Stability of Discrete Time-Variant Linear Delay Systems and Applications to Network Control

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Abstract

This note presents a stability condition for time-variant uncertain discrete delay systems, where the uncertainty set is finite and constraints on the allowable sequences of system matrices are imposed. The proposed stability condition is necessary and sufficient and can be viewed as a modified combination of two existing conditions for time-variant systems with polytopic uncertainties. The introduced stability test is especially useful in the area of network embedded control systems and network congestion control where time-variant delays with constraints occur.

Keywords: stability tests, time-varying systems, uncertain dynamic systems, time delay, communication networks.

1 INTRODUCTION

The problem of robust stability of uncertain, time-variant discrete time system with polytopic uncertainties has been the subject of several publications, for example see [1–3]. This type of system finds applications in fields such as fault tolerant, adaptive, switched and network embedded systems. The typical system investigated has a zero input state space representation, where the time-variant system matrix $A(n)$ is taken from a matrix polytope. In some problem settings, additional constraints on the rate of change of the system parameters (system matrix) were imposed [4, 5]. Asymptotic/exponential stability was shown using a number of conditions, three of which were shown to be equivalent in complexity as well as necessary and sufficient in nature [1–3, 6].

Motivated by the applications in network embedded control and congestion control systems, this paper investigates the problem of asymptotic stability of time-variant uncertain systems where the uncertainty set is finite and constraints on permissible matrix sequences exist. The derived stability condition is necessary and sufficient and can be viewed as a modified combination of two existing necessary and sufficient stability conditions on polytopic uncertain systems. The developed approach drastically reduces the computational complexity of the test. The gains over existing methods will be illustrated by several examples from the area of network control.

2 PROBLEM FORMULATION

Consider the linear, time-variant and uncertain zero input system of the form:

$$x(n+1) = A(n)x(n) \quad (1)$$
$$A(n) \in \mathcal{S},\ \mathcal{S} = \{A_1, A_2, \ldots, A_N\} \subset \mathcal{R}^{L\times L} \quad (2)$$
$$A(n) = A_i \Rightarrow A(n+1) \in \mathcal{S}_{A_i} \subset \mathcal{S} \quad (3)$$

where the $\mathcal{S}_{A_i}$ are non-empty subsets of $\mathcal{S}$ that are dependent only on the choice of the matrix $A(n)$ at time instant n, i.e. on A_i. Let there be a non-empty subset[1] $\mathcal{S}_{A_i}$ of $\mathcal{S}$ for each $A_i,\ i = 1, \ldots, N$. (This problem reduces to the typical robust stability problem for time-variant systems with polytopic uncertainties, if $\mathcal{S}$ is replaced by the convex hull of the matrices in $\mathcal{S}$ and condition (3) is dropped.)

The problem that is addressed in this note is to decide asymptotic stability of the system in (1) under the constraints (2),(3). It is well known [6] that this problem is NP hard even if the set $\mathcal{S}$ contains only two elements and condition (3) is dropped.

Two special cases deserve mentioning:

(a) If each $\mathcal{S}_{A_i},\ i = 1, \ldots, N$ is of cardinality 1, then the system is time-variant with a known coefficient trajectory. This system (except possibly for some initial states) is periodic and its stability is easily analyzed using the eigenvalues of the full period transition matrix.

(b) If $\mathcal{S}_{A_i} = \mathcal{S},\ i = 1, \ldots, N$ we have no additional constraints and the result obtained for this system also guarantees stability of the entire convex hull defined by the vertex matrices $A_1, \ldots, A_N$. [1, 2]

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[1] The exact definition for each of the $\mathcal{S}_{A_i}$ is application dependent and will be illustrated in the section 5

3 THE STABILITY TEST

The following theorem is a slight generalization of the results in [1] and provides a necessary and sufficient condition for time-variant systems with polytopic uncertainties.

Theorem 1 *The system*

$$x(n+1) = A(n)x(n), \quad A(n) \in conv(A_1, \ldots, A_N)$$

is exponentially stable, iff $\exists$ *a sufficiently large integer* k *such that*

$$\|A_{i_1} \cdot \ldots \cdot A_{i_k}\| \leq \gamma < 1 \ \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

where by $conv(A_1, \ldots, A_N)$ *we denote the convex hull (convex matrix polyhedron) of the set of constant matrices* $\{A_1, \ldots, A_N\}$ *and* $\|\cdot\|$ *is any vector induced matrix norm.*

Comments:

- Theorem 1 is more general than the main result in [1] due to generalization of the norm type [6] and of the polytope form. Exponential stability was explicitly shown in [6].
- The problem is NP hard and the test is often inconclusive due to the required k being very large.
- The theorem is only theoretically necessary and sufficient since instability cannot be determined in practice.

The practical implementation of the stability test implies testing the norms of all the combinations $A_{i_1} \cdot \ldots \cdot A_{i_k}$. The number of combinations is N^k which limits the test usability for large values of k. We may be able to use a lower product length k if instead of applying Theorem 1 to the original system we first transform it and then apply the theorem:

Corollary 2 *The system*

$$x(n+1) = A(n)x(n), \quad A(n) \in conv(A_1, \ldots, A_N)$$

is exponentially stable, iff $\exists$ *a sufficiently large integer* k *such that*

$$\|P^{-1}A_{i_1} \cdot \ldots \cdot A_{i_k}P\| \leq \gamma < 1 \ \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

where P *is some invertible matrix.*

Proof The proof is straight forward and follows from a similarity transformation on the polytope. Consider the following transformation:

$$x(n) = P\tilde{x}(n). \tag{4}$$

If P is invertible, equation (1) becomes with (4):

$$\tilde{x}(n+1) = P^{-1}A(n)P\tilde{x}(n). \tag{5}$$

Defining

$$P^{-1}A(n)P = \tilde{A}(n) \tag{6}$$

we now have an equivalent description of the system in (1):

$$\tilde{x}(n+1) = \tilde{A}(n)\tilde{x}(n), \quad \tilde{A}(n) \in \tilde{\mathcal{S}} \tag{7}$$

where $\tilde{\mathcal{S}} = conv(\tilde{A}_1, \ldots, \tilde{A}_N)$. Now applying the condition in Theorem 1 to the new vertex matrices $\tilde{A}_i = P^{-1}A_iP$ yields the desired result. □

If one would be able to find a similarity transformation which will *simultaneously* make all the norms $\|\tilde{A}_i\|$ smaller than one then Corollary 2 would be satisfied for $k = 1$. However, for a square matrix P this is not always possible [2].

Theorem 3 **[2]** *The system*

$$x(n+1) = A(n)x(n), \quad A(n) \in conv(A_1, \ldots, A_N),$$

with $\{A_1, \ldots, A_N\} \subset \mathcal{R}^{L\times L}$ *is exponentially stable, iff* $\exists$ *a matrix* $P = (p_1, \ldots, p_M)$ *where* p_i, $i = 1, \ldots, M$ *are column vectors with* $rank(P) = dim(A(n)) = L \leq M$ *such that the matrices* $\tilde{A}_i$, $i = 1, \ldots, N$, $\tilde{A}_i \in \mathcal{R}^{M\times M}$ *defined by:*

$$A_i^T P = P\tilde{A}_i^T, \ \ i = 1, \ldots, N$$

satisfy the condition:

$$\|\tilde{A}_i\|_\infty < 1 \ \ i = 1, \ldots, N$$

As we already mentioned Theorem 1 (and Corollary 2) cannot be used to prove instability. We will introduce a sufficient condition for instability in the next Lemma.

Lemma 4 *The system (1)-(3) is unstable, if there exists a sequence* $(i_1, i_2, \ldots, i_k) \in \{1, \ldots, N\}^k$ *such that the matrices* $A_{i_1}, \ldots, A_{i_k}$ *satisfy condition (3) and*

$$\rho\left(A_{i_k} \cdot \ldots \cdot A_{i_1}\right) > 1$$

where by $\rho(\cdot)$ *we denote the spectral radius of a matrix.*

Proof: The result follows immediately if one considers the periodic sequence $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ which results in an unstable full period transition matrix $A_k = A_{i_k} \cdot \ldots \cdot A_{i_1}$ if the condition of Lemma 4 is satisfied. □

The system (1)-(3) is different from the one considered in Theorem 1: it has a finite number of possible system matrices: $A_1, \ldots, A_N$ and it additionally imposes condition (3) on permissible matrix sequences. We will next present a Lemma which tailors Corollary 2 to the system (1)-(3).

Lemma 5 *The system (1)-(3) is exponentially stable, iff* $\exists$ *a sufficiently large integer* k *such that*

$$\|P^{-1}A_{i_k} \cdot \ldots \cdot A_{i_1}P\| \leq \gamma < 1 \ \ \forall (i_1, \ldots, i_k) \tag{8}$$

such that $A_{i_{j+1}} \in \mathcal{S}_{A_{i_j}}$ *where* P *is any invertible square matrix of appropriate dimension.*

Proof

Sufficiency: Since P is invertible we can use a similarity transformation as in equation (4). Applying equation (5) recursively we can write:

$$x(k+1) = P^{-1}A(k)A(k-1)\ldots A(1)Px(1).$$

If there is an integer k such that equation (8) is satisfied then the equation (3) is a contraction mapping which assures that the system (1)-(3) is asymptotically stable. Additionally, since $\|x(kn)\| \leq$

$\gamma^n \|x(0)\|$ it can be easily shown that it is exponentially stable.

Necessity: Assume that there exists no k such that equation (8) is satisfied. Therefore, given an arbitrary value for k, we may always satisfy:

$$\|P^{-1} A_{i_k} \cdot \ldots \cdot A_{i_1} P\| \geq 1 \qquad (9)$$

for some choice of a sequence $A(1), \ldots, A(k)$. Then there exists in initial vector $x(1)$ such that

$\|\tilde{x}(k+1)\| \geq \|\tilde{x}(1)\|$, independent of the choice of k

which contradicts the condition for globally asymptotic stability for the transformed system which in turn contradicts the stability condition for the original system. □

Comments

• In [1,2] it was shown that the stability of the vertex matrices $A_1, A_2, \ldots, A_N$ is necessary and sufficient for the stability of the entire convex hull $conv(A_1, A_2, \ldots, A_N)$.

• The cost of performing the test in Theorem 1 (or Corollary 2) is $O(N^k)$, i.e. it grows exponentially with k.

• The cost of performing the test in Lemma 5 depends on the cardinality of the sets $\mathcal{S}_{A_i}$. For example, if $card(\mathcal{S}_{A_i}) = Q \;\; \forall i = 1, \ldots, N$ then the complexity of the test is $O(Q^k)$.

• The stability conditions in Theorem 1, Corollary 2 and Lemma 5 are highly dependent on the chosen norm. Moreover, for some norms one might find a quadratic transformation matrix P which satisfies Lemma 5 for $k = 1$, while for others it may not be possible [2].

• A well chosen similarity transformation P in Lemma 2 or Lemma 5 can significantly decrease the necessary number of matrix products k, and since the complexity of the stability test increases exponential with k a good algorithm for computing P is warranted.

4 THE ALGORITHM

It was shown [7] that computing the generalized transformation P in the Theorem 3 is itself an NP hard problem. We will present an algorithm which is simple to implement but does not guarantees that it will reduce the necessary product length k to $k = 1$.

By assumption, all matrices $A_i \quad i = 1, \ldots, N$ have the eigenvalues in the unit circle. For any matrix (without repeated eigenvalues) we can find a transformation P_i which will diagonalize the matrix, such that $P_i^{-1} A_i P_i = D_i$ where $D_i = diag(\lambda_{i,1}, \ldots, \lambda_{i,N})$. Obviously $\|D_i\|_1 = \|D_i\|_\infty < 1$. Unfortunately the matrix P_i which results in $\|P_i^{-1} A_i P_i\|_1 < 1$ does not necessarily results in $\|P_i^{-1} A_j P_i\|_1 < 1$ for $j \neq i$. Therefore the following scheme is proposed:

Compute the weighted average A_0 of the matrices $A_i \;\; i = 1, \ldots, N$:

$$\mu_i = \|A_i\| \quad \forall i = 1, \ldots, N \qquad (10)$$

$$A_0 = \frac{\sum_{i=1}^N \mu_i A_i}{\sum_{i=1}^N \mu_i} \qquad (11)$$

Then compute the matrix P which will diagonalize the weighted average A_0 and use it as a similarity transformation to obtain the transformed system (7).

We can repeat the algorithm for the transformed system until the results do not further improve. In order to measure the "improvement" the following approach is taken:

Obviously, the best measure is the minimum product length k which will satisfy the condition in equation (8). This k needs to be minimized. But finding the minimum product length k is equivalent with performing the proposed stability test itself, which is NP hard. Hence one needs to find different, less ideal but easily computable, measures.

One candidate for such a measure is the lower bound on the necessary product length. This is computed as the minimum positive integer γ such that $\|A_i^\gamma\| < 1 \;\; \forall i = 1, \ldots, N$. Of course, one needs to minimize the lower bound γ on the necessary product length k in the hope that this will result in a lower necessary product length. Note that $\gamma \leq k$.

Another measure candidate is the number of the matrices $\tilde{A}_i$ with the norm smaller than 1: $\delta = card(\{\tilde{A}_i \mid \|\tilde{A}_i\| < 1\})$. δ should be as high as possible, primarily since it may enable us to avoid the test in equation (8): one can compute the matrix norms $\tilde{\mu}_i = \|\tilde{A}_i\|$ beforehand and if

$$\prod_{i \in \{i_1, \ldots, i_k\}} \tilde{\mu}_i < 1 \qquad (12)$$

then there is no need to compute the matrix multiplications and the norm in equation (8) since:

$$\|P A_{i_1} \cdot A_{i_2} \cdot \ldots \cdot A_{i_k} P^{-1}\| \leq \prod_{i \in \{i_1, \ldots, i_k\}} \tilde{\mu}_i < 1$$

The third measure candidate is the maximum of all the norm of the matrices $\tilde{A}_i$: $\nu = \max_i\{\|\tilde{A}_i\|\}$. As this maximum decreases the chance that the replacement test (12) will hold increases.

It was observed that the iterative application of the algorithm converges quickly (a few iterations) and that it does not always converges to the optimal solution. Therefore the choosing of the optimum number of iterations is divided in two phases: in the first phase, a fixed number of iterations (I_N) is performed and the measures for each iteration are recorded. In the second phase, after the recursion is stopped the optimum number of iterations o is chosen based on the performance measures recorded during the iteration phase. Notice that I_N is usually not the optimum number of iterations. Since we have more than one performance measure we will give priorities as follows: the highest priority is given to the lower bound γ on the product length k, the second highest priority is given to the number of matrices δ with norm smaller than 1, and finally, the maximum norm ν receives the lowest priority.

Therefore we will use the following procedure to choose the optimum number of iterations:

- First find the set I_1 of the number of iterations that result in the minimum lower bound γ. I_1 is a set because there might be more than one iteration which results in the minimum γ.

- Second find the set I_2, subset of I_1, of the number of iterations such that the number of matrices with the norm smaller than 1 is maximized.

- Third, find the set I_3, subset of I_2, of the number of iterations such that the maximum norm ν is minimized.

- Finally, choose the minimum element of the set I_3 to be the optimum number of iterations o.

$$I_1 = \{j | \gamma_j = \min_{i=1 \dots I_N} \{\gamma_i\}\} \quad (13)$$

$$I_2 = \{j | \delta_j = \max_{i \in I_1} \{\delta_i\}\} \quad (14)$$

$$I_3 = \{j | \nu_j = \min_{i \in I_2} \{\nu_i\}\} \quad (15)$$

$$o = \min_{j \in I_3} \{j\} \quad (16)$$

Example:

Assume that during the iteration phase the measures presented in Table 1 have been recorded. We denoted with γ_j, δ_j and ν_j the measures γ, δ and γ respectively, measured after the j^{th} iteration. Then choosing the optimum number of iterations according to equations (13)-(16) will result in: $I_1 = \{3, 4, 5, 6\}$, $I_2 = \{4, 5\}$, $I_3 = \{4\}$ and $o = 4$, i.e. one should use the transformation matrix P produced at the fourth iteration.

In the rare case that the weighted average A_0 cannot be diagonalized, one can use a different norm (which changes the coefficients μ_i), a different type of weighted average, or use a different system representation.

To conclude the description of the algorithm we present the pseudocode in Appendix A.

5 APPLICATIONS IN COMMUNICATION NETWORK CONTROL

To illustrate our results we will present two examples from the field of network congestion control. The objective of a congestion control system is to regulate the data flow through the network such that there are no packet losses (due to buffer overflows in the intermediate switches) and the network is fully utilized, i.e. there is always data to be sent in the buffers. One way to achieve these objectives is to design a set point control system for the buffers of the intermediate switches. Such a control system is shown in Figure 1 for the particular case of the Available Bit Rate (ABR) option in Asynchronous Transfer Mode (ATM) Networks [8].

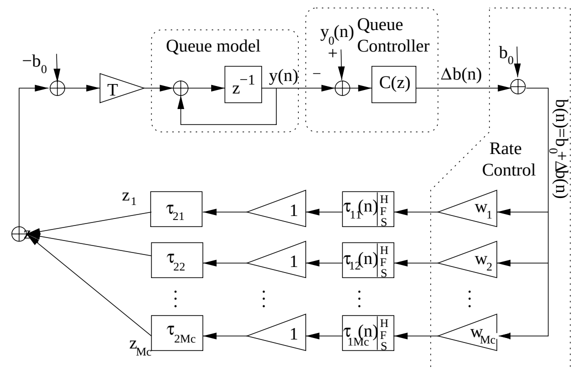

Figure 1: System model for an ATM congestion control system with time-variant delays: $C(z)$ is the rate controller, w_i are constant weights, M_c is the number of active connections, T is the controller sampling rate, $y(n)$ is the buffer occupancy level, y_0 is the set point for the buffer occupancy, b_0 is the bandwidth of the outgoing link and $\Delta b(n)$ is the total rate change as computed by the controller.

The congested switch controls the rates of the sources by providing explicit rate requests to the sources through the return path. The feedback data transmitted by the switch encounters time-variant delays as it propagates through the network. The sources adjust their transmission rate to the one specified in the most recently received rate request and continue to transmit at that rate until another request arrives. We assume that the time-variant delays on the return paths are bounded:

$$0 \leq \tau_{1,i}(n) \leq \bar{\tau}_{1,i} \quad i = 1, \dots, M_c. \quad (17)$$

Figure 2 depicts the model for a communication link with time-variant delays. The time-variant coefficients $\alpha_j(n)$ satisfy $\alpha_j(n) \in \{0, 1\} \; \forall j = 1, \dots, \bar{\tau}$ and at each time instant exactly one of the coefficients is not-zero, resulting in:

$$\sum_{j=0}^{\bar{\tau}} \alpha_j(n) = 1 \quad (18)$$

Since the source "holds" the same rate until it receives "fresh" information from the switch, we model the delays on the return path $\tau_{1,i}(n)$ as time-variant delays with a Hold Freshest Sample (HFS) interface (also called input variable delay in [8]). For

Iteration j	0	1	2	3	4	5	6	7	8	9	10
γ_j	237	9	5	4	4	4	4	5	5	5	5
δ_j	0	1	4	4	5	5	4	4	4	4	4
ν_j	4.45	1.52	1.43	1.37	1.40	1.54	1.48	1.45	1.46	1.46	1.46

Table 1: Example of recorded measures during the iteration phase

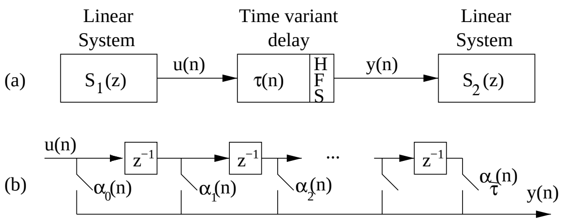

Figure 2: A network signal propagation/interface delay model. (2a) Signal propagation delay model interfaced with a linear system. (2b) Detailed block diagram of a signal propagation delay/interface.

an HFS interface the coefficients $\alpha_j(n)$ cannot vary arbitrarily from one time instant to another: the integer delay $\tau(n)$ is restricted by $\tau(n+1) \leq \tau(n)+1$, and hence we have:

$$\alpha_j(n) = 1 \Rightarrow \alpha_k(n+1) = 0 \;\; \forall k > j+1 \qquad (19)$$

In other words the used sample from the time-variant delay output maximally ages with time, but not faster.

Corresponding to each combination of the delays $\tau_{1,i}$ $i = 1, \ldots, M_c$ we have a different system matrix $A(n)$. We thus have $N = \prod_{i=1}^{M_c}(\bar{\tau}_{1,i}+1)$ matrices in the polytope $\mathcal{S}$. Since the delays on an HFS interface cannot vary arbitrarily the system matrix $A(n)$ cannot vary arbitrarily inside the polytope. Therefore the sets $\mathcal{S}_i$ in equation (3) are often smaller than the entire polytope $\mathcal{S}$.

The delays $\tau_{2,i}$ $i = 1, \ldots, M_c$ correspond to the delays encountered by the transmitted data on the forward path. For simplicity we assume they are constant.

5.1 The case of two sources

We will consider the case of a single congested switch with two sources ($M_c = 2$). We will use the following parameters for our system:

- $b_0 = 10^6$ cells/s.
- $T = 10^{-3}$s.
- $\bar{\tau}_{1,1} = \bar{\tau}_{1,2} = 2 \cdot 10^{-3}$ s, i.e. there are three possible delays: 0,1 and 2.
- $\tau_{2,1}(n) = \tau_{2,2}(n) = 10^{-3}$s
- $w_1(n) = w_2(n) = 0.5$.
- We used a proportional controller with a gain G.

A time invariant analysis of the system (i.e. analyzing stability of all the frozen time systems) yields the following (necessary) stability condition:

$$G \in (0,\ 445) \qquad (20)$$

The sufficient stability condition for the time-variant case introduced in [9] yields the following stability condition:

$$G \in (0,\ 77.06) \qquad (21)$$

Using Theorem 1 directly to determine stability is impractical due to the large product lengths k. In Table 2 the minimum product length that needs to be checked for $G = 1$ is shown for several matrix norms.

Using Lemma 5 results in a dramatic reduction of the necessary product length. By choosing the transformation P appropriately we managed to have the 2 norm of all matrices smaller than 1 for G up to 66.8, which corresponds to $k = 1$. In Table 3 the resulting bounds for the controller gain G are shown. It can be seen that the results are significantly less conservative than the sufficient condition presented in equation (21).

It should be mentioned that only the upper limit of the interval for G is obtained using the algorithm based on Lemma 5. The lower limit is known to be zero from [9].

Corresponding to each combination of the delays $\tau_{1,1}(n)$ and $\tau_{1,2}(n)$ we have a corresponding system matrix $A(n)$. Let us denote with $A_{i,j}(n)$ the system matrix corresponding to the delays $\tau_{1,1}(n) = i,\ \tau_{1,2}(n) = j;\ i,j \in \{0,1,2\}$. The entire matrix polytope $\mathcal{S}$ is given by $\mathcal{S} = \{A_{i,j} | 0 \leq i \leq 2,\ 0 \leq j \leq 2\}$. The matrix sets $\mathcal{S}_{A_{i,j}}$ are given by: $\mathcal{S}_{A_{i,j}} = \{A_{k,l} | 0 \leq k \leq \min\{2, i+1\},\ 0 \leq l \leq \min\{2, j+1\}\}$

For example, $\mathcal{S}_{A_{0,1}} = \{A_{0,0}, A_{0,1}, A_{0,2}, A_{1,0}, A_{1,1}, A_{1,2}\}$. For the results presented in Table 3, we used the phase variable representation for the system matrices $A(n)$.

The transformation matrix P corresponding to the product length of 6 and the gain $G = 366.4$ is for example given in Figure 3.

For this transformation matrix P, 7 out of the 9 $\tilde{A}_i$ matrices had a norm smaller than 1. This enabled use to use the replacement test in (12) in 58.8% of the cases. Moreover, only 26% of all the possible combinations are "HFS compliant", i.e. satisfy the HFS constraints in (19). Combining the two improvements, we performed the norm test in (8) only for 10% of the cases.

Used Norm	1	2	∞
Minimum Product Length $k \geq$	1387	696	1387

Table 2: Minimum product lengths in Theorem 1 for various matrix induced norms

Product length k	1	2	3	4	5	6
Controller gain $0 < G \leq$	66.8	118.3	254.3	279.1	322.7	366.4

Table 3: Provable stabilizing gain ranges for G as a function of k in Lemma 5 (two source case)

6 CONCLUSION

The article presented a new stability test for time-variant uncertain systems, where the uncertainty set is finite and additional constraints on the time-variance of the system exist. The main result is based on two previous results on stability of discrete time-variant systems with polytopic uncertainties. It is shown that the introduced necessary and sufficient stability condition is especially useful for systems with time-variant communication delays. These systems presently occur in network congestion control, networked control systems, tele-operation and sensor/actuator networks.

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A PSEUDOCODE FOR THE STABILITY TESTING ALGORITHM

In this appendix we provide the pseudocode of the stability testing algorithm based on Lemma 5 and described in section 4. In steps 2-8 the transformation matrix P is computed in constant time, while in the steps 9-13 the stability test presented in Lemma 5 is implemented. The pseudocode tests the condition in Lemma 5 for a given gain of the controller G and for a given product length k:

(1) Generate all vertex matrices $A_i^{(0)}(G) \quad \forall\ i = 1, \dots, N$;

(2) for $j = 1$ to I_N,

(3) $\mu_i^{(j)} \leftarrow \|A_i^{(j)}\| \ \forall\, i = 1, \dots, N$;

(4) $A_0^{(j)} \leftarrow \frac{\sum_{i=1}^N A_i^{(j)} \mu_i^{(j)}}{\sum_{i=1}^N \mu_i^{(j)}}$; (5) Compute P such that diagonalizes $A_0^{(j)}$ (i.e. $P^{-1} A_0^{(j)} P = diag(\lambda_{A_0^{(j)}}\ )\ )$

(6) $A_i^{(j+1)} \leftarrow P^{-1} A_i^{(j)} P \ \ \forall i = 1, \dots, N$;

(7) next j

(8) Choose the optimum iteration o as described in equations (13)-(16).

(9) Generate the first compliant index combination of length k: $(i_1,\ i_2, \dots, i_k)$;

(10) if $\prod_{l=1}^k \|A_{i_l}^{(o)}\| < 1$ then goto step 12 else goto step 11;

(11) if $\| \prod_{l=1}^k A_{i_l}^{(o)} \| < 1$ then goto step 12 else the test is inconclusive;

(12) Generate the next compliant index combination of length k: $(i_1,\ i_2, \dots, i_k)$;

(13) If all compliant index combinations have been tested conclude that the system is stable, else goto step 10

By compliant index combination we understand a index combination that satisfies the matrix sequence constraint from equation (3).

References

[1] P. H. Bauer, K. Premaratne, and J. Durán, "A necessary and sufficient condition for robust asymptotic stability of time-variant discrete systems," *IEEE Transactions on Automatic Control*, vol. 38, pp. 1427–1430, Sept. 1993.

[2] A. P. Molchanov, "Lyapunov functions for nonlinear discrete-time control systems," *Automat. Remote Control*, vol. 48, no. 6, pp. 728–736, 1987.

[3] R. Brayton and C. Tong, "Constructive stability and asymptotic stability of dynamical systems," *IEEE Trans. on Circuits and Systems*, vol. 27, pp. 1121–1130, 1980.

[4] K. Premaratne and M. Mansour, "Robust stability of time-variant discrete-time systems with bounded parameter perturbations," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, pp. 40–45, Jan. 1995.

[5] S. Dasgupta, G. Chockalingam, B. D. O. Anderson, and M. Fu, "Lyapunov functions for uncertain systems with applications to the stability of time varying systems," *IEEE Transactions on Circuits and Systems I*, vol. 41, pp. 93–106, Feb. 1994.

[6] A. Bhaya and F. Mota, "Equivalence of stability concepts for discrete time-varying systems," *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 725–740, 1994.

[7] J. N. Tsitsiklis and V. D. Blondel, "Lyapunov exponent and joint spectral radius of pairs of matrices are hard - when not impossible - to compute and

$$P = \begin{pmatrix} 0.3921 - 0.0065i & 0.3784 + 0.1031i & -0.0668 - 0.0231i & 0.0590 - 0.0388i \\ 0.3973 - 0.2136i & 0.3322 + 0.3051i & 0.1366 - 0.0866i & -0.1537 - 0.0502i \\ 0.2931 - 0.4280i & 0.1784 + 0.4872i & -0.0400 + 0.3679i & 0.1296 + 0.3467i \\ 0.0735 - 0.5921i & -0.0749 + 0.5919i & -0.6013 - 0.5966i & 0.4354 - 0.7266i \end{pmatrix}$$

Figure 3: The transformation matrix P corresponding to the product length of 6 and the gain $G = 366.4$

to approximate," *Mathematics of Control, Signals, and Systems*, vol. 10, no. 1, pp. 31–40, 1997.

[8] M. M. Ekanayake, *Robust Stability of Discrete Time Nonlinear Systems.* PhD thesis, University of Miami, 1999.

[9] P. H. Bauer, M. L. Sichitiu, and K. Premaratne, "Closing the loop through communication networks: The case of an integrator plant and multiple controllers," in *Proc. 38th IEEE Conference on Decision and Control*, pp. 2180–2185, Dec. 1999.