CONTROLLING AN INTEGRATOR THROUGH DATA NETWORKS: STABILITY IN THE PRESENCE OF UNKNOWN TIME-VARIANT DELAYS

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ABSTRACT

The subject of controlling an integrator via a communication network finds many applications in modern control systems. Stability conditions in closed form are derived for the arising feedback system, which consists of a controller, an integrator and time-variant links between integrator and controller. It will be shown, that these derived stability conditions are not conservative and easily applicable, even for systems with long delays. In order to draw from a recent result on time-variant discrete system stability, FIR controllers are assumed.

1. INTRODUCTION

In this paper we will provide simple sufficient conditions to test asymptotic stability of feedback systems, which connect controller and plant with communication links. The links are modeled with a time-variant delay.

Stability conditions for such a system exist [1, 2] in form of NP-hard tests [3]. Therefore, these tests are unsuitable for large delay uncertainties.

At first we will provide a simple model for the feedback system in difference equation form. For the sake of simplicity, only FIR controllers will be considered. All models will be represented in discrete time. The resulting characteristic equation will be derived. In the second step, the problem of asymptotic stability will be tackled. We will concentrate on the special case of an integrator plant, since this particular situation is of great importance in congestion control of ATM networks [4, 5] and many other areas where buffers or queues are a part of the feedback loop. In order to tackle the stability problem, a recent result on time-variant filters will be used [6].

This work was in part supported by the Alexander von Humboldt Foundation and by NSF grants ANI 9726253, ANI 9726247

2. PROBLEM DEFINITION AND MODEL

Figure 1 shows a single controller connected to an integrator plant via two time-variant delay links. Figure 1 depicts the zero input system model that is studied in this paper.

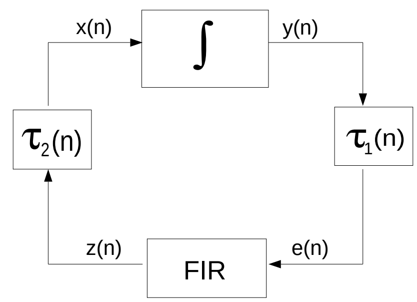

Figure 1: System model

System:

The dynamic behavior of the integrator is described by the linear time-invariant difference equation with real coefficient b_0:

$$y(n) = y(n-1) + b_0 x(n). \tag{1}$$

Controller:

For simplicity, we assume an FIR controller with real coefficients of the form:

$$z(n) = c_0 e(n) + \ldots + c_k e(n-k). \tag{2}$$

Delay 1:

The integer delay $\tau_1(n)$ describes the sum of all delays the signal $y(n)$ (plant output) encounters before it arrives at the FIR filter input:

$$e(n) = y(n - \tau_1(n)) \qquad (3)$$

or generally:

$$e(n-i) = y(n-i-\tau_1(n-i)), \quad i = 0, \ldots, k.$$

We also assume boundedness of the delay, i.e.

$$0 < \tau_{1_{min}} \le \tau_1(n) \le \tau_{1_{max}} \qquad (4)$$

Delay 2:

The integer delay $\tau_2(n)$ describes the sum of all delays the signal $z(n)$ (controller output) encounters before it arrives at the plant input.

$$x(n) = z(n - \tau_2(n)) \qquad (5)$$

Similarly as in the case of the first delay, boundedness of the delay is assumed:

$$0 < \tau_{2_{min}} \le \tau_2(n) \le \tau_{2_{max}} \qquad (6)$$

Although there are some restrictions on how the two delays can change if y and z are sent via a data communication network, in this work, we make no further assumptions on $\tau_{1,2}(n)$. Furthermore the case of $\tau_{1_{min}} = 0$ and $\tau_{2_{min}} = 0$ is not allowed, since it would create a delay free loop.

The resulting difference equation:

With (1-3) and (5) we obtain for the characteristic equation of the overall system:

$$\begin{aligned} y(n) &= y(n-1) + b_0[c_0 y(n - \tau_2(n) - \tau_1(n - \tau_2(n))) \\ &+ \ldots \\ &+ c_k y(n - k - \tau_2(n) - \tau_1(n - k - \tau_2(n)))] \end{aligned} \qquad (7)$$

The problem we will tackle in this paper is to find explicit closed form conditions, which ensure asymptotic stability of equation (7).

3. MAIN RESULT

There are a number of results in the literature [1]-[3] which could be applied to the problem at hand, after it is translated into a state space representation. The resulting tests, however, are of prohibitively high complexity, especially if the delay interval is large. In this paper, we pursue another avenue. We will derive a sufficient condition for global asymptotic stability of (7), which can easily be tested and is not very conservative.

Theorem:

The system (7) is globally asymptotically stable if the following three conditions are all satisfied:

(a) $b_0 c_i \le 0 \quad \forall\ i = 0, \ldots, k$

(b) The transfer function $H_\epsilon(z) = \frac{1}{1 - z^{-1} + \epsilon z^{-2} + \ldots + \epsilon z^{-L}} = \mathcal{Z}\{h_\epsilon(n)\}$, $L = k + \tau_{1_{max}} + \tau_{2_{max}}$ is stable for $\epsilon = |b_0| \sum_{i=0}^{k} |c_i|$ with $\epsilon > 0$

(c) $\sum_{n=0}^{\infty} |h_\epsilon(n)| \epsilon < \frac{1}{L-2}$

Proof:

Considering equations (4) and (6), equation (7) can be rewritten:

$$\begin{aligned} y(n) &= y(n-1) + b_0[c_0 y(n - T_0(n)) \\ &+ \ldots \\ &+ c_k y(n - T_k(n))] \end{aligned} \qquad (8)$$

where

$$T_i(n) \in [\tau_{1_{min}} + \tau_{2_{min}} + i \ , \ \tau_{1_{max}} + \tau_{2_{max}} + i] \quad 0 \le i \le k \qquad (9)$$

Define $L = \tau_{1_{max}} + \tau_{2_{max}} + k$;

Now consider the following time-variant system with uncertain, time variant parameters $\Delta a_i(n)$, $i = 2, \ldots, L$, $L \ge 3$:

$$\begin{aligned} y(n) = y(n-1) &- (\epsilon + \Delta a_2(n)) y(n-2) \\ &- (\epsilon + \Delta a_3(n)) y(n-3) \\ &- \ldots \\ &- (\epsilon + \Delta a_L(n)) y(n-L) \end{aligned} \qquad (10)$$

In what follows, we will formulate conditions on ϵ and $\Delta a_i(n)$ such that the set of systems described by (10) contains the system of equation (8), i.e. the time variant system (8) is an element of the set of uncertain time variant systems (10).

At first we choose:

$$\epsilon = -\sum_{i=0}^{k} b_0 c_i \qquad (11)$$

Notice that condition (a) in the hypothesis implies that $\epsilon > 0$ if we exclude the trivial cases where either $b_0 = 0$ or $c_i = 0 \ \forall\, i, \ 0 \le i \le k$

Imposing the condition

$$\sum_{i=2}^{L} \Delta a_i = -(L-2)\epsilon \tag{12}$$

on the uncertainties and letting

$$-\epsilon \le \Delta a_i \le 0 \quad , \, i = 2 \dots L \tag{13}$$

ensures that system (8) can be represented by (10). Notice that this guarantees that each coefficient $-(\epsilon + \Delta a_i(n)) \in [-\epsilon \ , \ 0] \ \ , \ \forall \ i = 2 \dots L$ and that the sum of these coefficients is exactly $-\epsilon = \sum_{i=0}^{k} b_0 c_i$.

From (12) and (13) we have:

$$\sum_{i=2}^{L} |\Delta a_i| = (L-2)\epsilon \tag{14}$$

From Theorem 1 in [6] it is known that the system (10) is asymptotically stable if

$$\sum_{n=0}^{\infty} |h_\epsilon(n)| \cdot \gamma < 1 \tag{15}$$

$$\gamma > \sum_{i=2}^{L} |\Delta a_i(n)| \tag{16}$$

Since by equation (14), in our case we have

$$\gamma = (L-2)\epsilon + \mu \tag{17}$$

with μ arbitrarily small and positive real. From (15), (16) we obtain for stability:

$$\sum_{n=0}^{\infty} |h_\epsilon(n)| \cdot [(L-2)\epsilon + \mu] < 1$$

and hence

$$\sum_{n=0}^{\infty} |h_\epsilon(n)| \epsilon < \frac{1}{L-2}$$

which is condition (c) in the theorem. Hence condition (c) together with (a) and (b) guarantees global asymptotic stability of the system.

Comments:

- Since $\sum_{n=0}^{\infty} |h_\epsilon(n)| > 1$, an upper bound for the values of ϵ which satisfy condition (c) is $\epsilon < \frac{1}{L-2}$, i.e. ϵ will have to decrease with increasing maximum delays $\tau_{1_{max}}, \tau_{2_{max}}$.
- Since b_0 is usually fixed and known, the theorem provides important guidelines for controller design by providing bounds on $\sum_{i=0}^{k} |c_i|$.
- The condition (c) in the theorem is close to being necessary since for $\sum_{n=0}^{\infty} |h_\epsilon(n)| \epsilon = \frac{1}{L-1}$, instability can be shown.
- The results derived can be directly applied to variable bit-rate control if we have a constant delay τ_2 i.e. $\tau_2(n) = \tau_2$ and more than one rate controller is included. If $\tau_2(n)$ is non-constant, equation (6) needs to be modified.

Now assume that ϵ is sufficiently small such that $h_\epsilon(n) \ge 0$ for all $n \ge 0$.

In this case

$$\sum_{n=0}^{\infty} |h_\epsilon(n)| = \sum_{n=0}^{\infty} h_\epsilon(n) = H_\epsilon(z)|_{z=1} = \frac{1}{L-1}$$

and condition (c) is satisfied. The existence of such an ϵ is guaranteed by the following lemma:

Lemma:

There exists $\epsilon_0 > 0$ such that $\forall \ \ 0 < \epsilon < \epsilon_0$ the impulse response of the system with the following Z-transform:
$H_\epsilon(z) = \frac{1}{1 - z^{-1} + \epsilon z^{-2} + \dots + \epsilon z^{-L}} = \mathcal{Z}\{h_\epsilon(n)\}$
is strictly positive.

Proof:

Denote $P_\epsilon(z) = z^L - z^{L-1} + \epsilon z^{L-2} + \epsilon z^{L-3} + \dots \epsilon z + \epsilon$, with $z_i \quad i = 1, \dots, L$ denoting the roots of $P_\epsilon(z)$ ordered such that $|z_i| \ge |z_{i+1}| \ \ \forall i = 1, \dots, L-1$

$P_0(z) = z^{L-1}(z-1)$ with $z_1 = 1$ and $z_i = 0 \quad \forall i = 2, \dots, L$.

Assume $P_\epsilon(z)$ has multiple roots and let α be such a multiple root. Then α will also be a root of the derivative $P'_\epsilon(z)$ and the two equations $P_\epsilon(\alpha) = 0$ and $P'_\epsilon(\alpha) = 0$ can be rewritten as:

$$2\alpha^L - (L+1)\alpha + L - 1 = 0 \tag{18}$$

$$\epsilon = \frac{(1-\alpha)^2 \alpha^{L-1}}{1 - \alpha^{L-1}} \tag{19}$$

From equation (18) it is apparent that there can only be L distinct locations of these multiple roots and these correspond through equation (19) to at most n real positive values of ϵ. Denote with ϵ_1 the smallest of these values. Note that equation (18) is having exactly one root in the interval $(0, 1)$ (which can be proved through the first derivative) and for this value the corresponding epsilon is between $(0, 1)$, thus the existence of ϵ_1 being ensured.

For $\epsilon < \epsilon_1$ we have:

$$
\begin{aligned}
h(n) &= \frac{1}{2\pi j}\oint H_\epsilon(z) z^{n-1} dz \\
&= \sum_{i=1}^{L} Res(H(z)\ z^{n-1}, z_i) \\
&= \sum_{i=1}^{L} \frac{z_i^{L+n-1}}{\prod_{k=1,k\neq i}^{L}(z_i - z_k)} \\
&= \sum_{i=1}^{L} C_i z_i^n \\
&= z_1^n \left(C_1 + \sum_{i=2}^{L} C_i \left(\frac{z_i}{z_1}\right)^n \right) \qquad (20)
\end{aligned}
$$

where

$$
\begin{aligned}
C_i &= \frac{z_i^{L-1}}{\prod_{k=1,k\neq i}^{L}(z_i - z_k)} \\
&= \frac{z_i^{L-1}}{P'_\epsilon(z_i)} \qquad i = 1, \ldots, L \qquad (21)
\end{aligned}
$$

and by $Res(f(z), z_i)$ we denote the residue of the function $f(z)$ at z_i.

C_1 is a continuous function of z_1 which is a continuous function of ϵ and $C_1|_{\epsilon=0} = 1$. Therefore there exists $\epsilon_2 > 0$ such that

$$C_1 > \frac{L-1}{L} \ \ \forall\, 0 < \epsilon < \epsilon_2 \qquad (22)$$

From equation (20)it follows that:

$$
\begin{aligned}
h(n) &\geq z_1^n \left(C_1 - \left| \sum_{i=2}^{L} C_i \left(\frac{z_i}{z_1}\right)^n \right| \right) \\
&\geq z_1^n \left(C_1 - \sum_{i=2}^{L} |C_i| \left|\frac{z_i}{z_1}\right|^n \right) \qquad (23)
\end{aligned}
$$

Also from the continuity of the roots of $P_\epsilon(z)$ and equation (21) there exists ϵ_3 such that if $0 < \epsilon < \epsilon_3$ we have

$$|C_i| < \frac{1}{L} \quad \forall\, i = 2, \ldots, L \qquad (24)$$

$$\epsilon_0 = \min\{\epsilon_1\ ,\ \epsilon_2\ ,\ \epsilon_3\} \qquad (25)$$

From equations (22-24) the lemma is proved with ϵ_0 defined as in equation (25).

Notice that if $\epsilon > 0$ small enough the root z_1 moves inside the unit circle (which can be shown by writing the polynomial $P(z) = \prod_{i=1}^{L}(z - z_i)$ and evaluating $P(z)$ at $z = 1$) thus resulting in a stable $H_\epsilon(z)$ system.

Therefore, given the upper bounds of the delay intervals $\tau_{1_{max}}, \tau_{2_{max}}$, there always exists an ϵ sufficiently small and positive real, such that condition (c) in the theorem is satisfied and H_ϵ in (b) is stable.

A stronger result which is yet to be proved is that the impulse response $h_\epsilon(n)$ is strictly positive for all ϵ such that $0 < \epsilon < \epsilon_0$ where ϵ_0 is given by equation (19) where we replace α by the unique solution of equation (18) that lies in the interval $(0, 1)$.

4. CONCLUSION

The theorem provides a condition to check stability for a feedback system in which controller and plant (integrator) are connected via time-variant communication links.

If the time-variance of the communication delays are rapid, then a time invariant analysis using roots of z-transform system polynomials is not valid anymore.

Even though condition (c) appears to be not very conservative, the model for the uncertainties given by (10) and (12) also includes systems which cannot occur in (7). Hence the condition will have to show some conservatism.

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