 ShBF \(_x\) – Query Phase

Given a query \( e \), for each \( 1 \leq i \leq k \), we first read \( c \) consecutive bits \( B[h_1(e)\%m], B[h_2(e)\%m + 1], \ldots, B[h_k(e)\%m + c - 1] \) in \( \lfloor \frac{c}{2} \rfloor \) memory accesses, where \( c \) is the maximum value of \( c(e) \) for any \( e \in S \). In these \( c \) arrays of \( c \) consecutive bits, for each \( 1 \leq j \leq c \), if all the \( k \) bits \( B[h_1(e)\%m + j - 1], \ldots, B[h_k(e)\%m + j - 1] \) are 1, we list \( j \) as a possible candidate of \( c(e) \). As the largest candidate of \( c(e) \) is always greater than or equal to the actual value of \( c(e) \), we report the largest candidate as the multiplicity of \( e \) to avoid false negatives. For the query phase, the number of memory accesses is \( k \lfloor \frac{c}{2} \rfloor \).

5.3 ShBF \(_x\) – Updating

5.3.1 ShBF \(_x\) – Updating with False Negatives

To handle element insertion and deletion, ShBF \(_x\) maintains its counting version denoted by CShBF \(_x\), which is an array \( C \) that consists of \( m \) counters, in addition to an array of \( B \) of \( m \) bits. During the construction phase, ShBF \(_x\) increments the counter \( C[h(e)\%m + o(e)] \) (for every \( 1 \leq i \leq k \)) by one every time it sets \( B[h(e)\%m + o(e)] \) to 1. During the update, we need to guarantee that one element with multiple multiplicities is always inserted into the filter one time. Specifically, for every new element \( e \) to insert into the multi-set \( S \), ShBF \(_x\) first obtains its multiplicity \( z \) from \( B \) as explained in Section 5.2. Second, it deletes the \( z \)–th multiplicity \( o(e) = z - 1 \) and inserts the \( (z + 1) \)–th multiplicity \( o(e) = z \). For this, it calculates the \( k \) hash functions \( h_i(e)\%m \) and decrements the \( k \) counters \( C[h_i(e)\%m + z - 1] \) by 1 when the counters are \( \geq 1 \). Third, if any of the decremented counters becomes 0, it sets the corresponding bit in \( B \) to 0. Note that maintaining the array \( C \) of counters allows us to reset the right bits in \( B \) to 0. Fourth, it increments the \( k \) counters \( C[h(e)\%m + z] \) by 1 and sets the bits \( B[h(e)\%m + z] \) to 1.

For deleting element \( e \), ShBF \(_x\) first obtains its multiplicity \( z \) from \( B \) as explained in Section 5.2. Second, it calculates the \( k \) hash functions and decrements the counters \( C[h(e)\%m + z - 1] \) by 1. Third, if any of the decremented counters becomes 0, it sets the corresponding bit in \( B \) to 0. Fourth, it increments the counters \( C[h(e)\%m + z - 2] \) by 1 and sets the bits \( B[h(e)\%m + z - 2] \) to 1.

Note that ShBF \(_x\) may introduce false negatives because before updating the multiplicity of an element, we first query its current multiplicity from \( B \). If the answer to that query is a false positive, i.e., the actual multiplicity of the element is less than the answer, ShBF \(_x\) will perform the second step and decrement some counters, which may cause a counter to decrement to 0. Thus, in the third step, it will set the corresponding bit in \( B \) to 0, which will cause false negatives.

5.3.2 ShBF \(_x\) – Updating without False Negatives

To eliminate false negatives, in addition to arrays \( B \) and \( C \), ShBF \(_x\) maintains a hash table to store counts of each element. In the hash table, each entry has two fields: element and its counts/multiplicities. When inserting or deleting element \( e \), ShBF \(_x\) follows four steps shown in Figure 5. First,
we obtain $e$'s counts/multiplicities from the hash table instead of ShBF$_x$. Second, we delete $e$'s $z$-th multiplicity from CShBF$_x$. Third, if a counter in CShBF$_x$ decreases to 0, we set the corresponding bit in ShBF$_x$ to 0. Fourth, when inserting/deleting $e$, we insert the $(z-1)$-th/$(z+1)$-th multiplicity into ShBF$_x$.

For multiplicity queries a false positive is defined as reporting the multiplicity of an element that is larger than its actual multiplicity. For any element $e$ belonging to multi-set $S_m$, ShBF$_x$ only sets $k$ bits in $B$ to 1 regardless of how many times it appears in $S_m$. This is because every time information about $e$ is updated, ShBF$_x$ removes the existing multiplicity information of the element before adding the new information. Let the total number of distinct elements in set $S_m$ be $n$. The probability that an element is reported to be present $j$ times is given by the following equation.

$$f_o \approx \left(1 - e^{-\frac{m}{n}}\right)^k$$  \hspace{1cm} (17)$$

We define a metric called correctness rate, which is the probability that an element that is present $j$ times in a multi-set is correctly reported to be present $j$ times. When querying an element not belonging to the set, the correctness rate $CR$ is given by the following equation.

$$CR = (1 - f_o)^e$$  \hspace{1cm} (18)$$

When querying an element with multiplicity $j$ ($1 \leq j \leq e$) in the set, the correctness rate $CR'$ is given by the following equation.

$$CR' = (1 - f_o)^{j-1}$$  \hspace{1cm} (19)$$

Note the right hand side of the expression for $CR'$ is not multiplied with $f_0$ because when $e$ has $j$ multiplicities, all positions $h_i(e) + j$, where $1 \leq i \leq k$, must be 1.

### 6. PERFORMANCE EVALUATION

In this section, we conduct experiments to evaluate our ShBF schemes and side-by-side comparison with state-of-the-art solutions for the three types of set queries.

#### 6.1 Experimental Setup

We give a brief overview of the data we have used for evaluation and describe our experimental setup.

**Data set:** We evaluate the performance of ShBF and state-of-the-art solutions using real-world network traces. Specifically, we deployed our traffic capturing system on a 10Gbps link of a backbone router. To reduce the processing load, our traffic capturing system consists of two parallel sub-systems each of which is equipped with a 10G network card and uses netmap to capture packets. Due to high link speed, capturing entire traffic was infeasible because our device could not access/write to memory at such high speed. Thus, we only captured 5-tuple flow ID of each packet, which consists of source IP, source port, destination IP, destination port, and protocol type. We stored each 5-tuple flow ID as a 13-byte string, which is used as an element of a set during evaluation. We collected a total of 10 million 5-tuple flow IDs, out of which 8 million flow IDs are distinct. To further evaluate the accuracy of our proposed schemes, we also generated and used synthetic data sets.

**Hash functions:** We collected several hash functions from open source web site [1] and tested them for randomness. Our criteria for testing randomness is that the probability of seeing 1 at any bit location in the hashed value should be 0.5. To test the randomness of each hash function, we first used that hash function to compute the hash value of the 8 million unique elements in our data set. Then, for each bit location, we calculated the fraction of times 1 appeared in the hash values to empirically calculate the probability of seeing 1 at that bit location. Out of all hash functions, 18 hash functions passed our randomness test, which we used for evaluation of ShBF and state-of-the-art solutions.

**Implementation:** We implemented our query processing schemes in C++ using Visual C++ 2012 platform. To compute average query processing speeds, we repeat our experiments 1000 times and take the average. Furthermore, we conducted all our experiments for 20 different sets of parameters. As the results across different parameter sets follow same trends, we will report results for one parameter set only for each of the three types of queries.

**Computing platform:** We did all our experiments on a standard off the shelf desktop computer equipped with an Intel(R) Core i7-3520 CPU @2.90GHz and 8GB RAM running Windows 7.

#### 6.2 ShBF$_M$ – Evaluation

In this section, we first validate the false positive rate of ShBF$_M$ calculated in Equation (1) using our experimental results. Then we compare ShBF$_M$ with BF and 1MemBF [13], which represents the prior scheme for answering

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<td>$\frac{2}{5}(1 - 0.5^2)$</td>
</tr>
<tr>
<td>false positives</td>
</tr>
<tr>
<td>YES</td>
</tr>
<tr>
<td>iBF</td>
</tr>
<tr>
<td>Optimal Memory</td>
</tr>
<tr>
<td>m$_1$ + m$_2$ = (n$_1$ + n$_2$ - n$_3$)k/ln2</td>
</tr>
<tr>
<td>#hash computations</td>
</tr>
<tr>
<td>2k</td>
</tr>
<tr>
<td>#memory accesses</td>
</tr>
<tr>
<td>k+2</td>
</tr>
<tr>
<td>Probability of a clear answer</td>
</tr>
<tr>
<td>$\frac{2}{5}(1 - 0.5^2)$</td>
</tr>
<tr>
<td>false positives</td>
</tr>
<tr>
<td>NO</td>
</tr>
</tbody>
</table>
Figure 6: Comparison false positive rates of ShBF_M and 1MemBF.

Figure 7: Comparison of number of memory accesses per query of ShBF_M and BF.

Figure 8: Comparison of query processing speeds of ShBF_M, BF, and 1MemBF.

Figure 9: Comparison of ShBF_A and iBF.

Figure 10: Comparison of ShBF_A, Spectral BF, and CM Sketch.
membership queries, in terms of FPR, the number of memory accesses, and query processing speed.

6.2.1 ShBF\textsubscript{M} – False Positive Rate
Our experimental results show that the FPR of ShBF\textsubscript{M} calculated in Equation (1) matches with the FPR calculated experimentally. For the experiments reported in this section, we set \( k = 8 \), \( m = 22008 \), \( w = 57 \), and vary \( n \) from 1000 to 1500. We first insert 1000 elements into ShBF\textsubscript{M} and then repeatedly insert 20 elements until the total number of elements inserted into ShBF\textsubscript{M} became 1500. On inserting each set of 20 elements, we generated membership queries for 7,000,000 elements whose information was not inserted into ShBF\textsubscript{M} and calculated the false positive rate. Figure 6(a) shows the false positive rate of ShBF\textsubscript{M} calculated through these simulations as well as through Equation (1). The bars in Figure 6(a) represent the theoretically calculated FPR, whereas the lines represent the FPR observed in our experiments. Our results show that the relative error between the FPRs of ShBF\textsubscript{M} calculated using simulation and theory is less than 3%, which is practically acceptable. Relative error is defined as \(|FPR\textsubscript{R} - FPR\textsubscript{E}| / FPR\textsubscript{R} \), where FPR\textsubscript{R} is the false positive rate calculated using simulation and FPR\textsubscript{E} is the false positive rates calculated using theory. The relative error of 3% for ShBF\textsubscript{M} is the same as relative error for BF calculated using simulation and the theory developed by Bloom et al. [3]. Using same parameters, the FPR of 1MemBF is over 5 to 10 times that of ShBF\textsubscript{M}. If we increase the space allocated to 1MemBF for storage to 1.5 times of the space of ShBF\textsubscript{M} calculated in Table 2 matches with the probability calculated in our experiments. Then we compare ShBF\textsubscript{A} with IBF in terms of FPR, memory accesses, and query processing speed.

6.3 ShBF\textsubscript{A} – Evaluation
In this section, we first validate the probability of a clear answer of ShBF\textsubscript{M} calculated in Table 2 using our experimental results. Then we compare ShBF\textsubscript{A} with IBF in terms of FPR, memory accesses, and query processing speed.

6.3.1 ShBF\textsubscript{A} – Probability of Clear Answer
Our results show that probability of clear answer for ShBF\textsubscript{A} calculated in Table 2 matches with the probability calculated experimentally. We performed experiments for both IBF and ShBFA using two sets with 1 million elements such that their intersection had 0.25 million elements. The querying elements hit the three parts with the same probability. While varying the value of \( k \), we also varied the value of \( m \) to keep the filter at its optimal. Note that in this case, IBF uses 1.7 times more memory than ShBFA. We observe from Figure 9(a) that the simulation results match the theoretical results, and the average relative error is 0.7% and 0.004% for IBF and ShBFA, respectively, which is negligible. When the value of \( k \) reaches 8, the probability of a clear answer reaches 66% and 99% for IBF and ShBFA, respectively.

6.3.2 ShBF\textsubscript{A} – Memory Accesses
Our results show that the average number of memory accesses per query of ShBF\textsubscript{A} is 0.66 times that of IBF. Figure 9(b) shows the number of memory accesses for different values of \( k \). We observed similar trends for different values of \( m \) and \( n \), but have not including the corresponding figures due to space limitation.

6.3.3 ShBF\textsubscript{A} – Query Processing Speed
Our results show that the average query processing speed of ShBF\textsubscript{A} is 1.4 times faster than that of IBF. Figure 9(c) plots the the query processing speed of ShBF\textsubscript{A} and IBF for different values of \( m \).

6.4 ShBF\textsubscript{\texttimes} – Evaluation
In this section, we first validate the correctness rate (CR) of ShBF\textsubscript{\texttimes} calculated in Equation (18). Then we compare ShBF\textsubscript{\texttimes} with spectral BF [8] and CM Sketches [9] in terms of CR, number of memory accesses, and query processing speed. The results for CM Sketches and Spectral BF are similar because their methods of recording the counts is similar.

6.4.1 ShBF\textsubscript{\texttimes} – Correctness Rate
Our results show that the CR of ShBF\textsubscript{\texttimes} calculated in Equation (18) matches with the CR calculated experimentally. Our results also show that on average, the CR of ShBF\textsubscript{\texttimes} is 1.6 times and 1.79 times of that of Spectral BF and CM Sketches, respectively. For the experiments reported in this section, we set \( c = 57 \), \( n = 100,000 \), and vary \( k \) in the range \( 8 \leq k \leq 16 \). For spectral BF and CM sketches, we set use 6 bits for each counter. For each value of \( k \), as ShBF\textsubscript{\texttimes} is more memory efficient, we use 1.5 times the optimal memory (i.e.,
as $k < 7$. Figure 10(b) plots the number of memory accesses of $\text{ShBF}_x$, CM Sketch, and spectral BF, calculated from the same experiments that we used to plot Figure 10(a) except that $k$ ranges from 3 to 18.

6.4.3 $\text{ShBF}_x$ – Query Processing Speed

Our results show that $\text{ShBF}_x$ is faster than spectral BF and CM Sketches when $k \geq 11$. We evaluate the query processing speed of $\text{ShBF}_x$, CM Sketch, and spectral BF using the same parameters as for Figure 10(b). Figure 10(c) plots the query processing speeds of $\text{ShBF}_x$ and spectral BF. We observe from this figure that when $k > 11$, the average query processing speed of $\text{ShBF}_x$ is over 3 Mqps.

7. CONCLUSION

The key contribution of this paper is in proposing Shifting Bloom Filter, a general framework to answer a variety of set queries, i.e., membership, association, and multiplicity queries. The key technical depth of this paper is in the analytical modeling of $\text{ShBF}$ for each of the three types queries, calculating optimal system parameters, and finding the minimum FPRs. We validated our analytical models through simulations using real world network traces. Our theoretical analysis and experimental results show that $\text{ShBF}$ significantly advances state-of-the-art solutions on all three types of set queries.

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9. REFERENCES


