Robust High Speed Normal Form Digital Oscillators *

Mihail L. Sichitiu
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
Mihail.L.Sichitiu.1@nd.edu

Peter H. Bauer
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
Peter.H.Bauer.2@nd.edu

Keywords: digital oscillators, normal form, frequency bounds, amplitude bounds

Abstract

A theoretical analysis of the properties of normal form digital oscillators is conducted. The effects of coefficient and state quantization on the frequency error, as well as amplitude and phase jitter, are investigated. Extensive simulations of the normal form oscillators are used to confirm the theoretical results and to obtain further information on the frequency quantization for a fixed wordlength.

1 Introduction

In modern high speed digital communication devices, it is very attractive to perform as many of the signal processing tasks as possible on the digital side. This includes modulators and mixers, which often need to work in or near GHz range. This requires high speed digital oscillators, which must be able to robustly work at very high frequencies. In cases where many frequencies need to be generated (some of which might be unknown *a priori*) a feedback structure is often preferable over a ROM solution [1].

In this paper we will focus our attention on normal form oscillators, since they are known to have very small coefficient sensitivity [2], are easy to implement and produce the quadrature component without additional hardware [3].

We will analyze the effect of the coefficient and state quantization on the frequency error, phase and amplitude jitter.

2 Formulation of the Oscillator Structure

Consider Figure 1 a, b, which illustrates the quantization lattice for the coefficients and the states of the normal form filter.

The ideal normal form oscillator can be described by

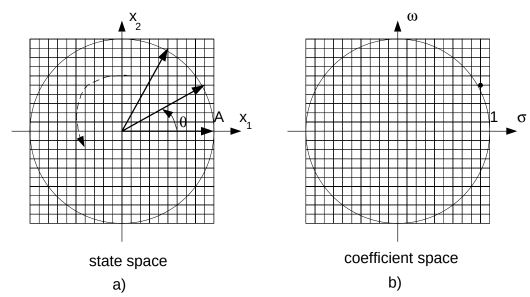

Figure 1: The quantization lattice with the coefficients and the state of the normal form filter

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{n+1} = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_n \tag{1}$$

where

$$\sigma^2 + \omega^2 = 1 \tag{2}$$

and the initial condition $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_0$ are chosen to obtain the desired amplitude A_d, where

$$x_{1_0}^2 + x_{2_0}^2 = A_d^2 \tag{3}$$

Considering the quantization process equation (1) becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{n+1} = Q\left[\begin{pmatrix} Q[\sigma] & Q[-\omega] \\ Q[\omega] & Q[\sigma] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_n\right] \tag{4}$$

where by $Q[\cdot]$ we denoted the quantization operator. Notice that we deal both with the coefficient quantization as well as the state quantization. From here on, we will express the quantized state in terms of multiples of the quantization step q. Therefore we will only deal with integers. Thus the maximum representable number on an N-bit realization will be $\left(2^{N-1}-1\right)q$ or simply $\left(2^{N-1}-1\right)$ and the minimum number will be $-\left(2^{N-1}-1\right)$ if a symmetric range around zero is assumed.

*This work was supported by the Alexander von Humboldt Foundation

The ideal coefficient range is the interval $[-1, +1]$ or considering the coefficient quantization, the range $\left[-\left(1-2^{-(N-1)}\right), \left(1-2^{-(N-1)}\right)\right]$.

Ideally we should choose the coefficients of the state transition matrix using the formula:

$$\sigma = \cos(2\pi \frac{f_d}{f_s}) \tag{5}$$

where f_d is the desired frequency and f_s is the sampling frequency; then the coefficient ω follows from equation (2):

$$\begin{aligned} \omega &= \sqrt{1-\sigma^2} \\ &= \sin(2\pi \frac{f_d}{f_s}) \end{aligned} \tag{6}$$

Unfortunately the coefficients, as well as the state, must be quantized. Usually the same number of bits is used for the state quantization and coefficient quantization. This number is typically equal to the processor's wordlength. This fact drastically reduces our choice of realizable frequencies since only 2^{2N} pairs of coefficients can be chosen for an N bit implementation. The number of coefficients is further reduced by one half if we consider that the lower half of the coefficient space produces the same frequencies as the symmetric coefficients in the upper half. (The only difference is in the direction of the rotation of the phasors.) Thus we can reduce the number of coefficient pairs to 2^{2N-1}.

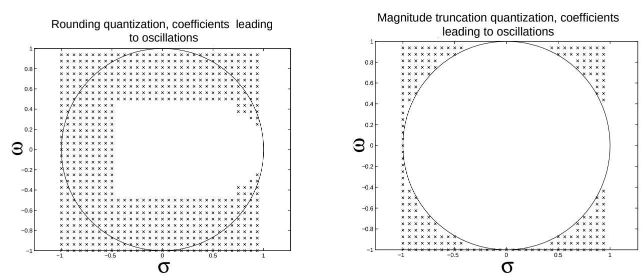

Figure 2: Coefficient maps for a) rounding quantization and b) magnitude truncation quantization

In the absence of state quantization only the coefficients which are exactly on the unit circle would produce oscillations. In practice, depending on the state quantization scheme, there are other pairs of coefficients that also work (i.e., produce limit cycles). Figure 2 a, b shows a map of the coefficients that lead to oscillations using two different quantization schemes. It can be shown [4] that for magnitude truncation any coefficients inside the unit circle lead to systems without limit cycles (which is needed for filters but is totally undesirable for oscillators). Thus for the remainder of this paper a rounding quantization scheme is assumed, which not only provides us with more options in choosing the coefficients, but also delivers a better signal to quantization noise ratio.

3 Frequency, Phase and Amplitude Bounds

According to Nyquist's theorem only frequencies between $[0, \frac{f_s}{2}]$ are realizable, but, as we will soon show, not all the frequencies in that range are realizable in practice.

Theorem 1 *(a) Using a given wordlength N a lower bound of the realizable frequencies is given by:*

$$f_{min_1} = \frac{f_s}{2\pi}\theta_{min} \tag{7}$$

where

$$\theta_{min} = \tan^{-1}\left(\frac{1}{2^{N-1}-1}\right). \tag{8}$$

(b) The frequency gap between the desired frequency and the closest realizable frequency is bounded from above by:

$$|f_i - f_{i+1}| = \Delta f_g \leq \frac{f_s}{2\pi}\sin^{-1}\left(\frac{\epsilon}{\sqrt{2}}\right). \tag{9}$$

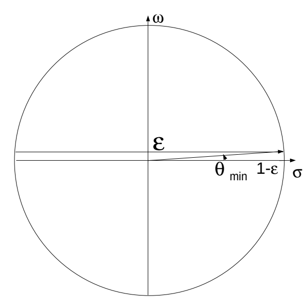

Figure 3: The smallest coefficient leading to oscillation

Proof

(a) First let us consider the two extreme cases:

- $\sigma = 1$ and $\omega = 0$ Consider the limit cycle described by equation 4. The states will be constant at all times, the limit cycle being degenerate.
- $\sigma = -1$ and $\omega = 0$ In this case the states will flip their sign at every step thus resulting in a two state cycle. This corresponds to the Nyquist limit frequency $\frac{f_s}{2}$

Figure 3 shows the coefficients space with the smallest coefficient leading to oscillation considering the quantization step for the coefficients $\epsilon = 2^{-(N-1)}$ and the maximum value for the coefficients being $1-\epsilon$. Thus the lowest bound of the realizable frequencies is given by the equation 7 where $\theta_{min} = \tan^{-1} \frac{\epsilon}{1-\epsilon}$.

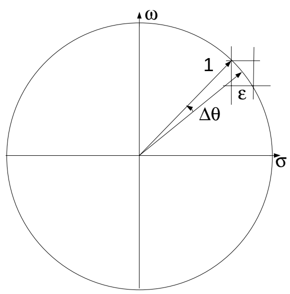

Figure 4: The gap between the desired and realizable frequency

(b) In Figure 4 the worst possible case for a frequency gap is shown in the coefficient space. This occurs at $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$ with $n = 0, 1, 2, 3, ...$ and the gap $\Delta\theta$ is given by

$$\Delta\theta = \sin^{-1}\left(\frac{\epsilon}{\sqrt{2}}\right) \tag{10}$$

which leads us to equation (9)

■

Now considering that the states are quantized as well, and considering the oscillator as a finite state machine with $2^N \text{x} 2^N$ states we obtain a maximum period of oscillation

$$T_{max} = 2^{2N} T_s \tag{11}$$

or

$$f_{min_2} = 2^{-2N} f_s \tag{12}$$

where by T_s, f_{min} and T_{max} we denoted the sampling period, the lower bound of the realizable frequencies and the corresponding largest period, respectively.

For $N \geq 2$, f_{min_1} is a better (or higher) lower bound than f_{min_2}.

Figure 5 shows the realizable frequencies for a normal form oscillator using a four bit and a six bit implementation, a rounding scheme and overflow with saturation. Notice that there are only a few frequencies that can be realized with a four bit implementation and as we increase the number of bits used for the realization, the number of realizable frequencies also increases while remaining discrete in nature. Notice that there are areas of dense realizable frequencies (like the ones around 0.1 and around 0.15) and areas where the realizable frequencies are more sparse. While the number of realizable frequencies increases with the increase of the number of bits in the implementation, the structure of dense - sparse realizable frequencies remains the same as it can be seen in Figure 5.

The fact that the states are quantized is reflected also in the phase and amplitude jitter of the oscillator. Indeed because the ideal phasor can "fall" between the points of the quantization lattice and be rounded to the nearest point on the lattice, a phase and an amplitude jitter could occur.

Theorem 2 *At each step the phase jitter is bounded by the expression:*

$$\Delta\theta \leq \sin^{-1}\left(\frac{1}{A\sqrt{2}}\right) \tag{13}$$

where by A we denote the amplitude of the oscillation given by the equation

$$A = \sqrt{x_1^2 + x_2^2} \tag{14}$$

Proof

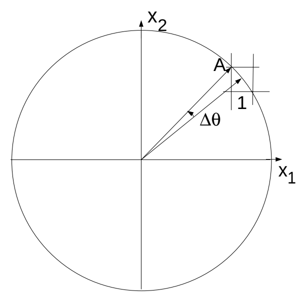

Figure 6: The frequency jitter

In Figure 6 the worst possible case in the state space is shown. This occurs at $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$ with $n = 0, 1, 2, 3, ...$

$$\Delta\theta = \sin^{-1}\left(\frac{1}{A\sqrt{2}}\right) \tag{15}$$

■

Notice that the phase jitter may be cumulative adding up to $M/2$ such phase jitters per cycle where M is the total number of states in a cycle and it is equal to $\frac{f_s}{f_r}$ where f_s and f_r are the sampling and realized frequency respectively. Statistically speaking, since a rounding scheme is employed, the phase jitter is having a zero mean but may cumulatively be adding up to $M/2$ times the bound in equation 13. Notice that the accumulated phase jitter can be larger when smaller frequencies are realized.

Inspecting equation 13 we notice that the phase jitter is also reduced by increasing the amplitude. Therefore we should always use the biggest amplitude possible, in order to reduce the signal to noise ratio, the phase jitter and the relative amplitude jitter.

Notice that the phase jitter due to state quantization and the frequency gap due to coefficient quantization have similar upper bounds. This is normal considering

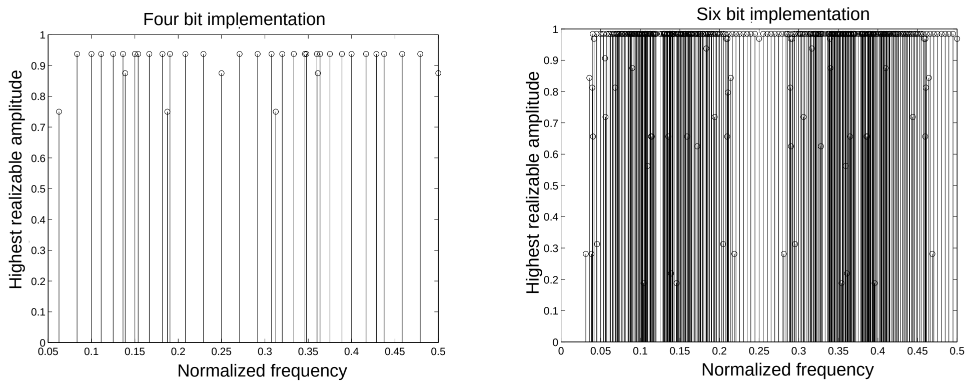

Figure 5: Realizable frequencies for a normal form oscillator using a four bit and a six bit implementation

that the quantization lattices of the state and coefficients are very similar. However, notice that the bounds are independent of the desired frequency. This implies that the relative errors are increasing with the decreasing of the desired frequency considering a fixed sampling frequency.

Lemma 1 *At each step the amplitude jitter can be bounded by*

$$\Delta A \leq \frac{1}{\sqrt{2}} \tag{16}$$

Proof

This is at the maximum distance from a point inside a unit square to the closest corner of the square. ■

Similarly with the phase jitter the amplitude jitter can be cumulative resulting up to a value of $\frac{M}{2\sqrt{2}}$ for an M steps cycle. As in the case of the phase jitter the mean value of the amplitude jitter is zero and the total amplitude jitter can be larger when lower frequencies are realized.

Notice that the relative amplitude jitter $\frac{\Delta A}{A}$ is decreasing with an increase of the amplitude; It is therefore preferable to choose an amplitude as large as possible. Notice from Figure 5 that not all the frequencies are realizable at the maximum amplitude permitted by the wordlength.

All the bounds presented so far are verified by detailed simulations of finite wordlength oscillators. In these simulations an exhaustive search is performed through all the limit cycles given all the possible combinations of coefficients and initial conditions for a given wordlength implementation.

The robustness of the oscillator can be improved using controlled rounding [5]. Also the performance of the oscillator can be further improved by means of error feedback [6].

4 Conclusion

The effects of coefficient and state quantization on the realizable frequencies as well as amplitude and phase jitter is investigated and confirmed through simulations.

References

[1] H. K. Kwan and W. C. Lee. Design of Low Sensitivities and Roundoff Noise Digital Oscillators. *Twentieth Asilomar Conference on Signals, Systems & Computers*, 191-195

[2] L. B. Jackson, A. G. Lindgren, and Y. Kim. Optimal synthesis of second-order state-space structures for digital filters *IEEE Trans. on Circuits and Systems* , CAS-26, 149-153, 1979

[3] N. J. Fliege and J. Wintermantel. Complex Digital Oscillators and FSK Modulators. *IEEE Trans. on Signal Processing*, SP-40:333-342, 1992

[4] A. Antoniou. Effects of finite wordlength in digital filters. *Digital Filters*, 376, 1993

[5] M. Miyata Highly Accurate Digital Sinusoidal Oscillators Using Controlled Rounding *Proc. IEEE Int. Sym. on Circuits and Systems* , 2213-2216, 1988

[6] A. I. Abu-El-Haija and M. M. Al-Ibrahim Improving Performance of Digital Sinusoidal Oscillators By Means of Error Feedback Circuits *IEEE Trans. on Circuits and Systems* , CAS-33:373-379, 1986