Coupled space-domain elastic migration in heterogeneous media
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SUMMARY
We develop a new space-domain elastic migration method for heterogeneous media based on downward continuation. This method uses the concepts of the recently developed Arbitrarily Wide Angle Wave Equations. The coupled one-way propagation of pressure and shear waves is modeled by attaching a virtual half-space to each depth step, which is represented by special complex-length finite elements. Although formulated using finite element concepts, the method is eventually implemented using an implicit finite difference method in the frequency-space domain. The impulse response of the proposed method illustrates that P, S and head wave fronts are captured accurately for almost the entire range of the dip angles. The effectiveness of the method is illustrated by imaging a synthetic heterogeneous exploding reflector model.

INTRODUCTION
Acoustic migration is not accurate when the effects of wave-mode conversion are significant, e.g., when the data offset is very large. Therefore, elastic migration that considers continuous coupling of wave modes is necessary. While migration can be performed using the full elastic wave equation (see e.g., Wapenaar et al., 1987), such a method suffers from the same drawbacks as the acoustic reverse time migration. Therefore, a migration method based on elastic one-way wave equations (OWWEs) is desirable. The development of space-domain elastic OWWEs is not as straightforward as their acoustic counterparts because the elastic wave equation cannot be factorized into upward and downward propagators in their original form. To circumvent this problem, dual domain methods have been proposed (see e.g., Wu, 1994). Recently, Guddati (2005) proposed a technique of factorizing the fully-coupled elastic wave equation, which results in so-called Arbitrarily Wide-angle Wave Equations (AWWE). Based on the concepts used in the derivation of AWWE, we have developed a downward continuation procedure for elastic migration. This method inherently factorizes the elastic wave equation by attaching a virtual half-space to each depth step, which ensures one-way propagation. The virtual half-space is then represented using a small number of complex-length finite elements. The method, however, is eventually implemented using the finite difference method. The proposed method requires special stabilization to suppress the growth associated with improper mapping of evanescent modes. We present the formulation, along with a brief overview of the stabilization process, followed by numerical examples illustrating the method’s accuracy. For the sake of simplicity, we limit our discussion to post-stack migration in two dimensions and note that the proposed method is extendible to pre-stack and 3D migration.

METHOD
Our approach to formulating an effective downward continuation procedure is based on the AWWE concepts and is summarized in Figure 1. We start with a two-dimensional domain in \((x, z)\) and discretize the domain in depth \(z\), as shown in panel (a). In order to extrapolate the wavefield from \(z_1\) to \(z_2\), we attach a virtual half-space to the \([z_1, z_2]\) layer to ensure one-way (downward) propagation, as shown in panel (b). At this point the downward continuation requires the solution of the governing equation in the physical layer \([z_1, z_2]\) along with the attached virtual half-space. To this end, the half-space
Coupled space-domain elastic migration in heterogeneous media

in panel (b) is discretized in depth using finite element layers. The discretization results in errors in the form of spurious upward reflections at the layer interfaces, as shown in panel (c). Following the concepts of AWWE (Guddati, 2005) we use midpoint integration to eliminate the spurious reflections. The resulting system, shown in panel (d), perfectly propagates the waves downward. Since the discretized half-space contains an infinite number of finite element layers, it must be truncated to be computationally tractable. This results in upward reflections at the truncation boundary, as shown in panel (e). It is shown in the development of AWWE (Guddati, 2005) that each element in the discretized half-space can perfectly absorb wave modes with a wavenumber of $2i/L_j$, where $L_j$ is the thickness of the $j^\text{th}$ finite element layer. It follows that real-length elements absorb evanescent modes which have imaginary wavenumbers. In order to make the discretization effective for absorbing the propagating waves, we choose imaginary lengths for the elements of the virtual half-space. Specifically, we choose $L_j = 2i/k_j$ for element $j$ in order to perfectly absorb propagating waves with wave number of $k_j$. Such a choice of element length (panel f) eliminates the reflections for the wave modes with the chosen wave number $k_j$, and significantly reduces the reflections for other wave modes. We call these imaginary-length layers AWWE layers. The system in panel (f) is approximately equivalent to the exact system in panel (b). The accuracy of AWWE discretization depends on the number of half-space elements as well as the corresponding reference wavenumbers $k_j$. The following provides the implementation details of the elastic-AWWE downward continuation procedure shown in panel (f).

We start with the coupled equations of elastodynamics in an isotropic homogeneous media, given by

$$\partial^2 \mathbf{u} + \frac{\partial}{\partial x} \mathbf{P} \frac{\partial}{\partial x} \mathbf{u} + \frac{\partial^2}{\partial x^2} \mathbf{u} - \rho \mathbf{I}_2 \frac{\partial^2}{\partial t^2} \mathbf{u} = 0,$$

where $\mathbf{u} = [u, v]^T$ is the displacement vector, $\mathbf{I}_2$ is the $2 \times 2$ identity matrix, and $\mathbf{P}_{1,2,3}$ are the $2 \times 2$ matrices of material properties defined as

$$\mathbf{P}_1 = \begin{bmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{bmatrix} , \quad \mathbf{P}_2 = \begin{bmatrix} 0 & \lambda + \mu \\ \lambda + \mu & 0 \end{bmatrix} , \quad \mathbf{P}_3 = \begin{bmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{bmatrix} ,$$

and $\lambda$ and $\mu$ are Lamé constants. Performing the Fourier transform in time, we obtain

$$\mathbf{P}_1 \frac{\partial^2 \mathbf{u}}{\partial t^2} + \left( \frac{\partial}{\partial x} \mathbf{P}_2 \frac{\partial}{\partial x} \mathbf{u} + \frac{\partial^2}{\partial x^2} \mathbf{u} + \rho \omega^2 \mathbf{I}_2 \right) \mathbf{u} = 0,$$

where $\omega$ is the frequency. Note that in the above expression, $\mathbf{u} = \mathbf{u}(x,z;\omega)$ is the transformed variable. Our goal is to obtain $\mathbf{u}_j = \mathbf{u}(z_j)$ from $\mathbf{u}_i = \mathbf{u}(z_i)$ using the discretization in Figure 1(f). This is achieved by writing the stiffness relationship of the discrete system in Figure 1(f) as

$$\mathbf{S}_j \mathbf{u}_j = 0,$$

where $\mathbf{u} = [u_1, u_2, u_3, \ldots, u_n]^T$ is the vector of field variables on the physical element ($\mathbf{u}_j$ and $\mathbf{u}_i$) and the AWWE elements ($\mathbf{u}'_1, \ldots, \mathbf{u}'_n$), and $\mathbf{S}$ is the global stiffness matrix, which is the assembly of the stiffness matrices of the physical element and the virtual half-space elements. The stiffness of the physical element, i.e., layer $[z_i, z_j]$, can be derived as

$$\mathbf{S}_j = \frac{1}{\Delta z} \mathbf{P}_1 \mathbf{P}^* + \frac{1}{2} \mathbf{P}_2 \mathbf{P}^* \left( \mathbf{I}_2 - \frac{1}{\rho \omega^2} \mathbf{P}_3 \right) \mathbf{I}_2,$$

where $\Delta z = z_j - z_i$ is the depth step, and the $^*$ operator multiplies the matrix on the left by each element of the matrix on the right. Noting that the AWWE layers in Figure 1(f) have imaginary lengths and the stiffness matrix is evaluated using the midpoint integration rule, we obtain the following expression for the stiffness of AWWE layers:

$$\mathbf{S}_j = i \omega c_j \mathbf{P}_1 \mathbf{P}^* + \frac{1}{2} \mathbf{P}_2 \mathbf{P}^* \left( \mathbf{I}_2 - \frac{1}{\rho \omega^2} \mathbf{P}_3 \right) \mathbf{I}_2.$$

Note that we have replaced the reference wavenumbers of the AWWE elements ($k_j$) with the reference phase velocities $c_j(= \omega/k_j)$. Assembling the stiffness matrices of the physical and half-space elements, and noting that $\mathbf{u}_i$ is known, we can write the elastic-AWWE downward continuation equation as follows:

$$\left[ \mathbf{P}_1 \mathbf{A}_1 + \mathbf{P}_2 \mathbf{A}_2 + \left( \rho \omega^2 \mathbf{I}_2 + \mathbf{P}_3 \frac{\partial^2}{\partial x^2} \right) \mathbf{A}_3 \right] \mathbf{u} = \mathbf{U},$$

where $\mathbf{U} = [u_3, u_4, \ldots, u_n]^T$, $\mathbf{U}' = [u_1, 0, \ldots, 0]^T$ and $\mathbf{A}_{1,2,3}$ are matrices given by

$$\mathbf{A}_i = \frac{i \omega}{2} \begin{bmatrix} \frac{2i}{\omega \Delta z} & \frac{1}{c_1} & -1/c_1 \\ \frac{1}{c_1} & \frac{1}{c_1 + c_2} & \cdots \\ \cdots & \cdots & \cdots \\ -1/c_n & 1/c_n + 1/c_{n+1} & \cdots \\ -1/c_n & 1/c_n + 1/c_{n+1} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$
Equation (7) is continuous in $x$ and needs to be discretized. We perform the discretization in the $x$ direction using the central-difference method. The final discretized form of the downward continuation procedure for laterally heterogeneous media can be written in a block-tridiagonal system of equations, as follows:

$$
\begin{bmatrix}
D_i & B_i' & B_i & B_i' & \cdots & B_i & B_i' \\
B_i & D_i & B_i' & B_i & \cdots & B_i & B_i' \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & B_i & D_i & B_i' & B_i \\
& & & & & \ddots & \ddots \ddots \\
& & & & & & B_i & D_i \\
& & & & & & & B_i \\
\end{bmatrix}
\begin{bmatrix}
U_{i1} \\
U_{i2} \\
\vdots \\
U_{in} \\
\vdots \\
\vdots \\
\end{bmatrix}
=
\begin{bmatrix}
R_{i1} \\
R_{i2} \\
\vdots \\
R_{in} \\
\vdots \\
\vdots \\
\end{bmatrix},
$$

(11)

where $U = [U_1, U_2, \ldots, U_n]^T$ is a vector containing $U$ vectors for all the grid points along the $x$ direction, and similarly, $U' = [U_1', U_2', \ldots, U_n']^T$, where $n_x$ is the number of grid points in $x$. The blocks in (11) are written as

$$D_i = P_i^r \ast \Lambda_i + \rho \omega I_x \ast \Lambda_i - \frac{2}{\Delta x^2} P_i^r \ast \Lambda_i,$$

(12)

$$B_i = -\frac{1}{2\Delta x} (P_i^r \ast \Lambda_i) + \frac{1}{\Delta x^2} (P_i^r \ast \Lambda_i),$$

(13)

and the right hand side coefficient matrix is given by

$$R = \frac{1}{\Delta x}
\begin{bmatrix}
P_i^r \\
P_i^2 \\
\vdots \\
P_i^{n_x} \\
\end{bmatrix}
+ \frac{1}{4\Delta x}
\begin{bmatrix}
0 & P_i^2 \\
-P_i^1 & 0 & P_i^2 \\
\vdots & \ddots & \ddots \\
-P_i^{n_x-1} & \cdots & 0 & P_i^{n_x} \\
\end{bmatrix}$$

(14)

Equation (11) is solved for each step of the downward continuation, and the wavefield is obtained for all the frequencies in the desired range. The migrated image, i.e., the wavefield at $t = 0$, is obtained by simply adding the downward-continued wavefields for all the frequencies.

**Stabilization**

The downward continuation procedure devised above is mathematically equivalent to one-way wave propagation in the elastic waveguide bounded by the left and right (vertical) boundaries of the aperture. It is well known that a finite number of propagating modes, along with an infinite number of evanescent and complex modes, exist in an elastic waveguide. An accurate OWWE should properly capture the wave modes that are propagating/decaying in the downward direction, i.e., the wavenumbers should lie either on the positive real line, or in the top half of the complex plane. If real $c_j$ are used in the AWWE approximation, the real (propagating) wavenumbers will be accurately mapped, but the imaginary/complex wavenumbers will be completely misrepresented. Figure 2(a) shows the mapping for six-layer AWWE with real $c_j$. It can be seen that some of the complex wavenumbers are mapped onto the lower half of the complex plane, thus indicating spurious exponential growth in the downward direction. Such a growth completely pollutes the downward-continued wavefield, rendering the computed image useless.

The spurious growth can be suppressed by the branch-cut rotation ideas proposed by Milinazzo et al. (1997) for elastic parabolic equations used in the context of underwater acoustics. Incorporating this concept, which translates into the use of complex phase velocities, we were able to suppress the spurious growth. Essentially, by choosing $c_j' = e^{i\alpha c_j}$ with $\alpha = 90^\circ$, we eliminated the spurious mapping onto the bottom half of the complex plane (see Heidari, 2005 for details), thus stabilizing the method. Figure 2(b) shows the mapping for the stabilized six-layer AWWE.
Coupled space-domain elastic migration in heterogeneous media

Accuracy

The accuracy of the stabilized elastic AWWE is examined using its impulse response. A four-layer AWWE is used to generate the impulse response shown in Figure 3, which shows that the pressure and shear wave fronts, as well as the adjoining head wave front, are clearly captured. The pressure and shear wave fronts match with the exact wavefronts (semi-circles) for almost the entire range of the dip angle, indicating that elastic AWWE migration is effective with very few AWWE layers.

NUMERICAL EXAMPLE

In order to demonstrate elastic-AWWE migration in laterally heterogeneous media, we consider the synthetic model, shown in Figure 4. The surface trace is obtained using the exploding reflector concept by applying vertical excitation at two interfaces using a Ricker pulse. The stabilized elastic-AWWE migration with six AWWE layers is used to migrate the surface trace. The resulting image (vertical displacement at $t = 0$) is shown in Figure 5. It can be seen that both the reflectors are imaged accurately.

CONCLUSIONS

We have developed a space-domain elastic migration method that accounts for the continuous coupling of wave modes in heterogeneous media. To the best of our knowledge, this is the first space-domain elastic migration method that considers continuous mode coupling. With a careful choice of approximation parameters, the method results in the accurate migration of pressure, shear and head wavefronts for almost the entire range of dip angles. Based on migration results from a synthetic laterally heterogeneous exploding reflector model, the method shows promise for accurate pre-stack migration of multi-component field data. Furthermore, the method is readily extendible to 3-D elastic migration. The extension to 3-D and pre-stack migration and the application to real data are subjects of future research.

REFERENCES


