General Mathematics II

—Sample exam on part 4a (differential equations)—

In preparation of the final exam on May 24, 2002 (2.00 p.m.)

General information

The final exam of this course will be written on May 24, 2002, at 2.00 p.m., in the Lecture Hall. You are supposed to demonstrate that you can apply the mathematical tools discussed in class.

Problem 1 will deal with multivariable calculus, problem 2 with integration, and problem 3 with basic linear algebra. These three problems will be of the same type and the same level as in the corresponding short exams. For further information see the general information sections of the sample exam sheets on parts 1–3.

Depending on the module you have chosen in part 4 of this course, you are expected to either solve problems 4b, 5b, 6b which will deal with advanced linear algebra topics (see the companion sample exam sheet for preparation), or solve 4a, 5a, 6a which will address differential equations. In order to deal with the latter three problems you should be able

- to solve first-order differential equations with separable variables,
- to transform the special types $y'(x) = f(\alpha x + \beta y + \gamma)$ and $y'(x) = f(y/x)$ to the separable variable case,
- to test whether first-order ODEs are exact, and to find the solution of an exact differential equation in implicit form $F(x, y) = C\ ,\ C \in \mathbb{R}$,
- to decide if a given non-exact differential equation has an integrating factor which depends only on one variable, and to determine such an integrating factor,
- to handle second-order ODEs of the type $y''(x) = f(y)$ (Newton's law),
- to find the general solution of a linear inhomogeneous first-order ODE, and to select the particular solution which satisfies a given initial condition,
- to deal with linear homogeneous ODEs of higher-order with constant coefficients (i.e., to compute roots of the characteristic polynomial and to find the corresponding solutions), in particular, to handle the second-order case.

Calculators, class notes, or textbooks must not be used during the final exam. Write your solutions into the blue books directly. So it's sufficient to bring a pen. No pencil, please.

The maximum time to work on the six problems is 120 minutes.

Please approach this sample exam sheet as if it was the differential equations part of actual final exam, and let us know how much time you needed to solve the problems.

Your friendly GM2 team.

Notation and useful formulas

Integrating factors

If an integrating factor $\phi = \phi(x)$ for the differential equation $p(x, y)\dot{x} + q(x, y)\dot{y} = 0$ exists, that means, if the differential equation $\phi p\dot{x} + \phi q\dot{y} = 0$ is exact, then

$$\frac{1}{q}\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = \frac{\phi'(x)}{\phi(x)} = (\ln\phi)'(x) \ .$$

If $\phi = \phi(y)$, then

$$\frac{1}{p}\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) = \frac{\phi'(y)}{\phi(y)} = (\ln\phi)'(y) \ .$$

Linear second-order ODEs with constant real coefficients

The general solution of a homogeneous linear second-order ODE with constant real coefficients

$$\ddot{y} + 2\Gamma\,\dot{y} + \Omega_0^2\,y = 0$$

($\Omega_0 > 0$) depends on the roots of the characteristic polynomial

$$p(\lambda) = \lambda^2 + 2\Gamma\lambda + \Omega_0^2 \ .$$

- Two distinct real roots λ_1, λ_2:

$$y(t) = A\,\mathrm{e}^{\lambda_1 t} + B\,\mathrm{e}^{\lambda_2 t} \ , \ A, B \in \mathbb{R} \ .$$

- Two complex roots $\lambda_{1,2} = \mu \pm \mathrm{i}\omega$:

$$y(t) = A\,\mathrm{e}^{\mu t}\cos\omega t + B\,\mathrm{e}^{\mu t}\sin\omega t \ , \ A, B \in \mathbb{R} \ .$$

- One real root λ:

$$y(t) = A\,\mathrm{e}^{\lambda t} + B\,t\mathrm{e}^{\lambda t} \ , \ A, B \in \mathbb{R} \ .$$

Problems on parts 1–3 of this course

For sample problems on parts 1–3 see the corresponding sample exam sheets.

1. Multivariable calculus

(50 points)

2. Integration

(50 points)

3. Basic linear algebra

(50 points)

Problems on part 4a (differential equations)

4a. Separable variables

(50 points)

Solve the following initial value problems.

(i) $y'(x) \; = \; \dfrac{2x}{\cos y} \; , \; y(0) = 0 \; .$

(ii) $y'(x) \; = \; \dfrac{y}{x} + \dfrac{y^2}{x^2} \; , \; y(1) = 1 \; .$

5a. Exact differential equations

(50 points)

(i) Show that the differential equation

$$\frac{xy + y^2 + y\sin x\cos x}{\cos^2 x}\,\dot{x} \;+\; \tan x\,(x+2y)\,\dot{y} \;=\; 0$$

is exact. Find the general solution in implicit form $F(x,y) \;=\; C,\; C \in \mathbb{R}$.

(ii) Show that the differential equation

$$2xy\,\dot{x} \;+\; (x^2y^2 + x^2)\,\dot{y} \;=\; 0$$

is not exact. Find an integrating factor which depends only on one variable. (You do not need to compute the solution of the differential equation.)

6a. Linear ODEs

(50 points)

(i) Determine the general solution of the linear first-order ODE

$$y'(x) \;+\; \frac{y}{x^2} \;=\; 2x\mathrm{e}^{1/x}\;.$$

(ii) For the linear second-order ODE

$$\ddot{y} \;-\; 4\dot{y} \;+\; 4y \;=\; 0$$

compute the roots of the characteristic polynomial and write down the general solution. Furthermore, find the particular solution which satisfies the initial conditions

$$y(0) \;=\; 0\;,\;\; \dot{y}(0) \;=\; 1\;.$$