Continued fraction absorbing boundary conditions for convex polygonal domains

Murthy N. Guddati1,*,† and Keng-Wit Lim2

1Department of Civil, Construction and Environmental Engineering, North Carolina State University, Raleigh, NC 27695-7908, U.S.A.
2Simpson Gumpertz & Heger Inc., Boston, MA, U.S.A.

SUMMARY

Continued fraction absorbing boundary conditions (CFABCs) are highly effective boundary conditions for modelling wave absorption into unbounded domains. They are based on rational approximation of the exact dispersion relationship and were originally developed for straight computational boundaries. In this paper, CFABCs are extended to the more general case of polygonal computational domains. The key to the current development is the surprising link found between the CFABCs and the complex co-ordinate stretching of perfectly matched layers (PMLs). This link facilitates the extension of CFABCs to oblique corners and, thus, to polygonal domains. It is shown that the proposed CFABCs are easy to implement, expected to perform better than PMLs, and are effective for general polygonal computational domains. In addition to the derivation of CFABCs, a novel explicit time-stepping scheme is developed for efficient numerical implementation. Numerical examples presented in the paper illustrate that effective absorption is attained with a negligible increase in the computational cost for the interior domain. Although this paper focuses on wave propagation, its theoretical development can be easily extended to the more general class of problems where the governing differential equation is second order in space with constant coefficients. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: absorbing boundary conditions; non-reflecting boundary conditions; perfectly matched layers; wave propagation; finite elements; finite differences

1. INTRODUCTION

Unbounded domains are ubiquitous in engineering and physics, especially in problems associated with wave propagation. In many such problems, the goal of the simulation is to obtain the field variable in a small bounded part of the domain, henceforth called the interior or the...
computational domain. Often, the interior is modelled using domain-based methods such as the finite element or finite difference methods, and special boundary conditions are applied on its boundary. These boundary conditions are expected to mimic the absorption of waves into the unbounded exterior, hence, referred to as transmitting, non-reflecting, or absorbing boundary conditions (ABCs). Due to their widespread applicability, ABCs have been the focus of significant research over the past 30 years. This paper develops a new ABC for transient wave propagation problems.

Most of the existing ABCs for transient wave propagation can be classified into two broad classes: differential equation-based and material-based [1]. Differential equation-based ABCs are obtained by factoring the wave equation and allowing only outgoing waves. Material-based ABCs, on the other hand, are realized by surrounding the computational domain with a lossy material that dampens the outgoing waves, thus reducing the reflections. Differential equation-based ABCs can be further classified into two sub-categories: global and local. Global ABCs often attempt to capture the exact absorption, with the help of Green's function of the exterior, and involve expensive convolution operations that are global in space and time. Global ABCs may be useful for small-scale wave propagation problems and lead to highly accurate results, but tend to be prohibitively expensive for large-scale wave propagation problems. For this reason, this research focuses on local ABCs and material ABCs. Furthermore, curved boundaries such as those treated in References [2, 3] are not considered; rather, discussion is limited to straight and polygonal computational boundaries.

The fundamental idea behind many of the local ABCs is the rational approximation of the exact impedance of the exterior. The exact impedance, or the associated dispersion relationship, involves the square root of a differential operator, which translates into a pseudo-differential operator that involves expensive convolution operations. Rational approximation of the square root function facilitates the conversion of the pseudo-differential operator into a differential operator, making the resulting boundary condition amenable to numerical computation. The use of rational approximations for unbounded domain modelling has been pioneered independently by Engquist and Majda [4, 5] and Lindman [6]. They made use of rational/continued fraction approximations to develop a series of ABCs of increasing accuracy. In spite of their theoretical potential for high accuracy, only low-order versions were used, mainly due to the fact that higher-order ABCs cannot be implemented into standard finite element settings. The multidirectional absorbers developed by Higdon [7], while based on a different idea, can be considered as generalizations of the Engquist–Majda boundary conditions and have the same limitations. The situation has changed in the past decade, with several researchers presenting practical approaches (based on auxiliary variables) to implement high-order local ABCs (see Reference [8] for a review). These ABCs are highly effective in modelling wave absorption into unbounded domains [9]. Most of these boundary conditions were developed for straight computational boundaries, with some of them extending to orthogonal corner regions. The extensions to corner regions are, however, not very transparent, and often require a special numerical implementation that may affect stability and accuracy. Thus, it is desirable to have a local ABC that provides an easy and transparent numerical implementation and that is also applicable to both orthogonal and oblique corners.

Material ABCs are based on the idea of artificially absorbing waves with the help of lossy material placed next to the computational boundary. While this idea has existed for more than 20 years [10], it was made robust in 1994 through Bérenger's perfectly matched layer (PML) [11]. The PML is designed so that the travelling waves are converted into evanescent waves.
when entering the lossy medium, yet create no reflection at the computational boundary. Other than the discretization error, the only approximation in the PML is that the lossy medium is truncated somewhere, creating reflections that eventually enter the computational domain. The ease of implementation of the PML and its performance has triggered explosive growth in the development of PML-based ABCs. Different interpretations of the PML were soon offered by various researchers in the field (e.g. References [12–14]). In particular, the co-ordinate stretching interpretation of the PML, provided by Chew et al. [12, 13], is one of the most widespread interpretations in PML literature. Various extensions and generalizations of the PML have been expounded subsequently (e.g. References [15–21]), and research into the optimization of PML parameters (to minimize the discretization and truncation errors) remains active. Despite the attractiveness of the PML, Hagstrom [9] observes that, due to discretization and truncation errors, its performance of PML is not as good as that of local ABCs, although local ABCs do have the drawback of limited applicability.

Considering the accuracy of local ABCs and the broader applicability of PMLs, it is desirable to obtain a boundary condition that combines the advantages of the two methods. While the two boundary conditions are based on completely different ideas, a surprising link between them has been identified. This paper presents this link and the development of new ABCs that combine the strengths of local ABCs and PMLs. This link was explored by Guddati [22] who observed similarities in the performance of the continued fraction ABCs (CFABCs) of Reference [23] and PMLs. Later, CFABCs were linked to the optimal finite difference discretization of PMLs by Asvadurov et al. [24], indicating that CFABCs should perform better than general PMLs. While the work reported in Reference [24] is the optimal local ABC, it is limited to the context of staggered-grid finite difference methods. Unfortunately, the ABCs in Reference [24] cannot be efficiently implemented in the standard Galerkin finite element method, and it is therefore desirable to obtain an equally effective ABC that fits in standard finite element settings.

In the current paper, an extremely simple link between CFABCs and PMLs is developed which translates into a simple and efficient implementation of the CFABCs in standard finite element settings. Specifically, when midpoint-integrated linear finite elements are used to discretize the PML with purely imaginary stretching, the boundary condition becomes equivalent to a CFABC with better absorption properties than the regularly discretized PML. Due to this simple interpretation, CFABCs can be immediately extended to orthogonal corner regions. This idea is extended further to develop CFABCs for oblique corners with angles less than 180°, thus making the ABCs applicable to convex polygonal interiors. As with the original CFABC formulation, the only unconventional aspect is that the spatially discrete equations are third-order evolution equations. A simple, systematic and stable time-stepping procedure is herein developed to solve these equations. The time-stepping procedure is based on a combination of the Crank–Nicholson method and Newmark method, and contains all the flexibility associated with the original Newmark method, including the option of explicit computation.

The outline of the rest of the paper is as follows. After a discussion of some preliminaries in Section 2, a new derivation of the CFABCs for straight boundaries is presented in Section 3. The link to the existing local ABCs and material ABCs are discussed in the same section. In Section 4, CFABCs are extended to oblique corners. Finite element discretization of the interior and the CFABCs is given in Section 5. Section 6 contains the new time-stepping techniques developed for solving the resulting evolution equations. In Section 7, several numerical examples are presented to illustrate the effectiveness of CFABCs. Finally, Section 8 provides a summary and conclusions.
2. PRELIMINARIES

This paper focuses on the development of CFABCs for scalar wave equations, which represent the propagation of anti-plane shear waves in solids as well as the acoustic wave propagation in fluids. In particular, consider the equation governing the anti-plane shear deformation of linearly elastic media in the \(x-z\) plane:

\[
-\mu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial^2 u}{\partial z^2} + \rho \frac{\partial^2 u}{\partial t^2} = 0
\]  

(1)

where \(\mu\) is the shear modulus and \(\rho\) is the density. While only two-dimensional problems are considered in this paper, inherent concepts are easily extended to three-dimensional scalar wave equations. For simplicity, the same notation is used for a function and its Fourier transform; the use of time, frequency or wave-number domains is always clear from the context.

To facilitate the presentation of the underlying concepts of CFABCs, a simple model problem is considered (Figure 1) where the objective is to replace the full-space by a left half-space and an ABC capturing the effect of the right half-space. Without any loss of generality, the computational (truncation) boundary is assumed to be in the vertical \((z)\) direction at \(x = x_0\). The exact ABC is the stiffness relationship for the right half-space, which can be written in the following form:

\[
\mu \frac{\partial^2 u}{\partial x} \bigg|_{x=x_0} + K_{\text{exact}} u_0 = 0
\]  

(2)

where \(K_{\text{exact}}\) is the (exact) stiffness of the right half-space and \(u_0\) is the displacement at \(x = x_0\). In order to obtain the exact stiffness, first the governing equation (1) is Fourier transformed in \(z\) and \(t\), resulting in a second-order differential equation in \(x\):

\[
\mu \left( -\frac{\partial^2 u}{\partial x^2} - k_x^2 u \right) = 0
\]  

(3)

Figure 1. Model problem: replacement of full-space with half-space augmented with an ABC.
In the above, \( k_x \) is the horizontal wave number given by the root of the dispersion relationship,

\[
k_x^2 + k_z^2 = \left( \frac{\omega}{c} \right)^2
\]

where \( k_z \) is the vertical wave number, \( \omega \) is the frequency, and \( c = \sqrt{\mu/\rho} \) is the wave velocity. The sign convention for the Fourier transform is such that the following dualities apply:

\[
\frac{\partial}{\partial x} \leftrightarrow ik_x, \quad \frac{\partial}{\partial z} \leftrightarrow ik_z, \quad \frac{\partial}{\partial t} \leftrightarrow -i\omega
\]

Noting that in the right half-space the displacement cannot grow in the positive \( x \) direction, \( k_x \) must have a non-negative imaginary part. In the event that the waves are purely propagating, the propagation direction should be in the positive \( x \) direction, making \( k_x \) a positive real number whenever \( \omega \) is positive. These conditions on \( k_x \) constitute the exact stiffness relationship (exact ABC) for the right half-space, i.e.

\[
\left. \frac{\partial u}{\partial x} \right|_{x=x_0} - ik_x u_0 = 0
\]

with \( k_x \) being the root of the dispersion relationship (4) lying on the positive real or the positive imaginary axes, i.e. it is the positive square root,

\[
k_x = \sqrt{\frac{\omega^2}{c^2} - k_z^2}
\]

Comparing (2) and (6), the exact stiffness of the right half-space is given by

\[
K_{\text{exact}} = -ik_x
\]

The exact stiffness relationship is simple in the Fourier domain. But due to the presence of the square root in (7), the stiffness relationship, when transformed back into the space domain, involves expensive pseudo-differential (convolution) operators. In order to facilitate efficient computation, the exact wave number in (7) is approximated as a rational function of \( k_z \) and \( \omega \), translating the stiffness relationship into a differential equation in terms of \( z \) and \( t \). This is the fundamental idea behind many of the local ABCs.

The CFABC [23] is one of such local ABCs. CFABCs utilize (rational) continued fraction expansions not only in deriving the boundary condition, but also in implementing it in the standard finite element and finite difference settings. CFABCs are shown to be generalizations of Engquist–Majda conditions and equivalent to Higdon’s multidirectional boundary conditions. The most attractive feature of CFABCs is that the implementation results in an absorbing mesh that is topologically equivalent to a standard finite element mesh. The superior absorption properties of CFABCs are illustrated in Reference [23]. The CFABCs, however, have a fundamental limitation; like the other local ABCs, they cannot be easily extended to corner regions. The rest of the paper presents an alternative derivation of CFABCs and their extension to orthogonal as well as oblique corners.

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3. CONTINUED FRACTION ABSORBING BOUNDARY CONDITIONS: A NEW DERIVATION

The new derivation of CFABCs is based on the idea of replacing the right half-space by a computationally tractable finite element mesh that efficiently absorbs the propagating waves. The first step in the mesh extension procedure is to replace the half-space \((x_0, \infty)\) by a finite element layer \((x_0, x_1)\) of length \(L = x_1 - x_0\) (discretized only in the \(x\) direction) and another half-space \((x_1, \infty)\) (see Figure 2). The displacement is assumed to vary linearly between the two edges, i.e.

\[
 u(x, z; t) = \left[ 1 - \frac{x}{L} \right] \frac{x}{L} \begin{bmatrix} u_0(z; t) \\ u_1(z; t) \end{bmatrix}
\]  

where \(u_0\) and \(u_1\) are the displacements at the left and right edges of the finite element layer, respectively.

The link between the stiffness of the original half-space and the stiffness of the smaller half-space is made starting with the variational statement of the governing equation for the layer:

\[
 \int_{x_0}^{x_1} \delta u \mu \left( -\frac{\partial^2 u}{\partial x^2} - k_x^2 u \right) \, dx = 0
\]  

where \(\delta u\) is the variation of \(u\). Performing integration by parts on the first term results in

\[
 \int_{x_0}^{x_1} \left( \frac{\partial \delta u}{\partial x} \mu \frac{\partial u}{\partial x} - \delta u \mu k_x^2 u \right) \, dx - \left[ \delta u \mu \frac{\partial u}{\partial x} \right]_{x=x_1} + \left[ \delta u \mu \frac{\partial u}{\partial x} \right]_{x=x_0} = 0 \quad \text{for any } \delta u
\]  

Figure 2. Replacement of the right half-space with a layer plus another right half-space.
Note from the definition of the half-space stiffness in (2) that
\[
\begin{bmatrix}
\frac{\partial u}{\partial x}
\end{bmatrix}_{x=x_0} = -K_0 u_0 \quad \text{and} \quad \begin{bmatrix}
\frac{\partial u}{\partial x}
\end{bmatrix}_{x=x_1} = -K_1 u_1
\] (12)

where \(K_0\) and \(K_1\) represent the stiffness of the original half-space and the smaller half-space, respectively. In addition, following the (Bubnov) Galerkin approach \(25\), \(\delta u\) is approximated in the same way as \(u\) in (9), i.e.

\[
\delta u(x) = \begin{bmatrix} 1 - \frac{x}{L} \end{bmatrix} \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial u_1}{\partial x} \end{bmatrix}
\] (13)

Substituting (9), (13), (12) in (11), we obtain, after some manipulation,

\[
\begin{bmatrix}
\delta u_0 \\ \delta u_1
\end{bmatrix} \begin{bmatrix}
\int_{x_0}^{x_1} \mu \begin{bmatrix}
1 \\ -1
\end{bmatrix} - k_x^2 \begin{bmatrix}
(1 - \frac{x}{L})^2 \\ (1 - \frac{x}{L}) \frac{x}{L}
\end{bmatrix} \begin{bmatrix}
(1 - \frac{x}{L}) \frac{x}{L}
\end{bmatrix}
\end{bmatrix} dx 
\times \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \delta u_1 K_1 u_1 - \delta u_0 K_0 u_0 \approx 0 \quad \text{for any} \ \delta u_0, \delta u_1
\] (14)

Note that the above equality is approximate due to the errors associated with the finite element discretization. Eliminating \(\{\delta u_0 \ \delta u_1\}\) from (14), we obtain a matrix form relating the stiffness of the original half-space with the smaller half-space:

\[
\begin{bmatrix}
K_0 u_0 \\ 0
\end{bmatrix} \approx \begin{bmatrix}
\bar{a} \\ \bar{b}
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\ 0 & K_1
\end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}
\] (15)

Essentially, the first \(2 \times 2\) matrix on the right-hand side is the dynamic stiffness matrix of the layer, which is obtained by evaluating the integral in (14). The elements of this matrix are

\[
\bar{a} = \mu \left( \frac{1}{L} - \frac{k_x^2 L}{3} \right) \quad \text{and} \quad \bar{b} = \mu \left( -\frac{1}{L} - \frac{k_x^2 L}{6} \right)
\] (16)

Condensing out \(u_1\) from (15), we obtain the (approximate) relationship between \(K_0\) and \(K_1\):

\[
K_0 \approx \frac{\bar{a}}{\bar{a} + K_1}
\] (17)

The non-exactness of the above equality can be verified by noting that if the exact stiffness is used for the smaller half-space \((K_1 = -i\mu k_x)\), the exact stiffness of the composite half-space \((K_0 = -i\mu k_x)\) is not recovered. The error in the approximation depends on the length of the finite element layer, which must be kept small in order to have accurate results.

The next step, which is based on the midpoint integration rule, is the key to the new formulation of CFABCs. Essentially, when the midpoint integration rule is used to calculate the
stiffness of the finite element layer, i.e. the integral in (14), the discretization error is eliminated in the calculation of the stiffness of the original half-space:

\[ K_0 = a - \frac{b^2}{a + K_1} \]  

(18)

where \( a \) and \( b \) are the elements of the layer stiffness matrix obtained by evaluating the integral in (14) using the midpoint rule. These elements are given by

\[ a = \mu \left( \frac{1}{L} - \frac{k_x^2 L}{4} \right) \quad \text{and} \quad b = \mu \left( -\frac{1}{L} - \frac{k_x^2 L}{4} \right) \]  

(19)

In contrast to (17), (18) is exact in that, if the exact stiffness is used for the smaller half-space \( (K_1 = -i\mu k_x) \), the exact stiffness of the original half-space \( (K_0 = -i\mu k_x) \) is recovered (for any element length \( L \) and wave number \( k_x \)). This is a powerful property, as it removes the limitation on the element length, thus increasing the efficiency of the discretization. Also, note that (18) is a continued fraction iteration for the stiffness, which eventually turns out to be a generalization of the continued fraction expansion used for local ABCs and, in particular, the original CFABCs.

The third step of the derivation is to apply the splitting recursively to replace the right half-space by an infinite number of midpoint-integrated finite element layers. This is an exact representation of the half-space stiffness, but is computationally intractable due to the presence of an infinite number of degrees of freedom. To make it tractable, the number of layers is truncated, and the Dirichlet boundary condition is applied on the right edge of the discretization, as shown in Figure 3. This truncation introduces error in the half-space stiffness, which can be measured with the help of the reflection coefficient (the ratio between the reflected and incident wave amplitudes at the computational boundary). For the sake of simplified presentation, the reflection coefficient is derived later in this section and is given by (28). Based on (28), note that perfect absorption is achieved when \( k_x = 2i/L \). This indicates that the mesh in Figure 3 is effective for evanescent waves, but completely reflects the propagating waves since \(|R| = 1\) for all real values of \( k_x \). Fortunately, the mesh can be made to absorb propagating waves.
waves by simply choosing imaginary element lengths; this is the final step of the CFABC derivation. Specifically, for frequency domain computations, the element lengths are chosen as 

\[ L_j = \frac{2i}{k_j}, \]

where \( k_j \) are the parameters of the CFABCs, called the reference wave numbers. With these lengths, waves are perfectly absorbed if the horizontal wave number, \( k_x \), coincides with one of the reference wave numbers, \( k_j \). For time-domain computations, these reference wave numbers are made frequency dependent, i.e. \( k_j = \frac{\omega}{c_j} \). Here, \( c_j \) are the reference phase velocities, which become the primary parameters of the time-domain versions of the CFABCs. Such choice of reference wave numbers is important to avoid any complex arithmetic in time-domain computations (as seen in Section 5, all the contribution matrices from the CFABC are real valued). The time-domain CFABCs absorb all the wave modes with phase velocities that match one of the reference phase velocities. Other wave modes, while not perfectly absorbed, are nonetheless well absorbed. As will be shown later, the reflection coefficient is significantly reduced as the number of CFABC layers is increased, indicating that the CFABCs are effective ABCs for general transient wave propagation.

Summarizing, a CFABC for a straight edge is constructed by: (a) choosing the number of absorbing layers, \( n \); (b) choosing the reference phase velocities, \( c_j \), for each layer; (c) replacing the half-space by \( n \) layers with lengths, \( L_j = \frac{2i c_j}{\omega} \); and (d) evaluating all the layer matrices using the midpoint rule in the direction perpendicular to the boundary. The subscript, \( x \), is removed to emphasize that the above procedure is applicable to any arbitrarily oriented straight computational boundary. Note that, due to the frequency-dependent element lengths, the entire formulation (including the finite element discretization) is first done in frequency domain and, at the very end, the contributions from various elements are transformed back into the time domain (see Section 5). Unlike the original CFABC in Reference [23], we no longer need to write the equation form of the CFABC, but just implement the special mesh extension technique. It is also important to note that the element lengths need not be purely imaginary; they can be chosen as complex numbers. Such a generalization is termed, the complex-layer CFABC, which may be utilized for accurately absorbing evanescent waves. However, complex-layer CFABCs are not implemented in this study since our primary objective is the absorption of propagating waves.

3.1. Reflection coefficient

The performance of absorbing conditions is often examined through reflection coefficients. This section provides the derivation of the reflection coefficient for an \( n \)-layer CFABC. For the sake of presentation, the numbering of the layers is reversed from left-to-right to right-to-left. Additionally, the half-space stiffness approximated by the \( n \)-layer CFABC is denoted by \( K_n \). Due to the approximation, when a rightward propagating incident wave of the form, \( Ae^{ik_x x} \), impinges onto the CFABC, a reflected wave of the form, \( RAe^{-ik_x x} \), is generated, with \( R \) being the reflection coefficient. To obtain \( R \), the total wave field, \( A(e^{ik_x x} + Re^{-ik_x x}) \), is substituted in the approximate boundary condition:

\[ \mu \frac{\partial u}{\partial x} \bigg|_{x=0} + K_n u \bigg|_{x=0} = 0 \]  

resulting in

\[ (i\mu k_x A - i\mu k_x RA) + K_n (A + AR) = 0 \]
Noting that the exact stiffness, \( K_{\text{exact}} = -i\mu k_x \), the above equation can be rearranged to result in the expression for the reflection coefficient:

\[
R_n = \frac{K_{\text{exact}} - K_n}{K_{\text{exact}} + K_n}
\]  

(22)

Since an \( n \)-layer CFABC is composed of the \( n \)th midpoint-integrated finite element layer and an \((n - 1)\)-layer CFABC, \( K_n \) can be written in terms of the stiffness of \((n - 1)\)-layer CFABC \( K_{n-1} \) (from (18)):

\[
K_n = \frac{b_n^2}{a_n + K_{n-1}}
\]  

(23)

One can also verify from (19) and (8) that

\[
a_n^2 - b_n^2 = K_{\text{exact}}^2
\]

(24)

Substituting (23) in (22) and utilizing (24), we obtain

\[
R_n = \left( \frac{a_n - K_{\text{exact}}}{a_n + K_{\text{exact}}} \right) \left( \frac{K_{\text{exact}} - K_{n-1}}{K_{\text{exact}} + K_{n-1}} \right)
\]

(25)

which is equivalent to the recursive relationship

\[
R_n = \left( \frac{a_n - K_{\text{exact}}}{a_n + K_{\text{exact}}} \right) R_{n-1}
\]

(26)

Noting that the magnitude of the reflection coefficient for the 0-layer CFABC (the Dirichlet boundary) is unity, and applying the above equation recursively, we obtain

\[
|R_n| = \prod_{j=1}^{n} \left( \frac{a_j - K_{\text{exact}}}{a_j + K_{\text{exact}}} \right)
\]

(27)

Substituting (19) in the above equation and rearranging, we obtain the reflection coefficient for a general CFABC (including the complex-layer CFABC):

\[
|R_n| = \prod_{j=1}^{n} \left( \frac{k_x - 2i/L_j}{k_x + 2i/L_j} \right)^2
\]

(28)

For the (imaginary-layer) time-domain CFABC, noting that \( L_j = 2i\omega / \omega \) and \( k_x = \omega / c_x^p \), where \( c_x^p \) is the phase velocity, the reflection coefficient takes the form:

\[
|R_n| = \prod_{j=1}^{n} \left( \frac{c_x^j - c_x^p}{c_x^j + c_x^p} \right)^2
\]

(29)

The reflection coefficients can also be written in terms of incident angle \( \theta \). Since \( c_x^p = c / \cos(\theta) \),

\[
|R_n| = \prod_{j=1}^{n} \left( \frac{\cos \theta_j - \cos \theta}{\cos \theta_j + \cos \theta} \right)^2
\]

(30)
where \( \theta_j = \cos^{-1}(c/c_j) \) are the incident angles corresponding to the reference phase velocities. The above expression is the square of the familiar reflection coefficient of Higdon’s multidirectional absorbers [7]. It is also identical to the reflection coefficient of the original CFABC [23], bringing us to examine the relationship with other boundary conditions.

3.2. Relationship to original CFABCs and other local absorbers

We first prove that the new (imaginary-layer) CFABCs are identical to the original CFABCs. To show this, we relate the reference phase velocities to the reference incident angles through

\[ c_j = c / \cos \theta_j, \]

resulting in

\[ L_j = 2i c / (\omega \cos \theta_j). \]

Substituting this equation into (19), we obtain

\[ a_j = \frac{i \omega}{2c} \left( -\cos \theta_j - \frac{\cos^2 \theta_j}{\cos \theta_j} \right) \]

and

\[ b_j = \frac{i \omega}{2c} \left( \cos \theta_j - \frac{\cos^2 \theta_j}{\cos \theta_j} \right) \]

which can easily be related to the entries of the original form of the CFABCs in Reference [23]. Furthermore, it is already noted in Reference [23] that CFABCs are similar to Higdon’s multidirectional absorbers and are generalizations of the Lindman and Engquist–Majda ABCs. Also, Kausel [26] noted the similarity between these ABCs as well as the other existing local ABCs. Thus, we conclude that the proposed CFABCs are similar to existing local ABCs and that they have the added advantage of ease of implementation and extendibility to more complex geometries, as described in the remainder of this paper.

3.3. Relationship to perfectly matched layers (PMLs)

CFABCs are closely related to the complex co-ordinate stretching ideas of the PML. Since the PML is equivalent to stretching the domain into the complex space [13], the discretized PML can be considered as augmenting the interior with discrete layers of complex thickness, but with the integration for the stiffness matrix evaluation performed in an exact manner. It follows that, if the midpoint integration rule is used to calculate these integrals, the discretized PML is equivalent to the complex-layer CFABC. Noting that the use of midpoint integration eliminates the reflections from the exterior even after discretization, the complex-layer CFABCs can be viewed as the perfectly matched discrete layers (PMDLs). The (imaginary-layer) CFABCs, implemented in this paper, can be considered as the limiting case of PMDLs, where the real parts of the element lengths are zero.

Based on Proposition 3.1 in Reference [24], which states that purely imaginary stretching is optimal for absorbing propagating waves, and also based on the numerical experiments presented in Reference [27], we note that the imaginary-layer CFABCs would be more accurate than the complex-layer CFABCs. Since complex-layer CFABCs are perfectly matched at the discrete level, they are expected to perform better than the discretized PMLs with exact integration. Thus, it is expected that the imaginary-layer CFABCs would perform better than the complex-layer CFABCs, which in turn would perform better than the regular PMLs. The above argument ignores the discretization error for the interior. Hence, our hypothesis should be verified with extensive numerical experiments, but such verification is outside the scope of this paper. However, it should be noted that Hagstrom [9] indicates that the rational approximation-based and Higdon’s ABCs (which are equivalent to imaginary-layer CFABCs) perform better than PMLs; this observation is indeed consistent with our hypothesis.
4. EXTENSION TO CORNER REGIONS

The treatment of corners for differential equation-based ABCs is not a new issue. Earlier work was done by Bamberger et al. [28] and Collino [29]. These approaches formulate the corner absorbers by deriving corner compatibility conditions, which are necessary to ensure the regularity of the solution of the problem when the data are smooth. More recently, Vacus [30] and Hagstrom and Warburton [31] also describe procedures for deriving corner compatibility conditions. All these approaches require special implementations and are limited to orthogonal corners. As highlighted by Hagstrom [9], a simpler approach for the treatment of orthogonal and oblique corners is desirable.

The new derivation of the CFABC provides just such corner ABCs. In light of the new physical representation of CFABCs, the extension to corners becomes straightforward, and it can also be made to include orthogonal as well as oblique corners. The corner region of the computational boundary is the intersection of two straight edges and a corner CFABC can be considered as the tensor product of the two-edge CFABCs. Such an idea is widely used in the PML literature, starting with the original paper [11], but most of the discussion is limited to orthogonal corners. Here, the tensor product idea is incorporated in a consistent way to derive effective CFABCs for the corner regions as well. As shown in Figure 4, the CFABC for a corner is obtained by taking a simple tensor product of the two-edge CFABCs, i.e. the tensor product of the two discretized lines perpendicular to the two intersecting boundaries at the corner. Such a treatment results in a rectangular absorbing mesh for orthogonal corners, but tends to produce an unconventionally shaped parallelogram mesh for oblique corners. To the best of our knowledge, there does not appear to be any similar development for oblique corners, perhaps because of the associated unconventional shape of the absorber. The choice of parallelogram-shaped absorbing elements may be justified using a more rigorous mathematical theory, but is not considered here. We simply note that the effectiveness of the parallelogram absorbers is verified by extensive numerical testing. (One of the examples is presented in Section 7.2.)

The lengths of the parallelogram elements are inherited from the associated edge CFABCs; hence, the length of each side is imaginary and is inversely proportional to the frequency.

![Figure 4. Parallelogram CFABC elements at the corner.](image-url)
As derived in the appendix, such a choice of the lengths makes the contribution from corner absorbers independent of frequency, making the numerical implementation of corner absorbers even simpler than edge absorbers. Similar to the implementation in Reference [24], since the contribution is frequency independent, the degrees-of-freedom interior to the corner absorbing region can be condensed out, thus reducing the number of degrees-of-freedom. (This idea is also similar to the suggestion in Reference [31].)

The development of CFABCs for oblique corners results in accurate and flexible ABCs that are applicable to convex polygonal domains. The requirement for convexity follows from standard causality arguments as well as from the fact that the angle of the parallelogram becomes negative whenever the angle at the corner is greater than 180°. Clearly, the use of polygonal computational boundaries could be utilized to reduce the extent of the interior, thus aiding in the reduction of the computational cost.

5. FINITE ELEMENT DISCRETIZATION

The discretization of a CFABC is performed in a manner consistent with the discretization of the interior (see Figure 5). The edge CFABCs are discretized into rectangular elements, with the length perpendicular to the boundary being imaginary ($L_j = 2i c_j / \omega$), and the length parallel to the boundary being real and consistent with the interior elements. The contribution from the edge absorbing elements takes the form (see Appendix A.1):

$$C^j_e \frac{\partial U^j_e}{\partial t} + R^j_e \int_0^t U^j_e \, d\tau$$

(32)

where the matrices, $C^j_e$ and $R^j_e$, are given by (A9) in the appendix. $U^j_e$ is the displacement vector associated with the $j$th element of the edge absorbing region. The parallelogram-shaped

![Figure 5. Finite element discretization of the CFABC. Note that midpoint integration is used for the corner absorbers and in the direction perpendicular to the boundary for edge absorbers.](image)
corner CFABCs are already in a discretized form, as discussed in the previous section. The contribution from a corner element is given by (see Appendix A.2):

\[ K_c^j U_c^j \]  

where \( K_c^j \) is given by (A15) and (A18) in the appendix. \( U_c^j \) is the element displacement vector for the \( j \)th corner absorbing element. The contribution from the interior element takes the standard form

\[ M_i^j \frac{\partial^2 U_i^j}{\partial t^2} + K_i^j U_i^j \]  

where the matrices, \( M_i^j \) and \( K_i^j \), are the mass and stiffness matrices of the \( j \)th element in the interior. Assembling all the interior elements as well as the edge and the corner absorbing elements results in the global system of evolution equations given by

\[ M_i^j \frac{\partial^2 U_i^j}{\partial t^2} + C_e \frac{\partial U_i^j}{\partial t} + (K_i + K_c) U_i^j + R_e \int_0^t U d\tau = F \]  

where \( F \) is the load vector resulting from the interior loading.

6. TIME STEPPING

The spatially discrete equation in (35) is effectively a system of third-order differential equations in time and is different from the second-order systems arising from standard dynamics problems. In Reference \[23\], Equation (35) is solved using the (implicit) extended constant-average acceleration (ECAA) scheme. In this paper, a more general procedure is developed that facilitates the option of efficient explicit computation.

The first step of the proposed time-stepping scheme is to rewrite the spatially discretized equation at \( t = t_{n+1} \) in a convenient form:

\[ M_i^j A_i^{n+1} + C_e V_e^{n+1} + K_i U_i^{n+1} + K_c U_c^{n+1} + R_e W_e^{n+1} = F^{n+1} \]  

where the field variables in the interior, edges and corners are viewed as separate variables, identified by the subscripts, \( i, e \) and \( c \), respectively. The notations, \( A, V \) and \( W \), are used, respectively, for acceleration, velocity and the integral of \( U \). Note that, irrespective of the subscript, these vectors have the same dimensions, equal to the total number of degrees-of-freedom of the discretized problem. That is, the vectors are expanded in that they have zero elements for the degrees-of-freedom that are not associated with the region corresponding to the subscript. Naturally, the coefficient matrices are also expanded in a similar manner. Note that this expansion is used only for the purposes of presentation. Actual implementation does not need or involve this expansion.

Since the lengths of the absorbing element are inversely proportional to \( \omega \), one can view the coupling of the edge CFABC and the interior as analogous to standard parabolic equations,

\[ \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} + (K + K_c) U + R_e \int_0^t U d\tau = F \]  

This can be seen clearly by differentiating (35).
where the first-order time derivative is coupled to the field variable. This observation motivates us to utilize the Crank–Nicholson method, or the trapezoidal rule, to link the edge absorber to the interior:

$$U_e^{n+1} = U_e^n + \frac{\Delta t}{2}(V_i^n + V_i^{n+1})$$  \hspace{1cm} (37)$$

where $\Delta t$ is the time increment. The above equation immediately translates to the following approximations of $W_e$ and $V_e$:

$$W_e^{n+1} = W_e^n + \frac{\Delta t}{2}(U_i^n + U_i^{n+1})$$  \hspace{1cm} (38)$$

$$V_e^{n+1} = V_e^n + \frac{\Delta t}{2}(A_i^n + A_i^{n+1})$$  \hspace{1cm} (39)$$

Noting that the corner absorber can be viewed as an absorber to the edge absorber, the trapezoidal rule is applied twice to link the corner variables to the interior:

$$U_c^{n+1} = U_c^n + \frac{\Delta t}{2}(V_e^n + V_e^{n+1})$$  \hspace{1cm} (40)$$

$$= U_c^n + \Delta t V_e^n + \frac{\Delta t^2}{4}(A_i^n + A_i^{n+1})$$

The substitution of (38) and (39) in (36) results in

$$\bar{M}A_i^{n+1} + \bar{K}U_i^{n+1} = \bar{F}^{n+1}$$

where

$$\bar{M} = M_i + \frac{\Delta t}{2}C_e + \frac{\Delta t^2}{4}K_e$$

$$\bar{K} = K_i + \frac{\Delta t}{2}R_e$$

$$\bar{F}^{n+1} = F^{n+1} - C_e \left( V_e^n + \frac{\Delta t}{2}A_i^n \right) - K_e \left( U_c^n + \Delta t V_e^n + \frac{\Delta t^2}{4}A_i^n \right) - R_e \left( W_e^n + \frac{\Delta t}{2}U_i^n \right)$$

Equation (40) is reminiscent of the second-order differential equation that arises in dynamics problems, and can be solved using standard time-stepping techniques. We also note that, based on extensive numerical experiments, it appears that the proposed Crank–Nicholson coupling does not impose any stability limitations. The stability conditions, if any, arise from the time-stepping procedure employed when solving (40).

In this paper, the Newmark method (see, e.g. Reference [32]) is used to solve (40). The displacement and velocity are related to acceleration through

$$U_i^{n+1} = U_i^n + \Delta t V_i^n + \frac{\Delta t^2}{2}[(1-2\beta)A_i^n + 2\beta A_i^{n+1}]$$  \hspace{1cm} (42)$$

$$V_i^{n+1} = V_i^n + \Delta t[(1-\gamma)A_i^n + \gamma A_i^{n+1}]$$
where $\beta$ and $\gamma$ are the parameters of the method. Substituting the above expressions into (40) results in

$$M_{\text{eff}} A_i^{n+1} = F_{\text{eff}}^{n+1}$$

(43)

where

$$M_{\text{eff}} = \bar{M} + \beta \Delta t^2 \bar{K}$$

$$F_{\text{eff}}^{n+1} = F^{n+1} - \bar{K} (U_i^n + \Delta t V_i^n + \Delta t^2 (\frac{1 \gamma }{2} - \beta) A_i^n)$$

(44)

It can easily be seen that when the constant-average acceleration method ($\beta = 1/4, \gamma = 1/2$) is used, the proposed time-stepping scheme is equivalent to solving (36) with the ECAA method proposed in the original CFABC development [23]. The proposed scheme is the generalization of ECAA and has all the flexibility associated with the Newmark method (such as the option of introducing artificial dissipation). We re-emphasize here that the combination of the Crank–Nicholson coupling and the Newmark time-stepping scheme appears to have the same stability conditions as those associated with solving the interior problem with the Newmark method.

Since the constant-average acceleration method and the Crank–Nicholson method are second-order accurate, ECAA would also be second-order accurate. However, ECAA results in implicit computation, which is useful for low-frequency problems such as those associated with soil-structure interaction, but is undesirable for large-scale high-frequency wave propagation problems. In order to obtain a more efficient computational method, an almost explicit time-stepping procedure based on the central difference method is presented.

Explicit versions of the Newmark method are typically obtained by choosing $\beta = 0$ (and $\gamma = 0.5$ for the central difference method and associated second-order accuracy). Unfortunately, in this case, such a choice does not result in a purely explicit computation. The reason is that the matrices, $C_e$ and $K_c$, occurring in the expression for $M$ in (41), contain contributions from the terms involving spatial derivatives and cannot be diagonalized (lumped). The computation, however, can be made almost as efficient as the explicit computation through partial lumping. Essentially, noting that $C_e$ does not have any derivatives in the direction of the boundary, lumping can be performed in that direction. Such lumping results in decoupling in the direction of the boundary, while the nodes in the direction perpendicular to the boundary remain coupled. The contribution from the corner absorbers ($K_c$) cannot be lumped, as it involves the derivatives in both directions of the parallelogram. For the interior, natural decoupling is performed by utilizing corner-point integration for the mass matrix. Corner-point integration is also employed for the stiffness matrix to be consistent with the integration in the exterior. (This is equivalent to using a two-dimensional 5-point stencil in the finite difference method.) The consistency of the integration appears to be crucial for stability of the method; further investigation of its stability is a subject of future research.

To summarize, the matrices for the interior finite elements are evaluated using the corner-point integration rule. The edge absorber contributions are evaluated using the midpoint integration rule in the direction normal to the boundary and using the nodal-point integration rule in the direction of the boundary. The corner absorber contributions are computed using the midpoint integration rule. This facilitates a semi-explicit implementation where: (a) the nodes in the interior are solved individually; (b) the nodes connected to the corner are solved simultaneously; and (c) each node line of the edge absorber perpendicular to the boundary is solved separately from the rest. Considering that in most cases very few absorbing layers are needed, the proposed
computational procedure is only minimally more expensive than fully explicit computation. Even if the number of layers is increased, only small tridiagonal matrices (with their size equal to the number of CFABC layers) need to be solved, keeping the increase in the computational cost low. Furthermore, if the medium is homogenous and all the reference phase velocities are chosen to be equal to the wave velocity, these matrices turn out to be diagonal, making the computation completely explicit. The semi-explicit scheme presented in this paragraph is named the **extended central difference method** (ECDM). Note that the stability conditions of the ECDM are the same as the central difference method, requiring $\Delta t \leq h/\sqrt{2c}$, where $h$ is the element size.

### 7. NUMERICAL EXAMPLES

Numerical experiments are conducted to examine the performance of the CFABCs in various settings: (a) square computational boundaries; (b) polygonal boundaries; (c) computational domains with high aspect ratios where there is a grazing incidence of waves; and finally, (d) heterogeneous media where there are interfacial wave modes. In all these examples, the focus is on the ability of CFABCs to provide engineering accuracy, as well as on the convergence properties of the CFABCs as the number of layers is increased. While not the optimal choice, the following physically motivated reference phase velocities are used:

$$c^j = \frac{c}{\cos\left(\frac{(j-1)}{n}\pi\right)}$$  \hspace{1cm} (45)

where $c$ is the wave velocity and $n$ is the total number of CFABC layers. In all the examples, the focus is the propagation of the wave front in full-space due to a Gaussian-type pulse applied at the origin. The form of the pulse is borrowed from Reference [33]:

$$f(r, t) = \begin{cases} 
-2\pi^2 f_0^2 (t - t_0) e^{-\pi^2 f_0^2 (t - t_0)^2} \left(1 - \frac{r^2}{a^2}\right)^3 & \text{if } t \leq 2t_0, \quad r \leq a \\
0 & \text{otherwise}
\end{cases}$$ \hspace{1cm} (46)

In the above, $r$ is the distance from the origin, $a = 5h$ is the radius of the disk, $h = 0.15$ m is the central frequency, $f_0 = c/(hN_L)$ is the central frequency, $t_0 = 1/f_0$, and $N_L = 20$ is the number of points per wavelength. In all the examples, except for the last one, the wave velocity is 2000 m/s. The time integration is performed using the ECDM method for reasons of computational efficiency.

#### 7.1. Square computational domain

In this problem, the interior is chosen as a 30 m $\times$ 30 m square, with its corners at $(-7.5$ m, $-7.5$ m), $(22.5$ m, $-7.5$ m), $(22.5$ m, $22.5$ m) and $(-7.5$ m, $22.5$ m). Such a choice is made so that a part of the computational boundary is close to the load (at 7.5 m). The interior is discretized using a uniform grid of 200 $\times$ 200 bilinear finite elements of size, 0.15 m $\times$ 0.15 m. CFABCs are applied all around the domain. The schematic of the analysis is shown in Figure 6(a). The time increment for the analysis is chosen as 0.05 ms.
Figure 6. The performance of CFABC for square computational domains. The contours are obtained using a 3-layer CFABC. Note that the contour levels are magnified by a factor of 30 and the relative magnitude of the reflections is less than 0.5%: (a) schematic of the problem; (b) convergence of the CFABC; (c) response at the lower left corner; (d) wave front at time $= 7$ ms; (e) wave front at time $= 10$ ms; and (f) wave front at time $= 15$ ms.

In order to examine the effectiveness of the CFABC, the response at the corner point closest to the load is observed, i.e. at $(-7.5\, m, -7.5\, m)$ for the first 15 ms. To obtain the reference solution, the wave propagation is analysed in a larger domain where the computational boundaries are at least 30 m from the load. The relative error in the solution is obtained by
the ratio of the maximum norm of the error and that of the reference solution over the period of interest.

Figure 6(b) shows the relative error in the solution as the number of CFABC layers is increased. It can be seen that the error rapidly decreases initially, but remains at approximately 0.03% after the number of layers exceeds six. This asymptote is attributed to the discretization error of the interior. Note that the CFABC converges to the exact impedance of the exterior, while the discretized interior approximates the impedance, thus resulting in some reflection at the computational boundary.

Figure 6(c) compares the response obtained from the 3-layer CFABC with the reference solution. It can be clearly seen that the results are practically exact, indicating that, for this example, three CFABC layers may be sufficient for engineering accuracy. Panels (d)–(f) contain the snapshots (contours of the velocity norm) at various instances, obtained using the 3-layer CFABC. The contours are magnified by approximately 30 times to show even the small reflections. Clearly, there are no significant reflections from the 3-layer CFABC.

In order to examine the effect of midpoint integration, we analysed the same problem, but with the exact (2-point) integration used for the CFABC computation. The results are shown in Figure 7. Panels (c)–(f) clearly indicate that there are significant reflections and errors. From panel (b), we observe that the error is more than 7% and does not reduce as the number of layers is increased. This is due to the fact that, unlike the midpoint rule, the 2-point integration rule does not eliminate the discretization error. Of course, if one is careful with the selection of the parameters, the errors will probably reduce as the number of layers is increased. This is one of the significant differences between the conventionally discretized PML and the CFABC; the PML requires more care with discretization, while the CFABC becomes more and more accurate as the number of layers is increased, even if no care is taken with respect to the choice of parameters.

7.2. Polygonal domain with oblique corners

In this example, the computational domain is chosen as a polygon with corners at (−7.5 m, 0), (15 m, −15 m), (30 m, 22.5 m) and (−7.5 m, 22.5 m). A structured 200 × 200 mesh of distorted bilinear elements is used, with CFABC layers laced around it with parallelogram-shaped absorbers used at the corners. The schematic is shown in Figure 8(a). The time increment is now changed to 0.025 ms due to increased stability constraints resulting from element distortion. The response at the lower left corner is examined, and the results are presented in a manner similar to the previous example. Note that the error converges to the discretization error more quickly than the square computational domain, not because of any higher accuracy of the polygonal CFABC, but because the interior discretization error is increased due to element distortion. From panels (c)–(f), one observes that the 3-layer CFABC is sufficient for engineering accuracy.

7.3. High-aspect-ratio computational domain

In order to examine the effectiveness of CFABCs for almost-grazing incidence, a thin 30 m × 3 m computational domain is chosen with the load applied at the centre (see Figure 9(a)). The domain is discretized using a 200 × 20 uniform mesh, and the time increment is chosen as 0.05 ms. The response at the lower left corner is examined to capture the errors due to reflections at high angles of incidence. As expected, the error in the CFABC is greater compared with the
Figure 7. Results for the square domain when midpoint integration is not used: (a) schematic of the problem; (b) convergence of the CFABC; (c) response at the lower left corner; (d) wave front at time $= 7$ ms; (e) wave front at time $= 10$ ms; and (f) wave front at time $= 15$ ms.

previous examples, with the error still not completely converged to the interior discretization error even after 20 layers. Figure 9 also shows that, as opposed to 3-layer CFABCs for the previous examples, 20 layers are required for just 1% error. It should be noted that this is a result of the *ad hoc* choice of the reference phase velocities given in (45). Improved accuracy, including exponential convergence, may be possible with careful optimization based on the ideas in Reference [24]; however, such a study is out of the scope of this paper.
Figure 8. The performance of CFABC for polygonal computational domains. The contours are obtained using a 3-layer CFABC. Note that the contour levels are magnified by a factor of 30 and the relative magnitude of the reflections is less than 1%: (a) schematic of the problem; (b) convergence of the CFABC; (c) response at the lower left corner; (d) wave front at time $= 7.5$ ms; (e) wave front at time $= 10$ ms; and (f) wave front at time $= 16$ ms.

7.4. Heterogeneous media

In the final example, the absorption of CFABCs is examined in the context of wave propagation along a bi-material interface. The wave velocity of the half-space above the $x$-axis is changed to 1000 m/s, while the wave speed of the lower half-space is kept at 2000 m/s. The
Figure 9. The performance of CFABC for almost-grazing incidence. Note that the contours are obtained using a 20-layer CFABC: (a) schematic of the problem; (b) convergence of the CFABC; (c) response at the lower left corner; (d) wave front at time = 2 ms; (e) wave front at time = 5 ms; (f) wave front at time = 8 ms; and (g) wave front at time = 11 ms.

The computational domain is chosen as a 30 m × 30 m square with corners at (−15 m, −15 m), (15 m, −15 m), (15 m, 15 m) and (−15 m, 15 m). Similar to the first example, a 200 × 200 mesh is used, and the time increment is chosen as 0.05 ms. The schematic is shown in Figure 10(a). The point of interest is chosen as the left edge of the interface in order to assess the absorption of the interface head waves. As seen from panels (b) and (c), the CFABC is highly accurate and converges very quickly to the discretization error. This is attributed to the fact that the angles of incidence at this point are fairly small. The wave front in panel (f) reveals that there are reflections in the upper half-space for the 3-layer CFABC; note that the contours are magnified and the reflections are rather small. These reflections did not reduce the accuracy of the solution.
Figure 10. The performance of CFABC for heterogeneous media. The contours are obtained using a 3-layer CFABC. Similar to previous figures, the contour levels are magnified by a factor of 30. Note that the reflections in the top half in panel (f) are related to the interior discretization and not to the CFABC (see the text): (a) schematic of the problem; (b) convergence of the CFABC; (c) response at the point of interest; (d) wave front at time $t = 7$ ms; (e) wave front at time $t = 12$ ms; and (f) wave front at time $t = 20$ ms.

Even when 10 CFABC layers were used, indicating that they are the result of the interior discretization error. The discretization error is higher in the top half because the time increment used for the analysis is much smaller than the critical time step, hence, more dispersion and reflections.
8. CONCLUSIONS

In this paper, a new formulation of continued fraction absorbing boundary conditions (CFABCs) is presented. We show that the CFABC is a perfectly matched discrete layer (PMDL) with pure imaginary stretching. The discretization error is eliminated with the help of midpoint integration. This observation, as well as other related observations made in an earlier paper [24], unifies the seemingly unrelated ideas underlying material-based PMLs and differential equation-based local ABCs. We have exploited this link to obtain efficient corner absorbing elements, and have extended the idea to oblique corners, thus making the CFABCs applicable to convex polygonal domains. Numerical examples are presented to illustrate the effectiveness of the CFABCs.

The new CFABCs are implemented in the standard finite element setting in a straightforward way. The resulting system of evolution equations is equivalent to third-order differential systems, which are different from conventional second-order systems encountered in standard wave propagation and dynamics problems. With the help of a Crank–Nicholson-type coupling, we have developed a systematic extension of the Newmark method to discretize these equations. Special cases of the extended Newmark methods (having second-order accuracy) are the extended constant-average acceleration (ECAA) scheme and the extended central difference method (ECDM). The ECAA is an unconditionally stable implicit scheme. The ECDM, on the other hand, is an almost explicit scheme, with a computational cost close to that of the simple central difference method (CDM) and a stability limit identical to that of the CDM.

The key idea presented in this paper, namely the use of the midpoint integration rule, is not limited to transient scalar wave propagation problems. It can be applied to modelling unbounded domains where the governing partial differential equation is second order in space with constant coefficients, a claim that is explicitly proven in a separate publication [27]. The idea is immediately applicable to static and time-harmonic analysis of acoustic (scalar), and elastic and electromagnetic (vector) media; preliminary tests (not reported here) are showing excellent results. CFABCs are already extended to dispersive wave propagation problems [34]. They can also be extended to time-domain analysis of elastic wave propagation, but require additional care in choosing the parameters and in devising semi-explicit time-stepping schemes.

We are currently in the process of developing such schemes and will report them in a future publication. Other related topics of ongoing and future investigations include: a priori optimal design of CFABCs based on given error tolerance; comprehensive continuous and discrete time stability analysis of the CFABCs; extension of CFABCs to poroelastic media; and possible extension of CFABCs to curved boundaries. Overall, it appears that the simple midpoint integration rule plays a fundamental role in the modelling of unbounded domains and should be explored further for various modelling problems.

APPENDIX A: DERIVATION OF CONTRIBUTION MATRICES FOR ABSORBING ELEMENTS

A.1. Edge absorbing elements

Since edge CFABC layers are finite elements with frequency-dependent dimensions \((L_j = 2c_j/\omega)\), the contribution matrices are derived in the frequency domain and then transformed...
back into the time domain in the final step. The governing equation (1), when written in
frequency domain, takes the form
\[-\mu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial^2 u}{\partial z^2} - \omega^2 \rho u = 0 \quad (A1)\]

The finite element contribution is evaluated by first obtaining the variational form and sub-
sequently using the approximate form of the displacement in the resulting variational form.
The variational form is obtained by multiplying the above equation by a virtual variable and
performing integration by parts in $x$ and $z$, resulting in
\[\int_{\Gamma_j} \int_{x=0}^{L_j} \partial u^T \frac{\partial u}{\partial x} \, dx \, dz + \int_{\Gamma_j} \int_{x=0}^{L_j} \frac{\partial u^T}{\partial z} \frac{\partial u}{\partial z} \, dx \, dz - \omega^2 \int_{\Gamma_j} \int_{x=0}^{L_j} \delta u^T \rho u \, dx \, dz \quad (A2)\]

where $\Gamma_j$ is the trace of the $j$th edge element on the boundary and $L_j$ is the length perpendicular
to the boundary. Using the discretization in (9) and midpoint integration, we obtain
\[\int_{\Gamma_j} \delta u^T \frac{\mu}{L_j} \left[\begin{array}{cc} +1 & -1 \\ -1 & +1 \end{array}\right] u \, dz + \int_{\Gamma_j} \frac{\partial u^T}{\partial z} \frac{\mu L_j}{4} \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right] \frac{\partial u}{\partial z} \, dz - \omega^2 \int_{\Gamma_j} \delta u^T \rho L_j \frac{1}{4} \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right] u \, dz \quad (A3)\]

where $u$ is the degree-of-freedom vector containing displacements at the left and right edges
of the CFABC layer. Noting that $L_j = 2i\epsilon^j/\omega$, the contribution takes the form
\[-i\omega \int_{\Gamma_j} \delta u^T \hat{C}^j u \, dz - \frac{1}{i\omega} \int_{\Gamma_j} \frac{\partial u^T}{\partial x} \hat{R}^j \frac{\partial u}{\partial x} \, dz \quad (A4)\]

where
\[\hat{C}^j = \frac{\mu}{2\epsilon^j} \left[\begin{array}{cc} +1 & -1 \\ -1 & +1 \end{array}\right] + \frac{\rho \epsilon^j}{2} \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right] \quad (A5)\]

We now discretize in the direction of the boundary, i.e.
\[u = N_\Gamma U_e^j \quad (A6)\]

where $N_\Gamma$ is the shape function matrix in the direction of the boundary and $U_e^j$ is the fully
discretized displacement of the $j$th edge absorbing element. Note that $N_\Gamma$ is of the form
\[N_\Gamma = \begin{bmatrix} N_1 & 0 & \cdots & N_k & 0 \\ 0 & N_1 & \cdots & 0 & N_k \end{bmatrix} \quad (A7)\]

where $k$ is the number of nodes in the direction of the boundary. Substituting (A7) into (A4),
we obtain the contribution of the edge CFABC element:
\[\begin{align*}
-i\omega C_e^j U^j - \frac{1}{i\omega} R_e^j U^j
\end{align*} \quad (A8)\]
where

\[ C_j^e = \int_{\Gamma_j} N_j^T \bar{C} N_j \, dz \]

\[ R_j^e = \int_{\Gamma_j} \frac{\partial N_j^T}{\partial z} \bar{R}^j \frac{\partial N_j}{\partial z} \, dz \]  \hspace{1cm} (A9)

Equivalently, in the time domain, we have:\§

\[ C_j^e \frac{\partial U_j^e}{\partial t} + R_j^e \int_0^t U_j^e \, d\tau \]  \hspace{1cm} (A10)

Observing that \( C_j^e \) is similar to a mass matrix, if nodal-point integration is performed for evaluating (A9), we would obtain a block-diagonal matrix, decoupling the degrees-of-freedom in the direction of the boundary. Note that decoupling cannot be achieved in the direction perpendicular to the boundary because \( \bar{C}^j \) cannot be diagonalized, even if the midpoint integration rule is abandoned.

\section*{A.2. Corner absorbing elements}

Isoparametric formulation [25] is used to derive the contribution matrices and define the shape functions in the natural \((r, s)\) co-ordinate system:

\[ u(r, s) = NU_c^j \]  \hspace{1cm} (A11)

where \( U_c^j \) is the element degree-of-freedom vector for the \( j \)th corner element, and \( N \) is the shape function matrix. Since the corner element is a bilinear element,

\[ N = \frac{1}{4} [(1 - r)(1 - s) (1 + r)(1 - s) (1 + r)(1 + s) (1 - r)(1 + s)] \]  \hspace{1cm} (A12)

Considering the geometry of the parallelogram element in Figure A1, it can be shown that the Jacobian of the transformation is given by

\[ J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} a & 0 \\ b \cos \alpha & b \sin \alpha \end{bmatrix} \]  \hspace{1cm} (A13)

\( \text{§Note that the lower limit of integration in (A10) is kept as zero since the displacement at the boundary (and the exterior) is considered to be zero for } t<0. \)
The contribution matrix is given by (see, e.g. Reference [25]):

$$\int_{-1}^{+1} \! \int_{-1}^{+1} \left[ \frac{\partial \mathbf{N}^T}{\partial \mathbf{r}} \mathbf{J}^{-T} \mu \mathbf{J}^{-1} \frac{\partial \mathbf{N}}{\partial \mathbf{r}} + \omega^2 \mathbf{N}^T \rho \mathbf{N} \right] \det(\mathbf{J}) \, dr \, ds$$  \hspace{1cm} (A14)

where $\frac{\partial \mathbf{N}}{\partial \mathbf{r}}$ is the matrix containing the derivatives of the shape functions with respect to the natural co-ordinates $(r, s)$. Employing midpoint $(1 \times 1)$ integration, the element contribution matrix becomes

$$\begin{bmatrix}
A & C & -A & -C \\
C & B & -C & -B \\
-A & -C & A & C \\
-C & -B & C & B
\end{bmatrix} + D \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}$$  \hspace{1cm} (A15)

where

$$A = \frac{\mu}{4ab \sin \alpha} (b^2 - 2ab \cos \alpha + a^2)$$

$$B = \frac{\mu}{4ab \sin \alpha} (b^2 + 2ab \cos \alpha + a^2)$$

$$C = \frac{\mu}{4ab \sin \alpha} (a^2 - b^2)$$

$$D = \frac{-\omega^2 \rho ab \sin \alpha}{4}$$

Noting that, for a corner absorbing element, the lengths take the form

$$2a = \frac{2ic_1^j}{\omega} \quad \text{and} \quad 2b = \frac{2ic_2^j}{\omega}$$  \hspace{1cm} (A17)

and the elements of the contribution matrix become:

$$A = \frac{\mu}{4 \sin \alpha} \left( \frac{c_1^j}{c_2^j} - 2 \cos \alpha + \frac{c_1^j}{c_1^j} \right)$$

$$B = \frac{\mu}{4 \sin \alpha} \left( \frac{c_2^j}{c_1^j} + 2 \cos \alpha + \frac{c_2^j}{c_2^j} \right)$$

$$C = \frac{\mu}{4 \sin \alpha} \left( \frac{c_1^j}{c_1^j} - \frac{c_2^j}{c_2^j} \right)$$

$$D = \frac{\rho c_1^j c_2^j \sin \alpha}{4}$$  \hspace{1cm} (A18)

Note that all the coefficients of the coefficient matrix are independent of $\omega$, indicating that the contribution from the corner takes the form

$$\mathbf{K}_c^j \mathbf{U}_c^j$$  \hspace{1cm} (A19)

where $\mathbf{K}_c^j$ is given by (A15) and (A18).

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