Stability of First Order Discrete Time Systems with Time-Variant Communication Delays in the Feedback Path

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Abstract

This paper addresses the problem of stability in 1st order systems with time-variant communication delays in the feedback path. Sets of necessary and sets of sufficient conditions are obtained and resulting stability regions are shown.

1 Introduction

Data networks play an increasingly important role in the development of technology in the 21^{st} century. With improved access to high speed data networks, many engineering systems will be embedded in networks that can support fast feedback loops. This paper takes a close look at the simple case of a first order system which is provided with proportional feedback through a communication network with time-variant delays. In particular the question of stability will be analyzed from several different perspectives.

The problem of stability in discrete time systems with time-variant delay has been addressed in only a few publications [1]-[4], even though its importance is rising rapidly. The derived conditions for stability are either conservative and relatively easy to test [1]-[4], or they are necessary and sufficient conditions but NP hard to test [6],[7],[9].

This paper attempts to close the gap between existing NP hard necessary and sufficient stability conditions and conservative sufficient stability conditions. This is achieved by deriving sets of sufficient conditions as well as sets of necessary conditions, thus tightly bounding the actual stability region.

2 Time variant delays in discrete-time systems

2.1 The nature of a discrete time-variant delay

One important characteristic of time-variant discrete delays is that it maps a countable sequence of input values into a countable sequence of sets of output values. In other terms, the output set of a time-variant discrete delay operator can be empty, or contain one or several values. For this reason, it is necessary to add an interface at the delay output, in order to restore a signal in the usual sense, that is a countable sequence of single values. We can distinguish between two components of a time-variant linear network connection: a delay introduced by the communication link, (i.e. the delay itself) and the interface that connects the network to the system at the receiving end.

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Figure 1 shows the two components of the time-variant linear network connection. The input and the output of the time-variant delay $d(n)$ are the sample sets $u(n)$ and $v(n)$. The symbol n describes discrete time with respect to an arbitrary sampling period. The sets $u(n)$ and $v(n)$ describe all input/output samples (scalar or otherwise) that are entering or leaving the block $d(n)$ within a sampling period, i.e. between time instant $n-1$ and time instant n. The output signal is a sequence of sets satisfying the following relation:

$$v(m) = \bigcup_{n+d(n)=m} u(n) \tag{1}$$

i.e. the set $v(m)$ is the union of all sets $u(n)$ which satisfy $n + d(n) = m$

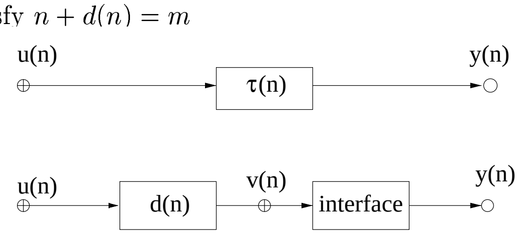

Figure 1: A discrete time-variant delay

We will use the following notation:

- $M(n)$: The cardinality of the input $u(n)$ at time n;
- $N(n)$: The cardinality of the output $v(n)$ at time instant n;
- $d(n)$: a function $d : Z_+ \longrightarrow Z_+$ where Z_+ is the set of nonnegative integers satisfying the inequality

$$0 \le \tau_{\min} \le d(n) \le \tau_{\max} < \infty. \quad (2)$$

The function $d(n)$ represents the delay the communication link introduces at time instant n for the input $u(n)$. The properties of the delays are completely described by the function $d(n)$. From (1) and the above introduced notation, the function $N(n)$ can be computed from the values of $d(n)$ with the following expression:

$$N(n) = \sum_{i \in U(n)} card(M(i)) \quad (3)$$

where by $card(A)$ we denote the cardinality of the set A and $U(n)$ is the set of time instants that produce an output at time instant n:

$$U(n) = \{i \mid i + d(i) = n, i \ge 0\}, \qquad n = n_0, n_0 + 1, ... \quad (4)$$

Assumig causality n_0 corresponds to the time instant where a sample is received for the first time. Notice that at some time instants the set $v(n)$ may contain more than one value while at other time instants the set may be empty. This is due to the discrete nature of time in the system. Intuitively, between two ticks of the system clock there might be one packet arriving from the communication link, more than one packet or none.

Equations (1) - (4) represent the most general description of a time-variant discrete delay. By nature these delay outputs must be sets (due to the finite and discrete time resolution). This model does not make any assumption on how the delay is interfaced with the linear system (input/output buffer, register, FIFO queues, etc...) and describes solely the nature of the delay itself. In fact an interface defines an operation on this set that results in a unique output value. Such an interface is therefore a necessity for a linear system connection with time-variant delays.

2.2 The delays interface

From the description of the time-variant delay operator above, it is clear that in general the output of such an operator cannot be interpreted as a signal. The output sequence $v(n)$ is a sequence of sets of values, and an operation on the output sets needs to be defined in order to generate a simple output sample $y(n)$ at every time instant n. Algorithms that choose a unique value from the output set can be designed depending on the nature of the problem, and are usually referred to as interfaces . In this paper we will use a Hold Freshest Sample interface, which chooses the freshest sample from all available sets $v(n)$, $v(n-1)$,... at every time instant n.

The instantaneous HFS delay value $\tau(n)$ is defined as follows:

$$\tau(n) = n - i_{\max}(n) \quad (5)$$

where,

$$\begin{aligned} i_{\max}(n) &= \max\left\{ \bigcup_{m=0}^{n} U(m) \right\} \\ &= \max(\{i_{\max}(n-1)\} \bigcup \{U(n)\}) \quad (6) \end{aligned}$$

Note that $i_{\max}(n)$ might not be defined for time instants n smaller than n_0. The sample set $U(m)$ is defined as in (4). Then the relation between the input and the output of the time-variable delay can be simply expressed as follows:

$$y(n) = u(n - \tau(n)) \quad (7)$$

From (2), (4) and (6) it follows that $i_{\max}$ is greater or equal to $n - \tau_{\max}$. Therefore, from (5), $\tau(n)$ is bounded: $\tau(n) \le \tau_{\max}$.

From (5), it can easily be shown that $0 \le \tau(n+1) \le \tau(n) + 1$, since $i_{\max}(n)$ is by (6) monotonically increasing. Throughout this sequel we distinguish two cases: (a) a known maximum delay $\tau_{\max}$ such that $0 \le \tau(n) \le \tau_{\max}$, (b) an unknown maximum delay $\tau_{\max}$. Results obtained for the case (b) are called "independent of delay". The case (a) is correspondingly denoted as delay dependent case. Hence we have:

$$0 \le \tau(n+1) \le \tau(n) + 1 \le \tau_{\max}. \quad (8)$$

3 The arising feedback system

The block diagram of a networked feedback system with a first order plant and a proportional controller is shown in Figure (2).

In order to perform a stability analysis of the system in Figure 2, we assume zero input condition (i.e. $u(n) \equiv 0$). Using the commutativity property of a time-variant delay with a proportional gain block, we can combine the two time-variant delays into one equivalent time variant delay $\tau(n)$:

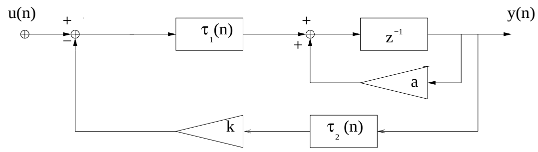

Figure 2: Proportional feedback control of a first order discrete time system via a data communication network

$$\tau(n) = \tau_1(n) + \tau_2(n - \tau_1(n)) \quad (9)$$

It can be shown that the slope condition (8) for HFS delays, also holds for the composite delay $\tau(n)$.

It should be noted, that while a time-variant delay and a linear system are in general not commutative [10], the case of a proportional controller is an exception. Also, note that in the general case two time variant delays are not commutative.

Stability analysis of time-variant discrete systems can be performed using recently developed analytical tools, such as a constructive stability algorithm by Brayton and Tong [7], a stability test based on matrix products by Bauer et al [6], or a test based on piecewise linear Lyapunov functions by Molchanov [9].

Even though these stability conditions are necessary and sufficient, they all suffer from the same drawback. The resulting computational complexity is NP (cf. [5]). Even in the case of a first order plant, a proportional controller, and small delay uncertainties, the algorithms in [6], [7], [9] often do not allow to determine stability in practice. An alternative approach to the stability problem is to derive tight sets of sufficient conditions as well as necessary conditions. This approach will be developed in the following subsections.

For a fixed delay τ the transfer function of the system is given by:

$$\frac{Y(z)}{\overline{U}(z)} = \frac{z^{-1}}{1 - az^{-\tau} + kz^{-\tau-1}} \quad (10)$$

$$\overline{u}(n) = u(n - \tau_1(n)) \quad (11)$$

or equivalently

$$y(n+1) = ay(n) - ky(n-\tau) + \overline{u}(n) \quad (12)$$

4 Necessary stability conditions using the frozen time systems

Using a Nyquist based argument we derived a necessary stability condition for the system with time-variant delays bounded by $\tau_{\max}$ that was described in the previous section. The details of the derivation are shown in [10]. The necessary stability condition can be expressed as follows:

$$\begin{cases} 1 - a\cos(\theta) + k\cos(\theta\tau_{\max}) & > & 0 \\ 1 - a + k & > & 0 \\ 1 + a - k & > & 0 \\ 1 + a + k & > & 0 \end{cases} \quad (13)$$

Where θ is the root of smallest magnitude of the equation:

$$\frac{\sin(\tau_{\max}\theta)}{\sin(\theta)} = \frac{a}{k}$$

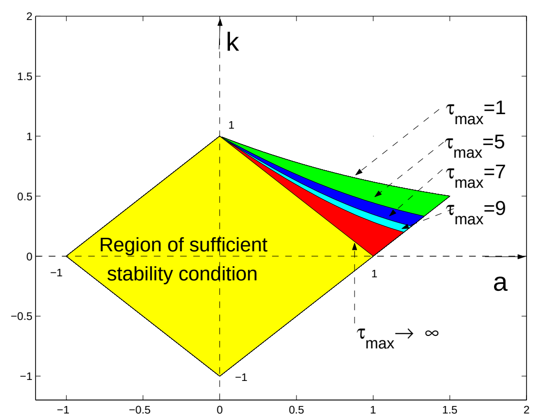

Figure 3: Stability region for a time-invariant system with $\tau \in [0, \tau_{\max}]$

The stability region for different values of τ_{max} is depicted in Figure 3. It illustrates that when the upper bound for the time-variant delay increases, the time-invariant system stability region becomes smaller. When $\tau_{\max}$ becomes infinite (which means the delays are unbounded) the first inequality of the stability condition (13) becomes [10]:

$$1 - a - k > 0$$

and the whole set of inequalities (13) becomes equivalent to:

$$|\, a \,| + |\, k \,| < 1 \quad (14)$$

We know that (14) is also a sufficient condition for stability of the time-variant system (see for example [4]).

Therefore, for the case of stability independent of delay (14) is a necessary and sufficient condition for both the stability of the time-invariant and time-variant system.

5 Necessary stability conditions using periodic delay trajectories

An additional set of necessary conditions is obtained using periodic delay trajectories $\tau(n)$, still satisfying the delay constraint: $0 \leq \tau(n) \leq \tau_{\max}$. Obviously the frozen time system case is a special case, where the period is equal to 1, resulting in constant delays.

The system in (12) with time-variant delays $\tau(n)$ and zero input takes the form:

$$y(n+1) = ay(n) - ky(n - \tau(n)) \tag{15}$$

Figure 4 shows the resulting stability region in the coefficient space for a periodic delay sequence.

5.1 Stability region for periodic delay ramps

We will consider the case when $\tau(n) = n \bmod (\tau_{\max} + 1)$, where by $x \bmod y$ we denote the remainder of the integer division of x by y, i.e.

$$x \bmod y = x - y \left\lfloor \frac{x}{y} \right\rfloor$$

By $\lfloor z \rfloor$ we denote the largest integer smaller or equal to z. This corresponds to the case of periodic delay ramps. Which are at most important in local area network connections. Access delays in general are of this nature.

Let us define the period of the delay ramp:

$$\tau^+ = \tau_{\max} + 1 \tag{16}$$

If $\tau(n) = n \bmod \tau^+$ equation (12) becomes:

$$y(n+1) = ay(n) - ky\left(\left\lfloor \frac{n}{\tau^+} \right\rfloor\right) + u(n) \tag{17}$$

We will analyze the system (17) under zero input. For $n \in [m\tau^+,\ (m+1)\tau^+ - 1]$ the system (17) has a constant feedback term, and therefore it can be described as a system with constant input:

$$y(n+1) = ay(n) - ky(m\tau^+) \tag{18}$$

$$y(n+1) = a^{n-m\tau^+ +1} y(m\tau^+) - y(ky(m\tau^+)) \sum_{i=0}^{n-m\tau^+} a^i \tag{19}$$

For $n = (m+1)\tau^+ - 1$ equation (19) becomes:

$$y((m+1)\tau^+) = \left(a^{\tau^+} - k\frac{a^{\tau^+} - 1}{a-1}\right) y(m\tau^+) \tag{20}$$

$$\Rightarrow y(m\tau^+) = \left(a^{\tau^+} - k\frac{a^{\tau^+} - 1}{a-1}\right)^m y(0) \tag{21}$$

The system (21) is stable if and only if

$$\left| a^{\tau^+} - k\frac{a^{\tau^+} - 1}{a-1} \right| < 1 \tag{22}$$

$$\Leftrightarrow \begin{cases} k > a - 1 \\ k < \frac{a^{\tau^+}+1}{a^{\tau^+}-1}(a-1) \end{cases} \tag{23}$$

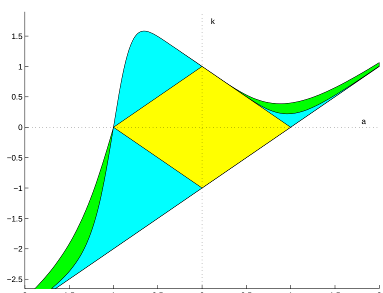

Figure 4: Stability region in the case of periodic ramp delays, for $\tau_{\max} = 4$ and $\tau_{\max} = 8$

The stability condition in (23) is also necessary and sufficient for stability in the case of non-periodic delay ramps as they appear in systems with local area connections.

Note that this system with time variant delays maybe stable, although some of its instantaneous frozen time system (for which the delays are kept constant) can be unstable. Thus a time invariant stability analysis cannot simply be extended to the stability analysis of time varying systems.

Alternatively, the system can be represented in state space form and one can formulate necessary conditions for any periodic delay trajectory using the standard techniques for periodic systems [11]. Results generally will not be possible in closed form.

6 Sufficient stability conditions

In this section we will present the general case: of the delay being both time-variant and uncertain.

The condition in (14)is known to be sufficient (but conservative) for the time-variant case (15). We will relax this sufficient condition using Theorem 1 presented in [8]. The following result is a generalization of a Theorem presented in [3].

Theorem

The system (15) with $\tau(n) \in [0, \quad \tau_{\max}]$ is globally asymptotically stable if the following two conditions are satisfied:

(a) The transfer function $H_\epsilon(z) = \frac{z^{-1}}{1-(a-\epsilon)z^{-1}+\epsilon z^{-2}+\ldots+\epsilon z^{-(\tau_{\max}+1)}} = \mathcal{Z}\{h_\epsilon(n)\}$, *is stable for some positive real* ϵ.

(b) $|k-\epsilon| < \frac{1}{\sum_{n=0}^{\infty}|h_\epsilon(n)|} - \tau_{\max}|\epsilon|$

proof:

Equation (15) can be written as an uncertain time variant system of the following form:

$$\begin{aligned} y(n+1) &= (a-(\epsilon+\Delta a_0(n)))y(n) - \\ &- (\epsilon+\Delta a_1(n)y(n-1) - \\ &- \ldots - (\epsilon+\Delta a_{\tau_{\max}}(n))y(n-\tau_{\max}) \end{aligned} \quad (24)$$

Define the nominal system as:

$$y_o(n+1) = (a-\epsilon)y_o(n) - \epsilon y_o(n-1) - \ldots \epsilon y_o(n-\tau_{\max}). \quad (25)$$

The uncertainties are $\Delta a_i(n) = k\alpha_i(n) - \epsilon \quad \forall i = 0, \ldots, \tau_{\max}$ where $\alpha_i(n) = 1$ if $\tau(n) = i$ and 0 otherwise. The size of the polytopic uncertainty is then:

$$\sum_{i=0}^{\tau_{\max}} |\Delta a_i| = \tau_{\max}|\epsilon| + |k-\epsilon| \quad (26)$$

The Theorem 1 in [8] shows that the system (15) is asymptotically stable if the nominal system (25) is stable and the following condition is satisfied:

$$\sum_{i=0}^{\tau_{\max}} |\Delta a_i(n)| < \frac{1}{\sum_{n=0}^{\infty}|h_\epsilon(n)|} \quad (27)$$

where $h_\epsilon(n)$ is the impulse response of the nominal system (25). Considering equation (26) condition (27) becomes equivalent to condition (b) in the hypothesis which proves the Theorem.

Of course one should maximize this bound as a function of ϵ and as a result maximize the stability region. It should be noticed however that despite the fact that it is an improvement over the stability condition independent of delay in (14), the above stability condition is often still conservative.

7 Conclusion

This paper derives a number of sufficient as well as necessary stability conditions for a first order discrete time system with a time-variant delay in the feedback path. These methods are shown to provide tight upper and lower bounds on the actual stability region of the system. The presented methods can easily be applied to systems of more general nature, i.e. plants and controllers of higher order.

References

[1] R. Krtolica, Ü. Özgüner, H. Chan, H. Goktas, J. Winkelman and M. Liubakka, "Stability of Linear Feedback Systems with Random Communication Delays", *International Journal of Control*, vol. 59, pp. 925-953, Apr. 1994.

[2] Peter H. Bauer, Mihail L. Sichitiu and Kamal Premaratne, "Closing the Loop through Communication Networks: The Case of an Integrator Plant and Multiple Controllers", *Proc. 38th IEEE Conference on Decision and Control*, pp. 2180-2185, Dec. 1999.

[3] Peter H. Bauer, Mihail L. Sichitiu and Kamal Premaratne, "Controlling an Integrator Through Data Networks: Stability in the Presence of Unknown Time-Variant Delays", *Proc. 1999 IEEE International Symposium on Circuits and Systems*, vol. 5, pp. 491-494, 1999.

[4] T. L. Hsien and C. H. Lee, "Exponential Stability of Discrete Time Uncertain Systems with Time-Variant Delay", *Journal of Franklin Institute*, vol. 332B, no. 4, pp. 479, Jul. 1995

[5] A. Bhaya and F. Mota, "Equivalence of Stability Concepts for Discrete Time-Varying Systems", *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 725-740, 1994.

[6] Peter H. Bauer, Kamal Premaratne and J. Durán, "A Necessary and Sufficient Condition for Robust Asymptotic Stability of Time-Variant Discrete Systems", *IEEE Transactions on Automatic Control*, vol. 38, pp. 1427-1430, Sep. 1993.

[7] R. Brayton and C.H. Tong, “ Constructive stability and asymptotic stability of dynamical systems”, *IEEE Trans. on Circuits and Systems*, vol. 27, pp 1121-1130, 1980,

[8] S. A. Yost and P H. Bauer, “Asymptotic Stability of Linear Shift-variant Difference Equations with Diamond Shaped Uncertainties”, *IEEE International Symposium on Circuits and Systems*, pp 785-788, 1995

[9] A. P. Molchanov, “Lyapunov Functions for Nonlinear Discrete-Time Control Systems”, *Automat. Remote Control*, vol. 48, pp 728-736, 1987

[10] Cédric Lorand, “Stabilitätsuntersuchung von diskreten Systemen mit zeitvariablen Totzeiten”, Diplom Thesis - Technishe Universität München, 1998

[11] P. J. Antsaklis and A. N. Michel “Linear Systems”, McGraw-Hill 1997.