

# ERGODIC PROPERTIES OF MULTIDIMENSIONAL BROWNIAN MOTION WITH REBIRTH<sup>†</sup>

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ABSTRACT. In a bounded open region of the  $d$  dimensional space we consider a Brownian motion which is reborn at a fixed interior point as soon as it reaches the boundary. The evolution is invariant with respect to a density equal, modulo a constant, to the Green function of the Laplacian with pole at the point of return. We determine the resolvent in closed form and prove the exponential ergodicity by Laplace transform methods using the analytic semigroup properties of the Dirichlet Laplacian.

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*Date:* September 2, 2004.

1991 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 60J50, 35K15 .

*Key words and phrases.* Absorbing Brownian motion, Green function, harmonic measure, collapsing paths.

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<sup>†</sup> The first author was supported in part by the University of Utah and the second author was supported in part by Northwestern University.

## 1. Introduction

This paper completes and generalizes in higher dimensions the results of [3]. Let  $\mathcal{R}$  be a bounded open region in  $R^d$  with a piecewise smooth boundary such that the origin  $O \in \mathcal{R}$ . For  $x \in R^d$ , let  $W_x = (w_x(t, \omega), \{\mathcal{F}_t\}_{t \geq 0})$  be a Brownian motion on  $R^d$  such that  $P(w_x(0, \omega) = x) = 1$ . On the region  $\mathcal{R}$ , for any  $x \in \mathcal{R}$ , we define a process  $\{z_x(t, \omega)\}_{t \geq 0}$  with values in  $\mathcal{R}$  which is identical to a standard  $d$  dimensional Brownian motion until the time  $T < \infty$  when it reaches the boundary, then instantaneously returns to the origin  $O$  at  $T$  and repeats the same evolution indefinitely. This is the multidimensional version of the problem described in [3], which may be called *Brownian motion with rebirth*, since after emulating the Brownian motion with absorbing boundary conditions (in other words, *killed* at the boundary) it is reborn at the origin. Naturally, since the state space can be shown to be compact (the case  $d = 1$  is explained in [3]), the dynamics has an invariant measure given, modulo a constant factor, by the Green function for the Laplacian with pole at  $\xi = 0$  (Theorem 2). In section 7.4 of [5] the average time a Brownian motion starting at  $x$  spends in the set  $B \subset \mathcal{R}$  before hitting the boundary is identified as  $2 \int_B G(x, y) dy$ . Our particle will repeat the trip from the origin to the boundary indefinitely and will stabilize in time, by ergodicity, towards the measure which gives the mean value over all configurations.

A way to generalize the current process is to change the distribution of the location where the particle is reborn into a time-dependent measure-valued process  $\mu(t, dx)$ . The tagged particle process from [4] is an example in the case when  $\mu(t, dx)$  is the deterministic macroscopic limit of the empirical measures of a large system of Brownian particles with branching confined to the region  $D$ . In particular, in equilibrium, the updating measure

$\mu(t, dx)$  is constant in time, being equal to  $\mu(dx) = \Phi_1(x)dx$ , the probability measure with density equal to the first eigenfunction of the Dirichlet Laplacian (normalized). In this context, the Brownian motion with rebirth can be regarded as the case  $\mu(dx) = \delta_0$ . The renewal mechanism is captured by the closed formula (2.19). The degeneracy of the update distribution at the origin gives a local (pointwise) character to the problem. The estimates needed for the Laplace transform inversion formula (for a general sufficient result see [7]) are easier to obtain in an  $L^p$  norm than in the uniform norm. In that sense, one needs the analytic semigroup results from Stewart ([8], [9]).

## 2. Results

We shall denote by  $(\Omega, \mathcal{F}, P)$  a probability space supporting the law of the family of  $d$ -dimensional coupled Brownian motions indexed by their starting point  $x \in \mathcal{R}$ . Let  $\mathcal{A}$  be an open region in  $R^d$  and  $x \in \mathcal{A}$ . In general we shall use the notation

$$(2.1) \quad T_x(\mathcal{A}) = \inf\{t > 0 : w_x(t, \omega) \notin \mathcal{A}\},$$

the *exit time* from the region  $\mathcal{A}$  for the Brownian motion starting at  $x$ . Occasionally we shall suppress either  $x$  or the set  $\mathcal{A}$  if they are unambiguously defined in a particular context. We shall define inductively the increasing sequence of stopping times  $\{\tau_n\}_{n \geq 0}$ , together with a family of adapted nondecreasing processes  $\{N_x^b(t, \omega)\}_{t \geq 0}$ , indexed by  $b \in \partial\mathcal{R}$  and the process  $\{z_x(t, \omega)\}_{t \geq 0}$ , starting at  $x \in \mathcal{R}$ . Let  $T_x = \tau_0 = \inf\{t : w_x(t, \omega) \notin \mathcal{R}\}$ , while for  $t \leq \tau_0$  and all  $b \in \partial\mathcal{R}$  we set  $N_x^b(t, \omega) = 1_{\{b\}}(w_x(t, \omega))$  and  $z_x(t, \omega) = w_x(t, \omega) - \sum_{b \in \partial\mathcal{R}} b N_x^b(t, \omega)$ . By

induction on  $n \in \mathbb{Z}_+$

$$(2.2) \quad \tau_{n+1} = \inf\{t > \tau_n : w_x(t, \omega) - \sum_{b \in \partial \mathcal{R}} b N_x^b(\tau_n, \omega) \notin \mathcal{R}\}$$

which enables us to define

$$(2.3) \quad N_x^b(t, \omega) = N_x^b(\tau_n, \omega) + 1_{\{b\}}(z_x(t, \omega)),$$

as well as

$$(2.4) \quad z_x(t, \omega) = w_x(t, \omega) - \sum_{b \in \partial \mathcal{R}} b N_x^b(t, \omega)$$

for  $\tau_n < t \leq \tau_{n+1}$ . We notice that  $z_x(t, \omega) = 0$  for all  $t = \tau_n$ . The construction and the summations present in (2.2) and (2.4) are finite due to the following result.

**Proposition 1.** *The sequence of stopping times  $\tau_0 < \tau_1 < \dots < \tau_n < \dots$  are finite for all  $n$  and  $\lim_{n \rightarrow \infty} \tau_n = \infty$ , both almost surely. Also, for any  $b \in \partial \mathcal{R}$ , the integer-valued processes  $N_x^b(t, \omega)$  defined for  $t \geq 0$  have the properties*

- (i) *they are nondecreasing, piecewise constant, predictable and right-continuous,*
- (ii)  $P(N_x^b(t, \omega) < \infty) = 1$ .

*Proof.* By monotonicity, since the expected value of the first exit time from a ball centered at the origin is finite in any dimension  $d$  (for example, in [10]), we deduce that  $E[T] < \infty$ . As a consequence,  $T = T_x < \infty$  a.s.. The time intervals between  $\tau_n$  and  $\tau_{n+1}$  (we include  $\tau_{-1} = 0$ ), for any  $n \geq -1$  are either  $T_x$  for the first exit time or independently identically distributed as  $T_0$  for all the rest. Since  $P(T_x = 0) = 0$  for any  $x \in \mathcal{R}$  the sequence is strictly increasing. Moreover,  $E[\tau_n] < \infty$ , which implies  $P(\tau_n < \infty) = 1$ . In the same

time, if  $N > 0$  is fixed,  $P(\lim_{n \rightarrow \infty} \tau_n \leq N) \leq P(T_0^1 + T_0^2 + \dots < N)$  for a sequence of i.i.d.  $T_0^i \sim T_0$ . If the sum  $T_0^1 + T_0^2 + \dots$  is finite we must have elements in the summation arbitrarily small, for instance  $T_0^k < \epsilon$ , for an infinite sequence of increasing ranks  $k$ . We can find a value  $\epsilon$  such that  $P(T_0 < \epsilon) < 1$ . From the independence condition, we derive that  $P(\lim_{n \rightarrow \infty} \tau_n \leq N) = 0$ . But  $\{\lim_{n \rightarrow \infty} \tau_n < \infty\}$  is the union of these events when  $N \rightarrow \infty$ .

The processes  $N_x^b(t, \omega) \geq 0$  are clearly nondecreasing, integer-valued and piecewise constant. They are right-continuous by construction (2.3) preserving the same value until the next boundary hit. Predictability is a consequence of the fact that the first exit times  $\{\tau_n\}$  are stopping times.

The summations over boundary elements  $b \in \partial\mathcal{R}$  from equations (2.2) and (2.4) are actually finite, since we have shown that, with probability one, the number of times we can hit the boundary if we start each time from an interior point of the region is finite.  $\square$

Let  $A \in \mathcal{B}(R^d)$  and  $p_{abs}(t, x, y)$  denote the *absorbing Brownian kernel*

$$(2.5) \quad \int_A p_{abs}(t, x, y) dy = P\left(w_x(t, \omega) \in A, t < T_x(\mathcal{R})\right).$$

The operator  $\Delta$  with Dirichlet boundary conditions on  $\partial\mathcal{R}$  has a countable spectrum  $\{\lambda_i^{abs}\}_{i \geq 1}$

$$0 > \lambda_1^{abs} \geq \lambda_2^{abs} \geq \dots$$

with corresponding eigenfunctions  $\{\Phi_n(x)\}$  and

$$(2.6) \quad p_{abs}(t, x, y) = \sum_{n=1}^{\infty} \exp\left(\frac{\lambda_n^{abs} t}{2}\right) \Phi_n(x) \Phi_n(y)$$

(as in [5]). The functions  $\{\Phi_n(x)\}$  are smooth (Proposition 3) and form an orthonormal basis of  $L^2(\mathcal{R})$  ([1, 6.5]). The resolvent of the absorbing Brownian motion will be denoted by

$$(2.7) \quad R_\alpha^{abs} f(x) = \int_0^\infty \int_{\mathcal{R}} e^{-\alpha t} p_{abs}(t, x, y) f(y) dy dt.$$

In the following, the Laplace transform of the first exit time  $T_x(\mathcal{R})$  from the domain  $\mathcal{R}$  of a Brownian motion starting at  $x$  will be denoted by

$$(2.8) \quad \widehat{h}^x(\alpha) = E_x[e^{-\alpha T_x(\mathcal{R})}].$$

The Laplace transform (2.8) exists on the complex plane for all  $\alpha$  with  $\Re(\alpha) > \lambda_1^{abs}$ . (see, in that sense, the remark following Theorem 1.)

The law of the process  $\{z_x(t, \omega)\}_{t \geq 0}$ , adapted to  $\{\mathcal{F}_t\}_{t \geq 0}$  will be denoted by  $Q_x$  and the family of processes  $\{Q_x\}_{x \in \mathcal{R}}$  will be denoted simply by  $\{Q\}$ . The construction described by equations (2.2) through (2.4) can be made deterministically for any  $x \in \mathcal{R}$  and each path  $w_x(\cdot) \in C([0, \infty), R^d)$  resulting in a predictable mapping

$$(2.9) \quad \Phi(w_x(\cdot)) = w_x(t) - \sum_{b \in \partial \mathcal{R}} b N_x^b(t).$$

With this notation  $\Phi : C([0, \infty), R^d) \rightarrow D([0, \infty), \mathcal{R})$  and  $Q_x = W_x \circ \Phi^{-1}$  is the law of the process  $\{z_x(t, \omega)\}_{t \geq 0}$  with values in the region  $\mathcal{R}$ , a measure on the Skorohod space  $D([0, \infty), \mathcal{R})$ .

Let  $m \in \mathbb{Z}_+$ ,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}_+^d$  be a  $d$  dimensional multi-index vector and we write

$|\alpha| = \sum_{i=1}^d \alpha_i$ . If  $D \subseteq \mathbb{R}^d$  and  $f : D \rightarrow \mathbb{R}$ , we denote

$$\partial^{(\alpha)} f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}(x)$$

if the derivatives exists. Naturally  $C^m(D)$  is the set of functions for which all derivatives with multi-indices  $\alpha$  such that  $|\alpha| \leq m$  exist and are continuous.

We recall that the process  $\{z_x(t, \omega)\}_{t \geq 0}$  is adapted to the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  corresponding to the underlying standard  $d$ -dimensional Brownian motion.

**Proposition 2.** *If  $f \in C^2(\bar{\mathcal{R}})$ , then*

$$(2.10) \quad f(z_x(t, \omega)) - f(x) - \int_0^t \frac{1}{2} \Delta f(z_x(s, \omega)) ds - \sum_{b \in \partial \mathcal{R}} \int_0^t (f(0) - f(z_x(s, \omega))) dN_x^b(s, \omega)$$

is a  $\mathcal{F}_t$  - martingale with respect to  $Q_x$ .

*Proof.* The proof is identical with the  $d = 1$  case from [3]. □

Let

$$(2.11) \quad \mathcal{D}_0 = \left\{ f : \forall |\alpha| \leq 2, \partial^{(\alpha)} f \in C(\bar{\mathcal{R}}), \forall b \in \partial \mathcal{R}, f(0) = f(b) \right\}.$$

**Corollary 1.** *If  $f \in \mathcal{D}_0$  then*

$$(2.12) \quad f(z_x(t, \omega)) - f(x) - \int_0^t \frac{1}{2} \Delta f(z_x(s, \omega)) ds$$

is a  $\mathcal{F}_t$  - martingale with respect to  $Q_x$ .

The next result allows us to regard  $\{z_x(t, \omega)\}_{t \geq 0}$  as a process with continuous paths on the compact state space  $X$ .

Let  $X = \mathcal{R}$  with the topology  $\mathcal{T}$  generated by the neighborhood basis

$$\begin{aligned} V_{x,r} &= \left\{ B(x, r) : \forall r > 0 \text{ such that } B(x, r) \subset \mathcal{R} \setminus \{0\} \right\} & \text{if } x &\neq 0 \\ V_{0,r} &= \left\{ B(0, r) \cup \left( \cup_{b \in \partial \mathcal{R}} (B(b, r) \cap \mathcal{R}) \right) : r \in (0, \frac{1}{2}d(x, \partial \mathcal{R})) \right\} & \text{if } x &= 0. \end{aligned}$$

with the usual topology. *Remark:* The space  $(X, \mathcal{T})$  is compact.

We define the class of functions of class  $C^2$  up to the boundary

(2.13)

$$\mathcal{D}(X) = \{f \in C^2(X \setminus \{0\}) : \lim_{x \rightarrow r} f^{(\alpha)}(x) \text{ exists and is finite, } 0 \leq |\alpha| \leq 2, r \in \{0\} \cup \partial \mathcal{R}\}$$

where the lateral limit  $\lim_{x \rightarrow r} g(x)$  is defined as  $\lim_{x \rightarrow r} g(x)$  in the topology inherited from  $R^d$  by the set  $B(0, r) \subseteq \mathcal{R}$ ,  $r > 0$ , in the case of the origin and  $X \cap B(b, r)$ , if  $b \in \partial \mathcal{R}$ .

The inclusion mapping  $\mathcal{I} : \mathcal{D}(X) \rightarrow \mathcal{D}_0$  is defined as  $\mathcal{D}(X) \ni f \rightarrow \mathcal{I}(f) \in \mathcal{D}_0$ , where  $\mathcal{I}(f)(x) = f(i(x))$  and  $i(x) = x$  is the identification mapping from  $\mathcal{R}$  to  $X$ .

Under the inclusion mapping  $\mathcal{I} : \mathcal{D}(X) \rightarrow \mathcal{D}_0$  the domain (2.11) is equal to

$$(2.14) \quad \mathcal{D}_0(X) = \left\{ f \in \mathcal{D}(X) : \forall b \in \partial \mathcal{R} \quad \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow b} f(x) \right\}.$$

**Corollary 2.** Let  $\widehat{Q}_x = Q_x \circ i^{-1}$  be the measure induced on  $C([0, \infty), X)$  by  $i : \mathcal{R} \rightarrow X$ .

Then,  $\widehat{Q}_x$  solves the martingale problem for the Markov pregenerator

$$(2.15) \quad \mathcal{L} = \left( \frac{1}{2} \Delta, \mathcal{D}_0(X) \right).$$

*Proof.* The argument does not change with  $d > 1$  and is presented in [3]. We refer to [6] for the definition of a Markov pregenerator. The properties of  $f \in \mathcal{D}_0(X)$  ensure that  $\overline{\mathcal{D}_0(X)} = C(X)$ . All the other properties are local, and follow from the maximum principle for the Laplacian. At  $x = 0$  we can apply the standard argument which shows that  $\nabla f(0) = 0$  because it only depends on the ball  $B(0, r)$ , which is a subset of a neighborhood of the origin in  $(X, \mathcal{T})$  as well. The rest is immediate from Proposition 2.  $\square$

We remind that an open bounded region  $\mathcal{R}$  has the *exterior cone condition* (see [2], page 193) if for any  $x_0 \in \partial\mathcal{R}$  there exists a circular cone  $V_{x_0}$  with vertex at  $x_0$  such that  $\overline{\mathcal{R}} \cap V_{x_0} = \{x_0\}$ . It is clear that many regions with piecewise smooth boundaries, including nonconvex examples and certainly all polyhedra satisfy the condition. In the following we shall use the notation  $\|f\|_{C(\overline{\mathcal{R}})}$  for the supremum norm of the bounded function  $f$ . The next theorem is valid for domains  $\mathcal{R}$  with boundary  $\partial\mathcal{R} \in C^2$ .

**Theorem 1.** *Let  $P(t, x, dy)$  be the transition probability for the process  $\{Q_x\}_{x \in \mathcal{R}}$ . For any  $t > 0$  the measure  $P(t, x, dy)$  is absolutely continuous with respect to the Lebesgue measure on  $\mathcal{R}$  and, if  $N_x(t) = \sum_{b \in \partial\mathcal{R}} N_x^b(t)$  is the total number of visits to the boundary up to time  $t > 0$ , its probability density function  $p(t, x, y)$  is given by*

$$(2.16) \quad p(t, x, y) = p_{abs}(t, x, y) + \int_0^t p_{abs}(t - s, 0, y) dE[N_x(s)]$$

and satisfies the properties:

(i) for  $f \in C(X)$ , the contraction semigroup

$$(2.17) \quad S_t f(x) = \int_{\mathcal{R}} p(t, x, y) f(y) dy$$

maps continuous functions into continuous functions (generating a Feller process) and there exist  $\phi \in (\frac{\pi}{2}, \pi)$ ,  $R > 0$  and  $M > 0$  such that the resolvent set  $\rho(\mathcal{L})$  of the infinitesimal generator of (2.17) includes the union of  $(\lambda_1^{abs}, \infty) \setminus \{0\}$ , the right half-plane  $\Re(\alpha) > 0$  and the truncated sector

$$(2.18) \quad U_0^R = \left\{ \alpha : |\arg(\alpha)| < \phi, |\alpha| > R \right\}$$

and its resolvent  $R_\alpha f = \int_{\mathcal{R}} e^{-\alpha t} S_t f dt$  is a meromorphic function on the resolvent set of the Dirichlet Laplacian with a simple pole at  $\lambda_0 = 0$

$$(2.19) \quad R_\alpha f(x) = R_\alpha^{abs} f(x) + R_\alpha^{abs} f(0) \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)}$$

satisfying

$$(2.20) \quad \|R_\alpha f\|_{C(\overline{\mathcal{R}})} \leq \frac{M}{|\alpha|} \|f\|_{C(\overline{\mathcal{R}})} \quad \forall \alpha \in U_0^R,$$

(ii) the residue at  $\alpha = 0$  has kernel

$$(2.21) \quad \rho(y) = \frac{G(0, y)}{\int_{\mathcal{R}} G(0, y) dy}$$

where  $G(x, y)$  is the Green function of the Laplacian with Dirichlet boundary conditions and

(iii) if  $\alpha^*$  is the nonzero element of the spectrum  $\sigma(\mathcal{L})$  with maximal real part, then

$$\sup_{\alpha \in \sigma(\mathcal{L}) \setminus \{0\}} \Re(\alpha) = \Re(\alpha^*) < 0$$

and, for any  $f \in \mathcal{D}_0(X)$

$$(2.22) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left( \sup_{\|f\|_{C(\overline{\mathcal{R}})} \leq 1} \|S_t f(x) - \int_{\mathcal{R}} \rho(x) f(x) dx\|_{C(\overline{\mathcal{R}})} \right) \leq \Re(\alpha^*).$$

**Corollary 3.** *The process  $\{Q\}$  is ergodic.*

*Remark:* The function defined by (2.8) has an analytic continuation on the resolvent set of the Dirichlet Laplacian, and can be re-written directly in terms of the resolvent  $R_\alpha^{abs}$  as shown in equations (3.1)-(3.2).

The proof of Theorem 1 depends on the uniform estimates on the resolvent of the Dirichlet Laplacian, which are available for domains with smooth boundary (in Stewart [8], [9]). The following theorem identifies directly the invariant measure as the Green function at the point of return for piecewise smooth domains.

**Theorem 2.** *If  $\mathcal{R}$  is an open bounded region of  $R^d$  with piecewise smooth boundary satisfying the exterior cone condition, then the process  $\{Q\}$  has a unique invariant measure  $\nu(dx)$ , absolutely continuous with respect to the Lebesgue measure on  $\mathcal{R}$ , with density  $\rho(x)$  equal to the Green function for the Dirichlet problem with pole at  $\xi = 0$  and normalized to have integral equal to one.*

### 3. Proof of Theorem 1

*Proof.* (i) The derivation of (2.16) does not depend on the dimension  $d \in Z_+$  henceforth we can refer to the proof of Theorem 1 in [3] directly.

By definition, the Laplace transform of a function  $g(t)$  is equal to  $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha t} g(t) dt$  whenever the integral converges. From equation (2.5)  $P(T^x > t) = \int_{\mathcal{R}} p_{abs}(t, x, y) dy$  we see that

$$(3.1) \quad \widehat{h^x}(\alpha) = E \left[ e^{-\alpha T^x} \right] = - \int_0^\infty e^{-\alpha t} dP(T^x > t).$$

For  $\Re(\alpha) > 0$ , we derive

$$(3.2) \quad \widehat{h}^x(\alpha) = - \int_{\mathcal{R}} e^{-\alpha t} p_{abs}(t, x, y) dy \Big|_0^{\infty} - \alpha R_{\alpha}^{abs} \mathbf{1}(x) = 1 - \alpha R_{\alpha}^{abs} \mathbf{1}(x)$$

where  $\mathbf{1}(x)$  is the constant function equal to 1 and  $R_{\alpha}^{abs}$  is the resolvent of the half Laplacian with Dirichlet boundary conditions (the infinitesimal generator of the absorbing Brownian motion) from (2.7). For  $\Re(\alpha) > 0$  we immediately have  $\|\widehat{h}^x(\alpha)\|_{C(\overline{\mathcal{R}})} < 1$ . For any  $f \in \mathcal{D}_0(X)$  the resolvent  $R_{\alpha}^{abs} f(x)$  is analytic on  $C \setminus \{\lambda_n : n \geq 1\}$  (for example, [11], page 211, applied to the generator of a semigroup). With this in mind, for  $\Re(\alpha) > 0$ , from (2.16) we obtain

$$\begin{aligned} \int_{\mathcal{R}} \widehat{p}(\alpha, x, y) f(y) dy &= \int_{\mathcal{R}} \widehat{p}_{abs}(\alpha, x, y) f(y) dy + \\ &+ \int_{\mathcal{R}} \widehat{p}_{abs}(\alpha, 0, y) f(y) dy \left( \sum_{n=1}^{\infty} \widehat{(h^x * (h^0)^{*, n-1})} \right) (\alpha) \\ &= \int_{\mathcal{R}} \widehat{p}_{abs}(\alpha, x, y) f(y) dy + \int_{\mathcal{R}} \widehat{p}_{abs}(\alpha, 0, y) f(y) dy \left( \sum_{n=1}^{\infty} \widehat{h}^x(\alpha) (\widehat{h}^0(\alpha))^{n-1} \right) \end{aligned}$$

which proves (2.19) on  $\{\alpha : \Re(\alpha) > 0\}$  in the form

$$(3.3) \quad R_{\alpha} f(x) = R_{\alpha}^{abs} f(x) + R_{\alpha}^{abs} f(0) \frac{1 - \alpha R_{\alpha}^{abs} \mathbf{1}(x)}{\alpha R_{\alpha}^{abs} \mathbf{1}(0)}.$$

For  $\phi \in (\frac{\pi}{2}, \pi)$  we denote by  $U_0(\phi) = U_0$  the sector of the complex plane containing the positive real axis and bounded by the two half-lines  $(x, \pm \tan(\phi)x)$  for  $x \leq 0$ . We shall use the results on analytic semigroups generated by strongly elliptic operators under Dirichlet boundary condition from [8] and [9]. The domain of Dirichlet Laplacian is not dense in  $C(\overline{\mathcal{R}})$  in the uniform convergence norm, yet there exists a  $\phi \in (\frac{\pi}{2}, \pi)$  for which  $C(\overline{\mathcal{R}})$  belongs to the domain of the resolvent operator for any  $\alpha$  in  $U_0$ . Moreover, there exist  $R_0 > 0$  and  $M^{abs} > 0$  such that the main estimate for analytic semigroups

$$(3.4) \quad \|R_\alpha^{abs} f\|_{C(\overline{\mathcal{R}})} \leq \frac{M^{abs}}{|\alpha|} \|f\|_{C(\overline{\mathcal{R}})} \quad \text{for all } \alpha \in \{\alpha \in C : |\arg(\alpha)| \leq \phi, |\alpha| \geq R_0\}$$

is valid.

We want to extend the estimate (3.4) to the resolvent (3.3) to obtain (2.20). We prove that  $\alpha R_\alpha f(x)$  stays bounded for  $\alpha \in U_0^R$ . The resolvent identity applied to the constant function  $\mathbf{1}$  for  $\alpha, \beta \in \varrho(\mathcal{L})$  reads

$$R_\beta^{abs} \mathbf{1} - R_\alpha^{abs} \mathbf{1} = (\alpha - \beta) R_\beta^{abs} (R_\alpha^{abs} \mathbf{1})$$

and implies

$$\left(I - \left(1 - \frac{\beta}{\alpha}\right) \alpha R_\alpha^{abs}\right) (\beta R_\beta^{abs} \mathbf{1} - \mathbf{1}) = \alpha R_\alpha^{abs} \mathbf{1} - \mathbf{1}.$$

Let  $\beta = |\alpha|$ . Since we have  $\|\alpha R_\alpha^{abs}\| \leq M^{abs}$  in the operator norm from  $C(\overline{\mathcal{R}})$  to  $C(\overline{\mathcal{R}})$ , then for all  $\alpha$  in the sector  $U_0^R$ ,

$$\left\| \left(I - \left(1 - \frac{\beta}{\alpha}\right) \alpha R_\alpha^{abs}\right) \right\| \leq (1 + 2M^{abs}) = M'.$$

Therefore,

(3.5)

$$\|\alpha R_\alpha^{abs} \mathbf{1} - \mathbf{1}\|_{C(\overline{\mathcal{R}})} = \left\| \left(I - \left(1 - \frac{\beta}{\alpha}\right) \alpha R_\alpha^{abs}\right) (\beta R_\beta^{abs} \mathbf{1} - \mathbf{1}) \right\|_{C(\overline{\mathcal{R}})} \leq M' \|\beta R_\beta^{abs} \mathbf{1} - \mathbf{1}\|_{C(\overline{\mathcal{R}})}.$$

Now we note that  $\lim_{\beta \rightarrow \infty} \|\beta R_\beta^{abs} \mathbf{1} - \mathbf{1}\|_{C(\overline{\mathcal{R}})} = 0$ . This gives the uniform lower bound on  $\|\alpha R_\alpha^{abs} \mathbf{1}\|_{C(\overline{\mathcal{R}})}$  away from 0, as  $|\alpha| \rightarrow \infty$  in the sector  $U_0$ , which proves (2.20).

On the real axis, the function  $\widehat{h^0}(\alpha)$  is the Laplace transform of the first hitting time of the boundary, equal to 1 at  $\alpha = 0$  and non-increasing on  $(\lambda_1^{abs}, \infty)$ . The function is analytic

wherever  $R_\alpha^{abs}$  is analytic, henceforth  $1 - \widehat{h^0}(\alpha)$  has no other zeros on a neighborhood of  $(\lambda_1^{abs}, \infty)$ .

Since  $R_\alpha^{abs} f$  is analytic in the union of  $U_0$  with the right half-plane  $\Re(\alpha) > \lambda_1^{abs}$ , the denominator  $1 - \widehat{h^0}(\alpha)$  from (3.3) has only isolated zeros. We conclude that all singularities of the resolvent  $R_\alpha$  contained in the resolvent set of the Dirichlet Laplacian are poles coinciding with the zeros of the denominator.

(ii) and (iii). We can compute the residue at  $\alpha = 0$ . Multiplying (2.19) by  $\alpha$ , we get

$$\alpha R_\alpha f(x) = \alpha R_\alpha^{abs} f(x) + \frac{\alpha R_\alpha^{abs} f(0)}{\alpha R_\alpha^{abs} \mathbf{1}(0)} (1 - \alpha R_\alpha^{abs} \mathbf{1}(x))$$

Since  $R_\alpha^{abs} f$  is analytic in a neighborhood of  $\alpha = 0$ , it is enough to figure out the limit of  $\alpha R_\alpha f(x)$  as  $\alpha \rightarrow 0$  along the positive real axis. By dominated convergence theorem, or directly from the continuity of the resolvent at  $\alpha = 0$ , we see that  $\lim_{\alpha \rightarrow 0+} \alpha R_\alpha^{abs} f(x) = 0$ , and that

$$\lim_{\alpha \rightarrow 0+} \frac{\alpha R_\alpha^{abs} f(0)}{\alpha R_\alpha^{abs} \mathbf{1}(0)} = \frac{\int_{\mathcal{R}} G(0, y) f(y) dy}{\int_{\mathcal{R}} G(0, y) dy} = \int_{\mathcal{R}} \rho(y) f(y) dy$$

where  $\rho(y) = G(0, y) (\int_{\mathcal{R}} G(0, y) dy)^{-1}$ .

It is easy to see that all singularities, with the exception of zero, have negative real part. Let  $\alpha^*$  be the nonzero element of the spectrum with maximum real part. Let  $\epsilon$  be a positive number less than  $|\Re(\alpha^*)|/2$ . There exists a positive constant  $R_0^* > \max(R_0, 2\Re(\alpha^*)/\cos \phi^*)$  and an angle  $\phi^* \in (\frac{\pi}{2}, \phi)$  such that we can construct a piecewise smooth, continuous non-intersecting contour  $L_\epsilon$  with the following properties:

- (1)  $L_\epsilon$  coincides with the half-lines  $\{(x, \pm \tan(\phi^*)x) : x < 0\}$  for  $|\alpha| > R_0^*$
- (2) the spectrum  $\sigma(\mathcal{L})$  is entirely on the left hand side of  $L_\epsilon$

$$(3) \sup_{\alpha \in L_\epsilon} \Re(\alpha) \leq \Re(\alpha^*) + \epsilon$$

(4) there exists a constant  $C > 0$ , independent of  $\epsilon$ , such that the length of the contour segment  $L_\epsilon \cap B(0, R_0^*)$  is bounded above by  $CR_0$ .

Theorem 4 provides an inversion formula for the resolvent  $R_\alpha$ . For  $\alpha_0 > 0$

$$(3.6) \quad S_t f(x) = \frac{1}{2\pi i} \int_{\alpha_0 - i\infty}^{\alpha_0 + i\infty} e^{\alpha t} R_\alpha f(x) d\alpha.$$

The proof of part (iii) is analogous to that of part (iii) in Theorem 1 in [3]. Let  $R > R_0^*$ . We denote by  $A_\pm = (R \cos(\phi^*), \pm R \sin(\phi^*))$  and  $B_\pm = (\alpha_0, \pm R \sin(\phi^*))$ . Then, from Cauchy formula,

$$(3.7) \quad |S_t f(x) - Res(0; e^{\alpha t} R_\alpha f(x))| \leq \frac{1}{2\pi} \left| \int_{A_- A_+} e^{\alpha t} R_\alpha f(x) d\alpha \right| + \mathcal{E}(R) \|f\|.$$

The error term is the sum of the integrals over the horizontal segments  $A_+ B_+$  and  $A_- B_-$  (which approach zero for large  $R$  as in the proof of Theorem 1 in [3]) and the error corresponding to the principal value in (5.2). Due to properties (1) - (4), the integral over  $A_- A_+$  from (3.7) is bounded above by

$$(3.8) \quad e^{(\epsilon + \Re(\alpha^*))t} \|f\| (M_1(\epsilon, t) + M_2(\epsilon, t))$$

where

$$(3.9) \quad M_1(\epsilon, t) = CR_0 \left( \sup_{\alpha \in B(0, R_0^*) \cap L_\epsilon} \|R_\alpha\| \right)$$

and

$$(3.10) \quad M_2(\epsilon, t) = \frac{M^{abs}}{\pi R_0^*} \int_{R_0^*}^{\infty} e^{[r \cos \phi^* - (\epsilon + \Re(\alpha^*))]t} dr \leq \frac{2M^{abs}}{\pi R_0^* |\cos \phi^*| t} e^{-\frac{R_0^* |\cos \phi^*| t}{2}}$$

For  $t > t_0$ , we can further bound  $M_2(\epsilon, t)$  by  $M_2(t_0)$ , independent from  $\epsilon$ . We notice that  $M_1(\epsilon, t) \leq M_1(\epsilon)$ , a bound independent of the time variable. Finally, if  $M(\epsilon, t_0) = M_1(\epsilon) + M_2(t_0)$ , we have shown that

$$(3.11) \quad \left| S_t f(x) - \int_{\mathcal{R}} \rho(y) f(y) dy \right| \leq e^{(\epsilon + \Re(\alpha^*))t} M(\epsilon, t_0) \|f\| + \mathcal{E}(R) \|f\|.$$

This concludes the proof of the theorem. □

*Remark:* The conditions needed to carry out the contour integration in the complex plane in order to calculate the inverse Laplace transform (based on Proposition 4) are the same as the sufficient conditions for an analytic semigroup from Theorem 7.7 in [7], applied to the Dirichlet Laplacian.

#### 4. Proof of Theorem 2.

*Proof.* Since the space  $(X, \mathcal{T})$  is compact,  $\{\widehat{Q}\}$  has an invariant measure, possibly not unique (see [6]), denoted by  $\nu(dx)$  which must solve the equation

$$(4.1) \quad \int_{\mathcal{R}} \Delta f(x) \nu(dx) = 0 \quad \forall f \in \mathcal{D}_0(X).$$

Without loss of generality we shall operate on  $R^d$  and not on  $X$ . All along the proof we shall assume the existence of the Green function of the Dirichlet problem for the Laplacian  $G(x, y)$  which will be smooth up to the boundary  $\partial\mathcal{R}$ . This fact is proven in the Appendix.

*Absolute continuity.* It is clear that we can take  $f \in C_0^2(\mathcal{R} \setminus \{0\}) \subseteq \mathcal{D}_0$  and derive that  $\Delta \rho = 0$  in the punctured domain  $\mathcal{R} \setminus \{0\}$  in the sense of distributions, which implies that  $\nu(dx)$  has a density  $\rho(x)$  on  $\mathcal{R} \setminus \{0\}$ . There exist two nonzero and nonnegative smooth functions

$q_0(t)$  and  $q_1(t)$  on  $[0, \infty)$ , bounded by one, with the properties  $q_0(0) = 1$ ,  $\text{supp}(q_0) \subseteq [0, 1]$  and  $\text{supp}(q_1) \subseteq [2, 3]$ . Naturally

$$(4.2) \quad C_1 = \int_{[0,1]} tq_0(t)dt > 0 \quad \text{and} \quad C_0 = \int_{[2,3]} tq_1(t)dt > 0.$$

We define

$$(4.3) \quad v_\epsilon(x) = C_0(\epsilon)q_0(\epsilon^{-1}r) - C_1(\epsilon)q_1(\epsilon^{-1}r) \quad \text{and} \quad f_\epsilon(x) = \int_{\mathcal{R}} G(x, y)v_\epsilon(y)dy$$

with the notation  $r = \|x\|$  and constants  $C_0(\epsilon) = C_0 + O(\epsilon)$  and  $C_1(\epsilon) = C_1 + O(\epsilon)$  which will be determined in the following. By construction,  $\Delta f_\epsilon(x) = v_\epsilon(x)$  and, since the functions  $q_0$  and  $q_1$  are smooth,  $f_\epsilon(x) \in C^2(\overline{\mathcal{R}})$  and  $f_\epsilon(x) = 0$  on the boundary  $\partial\mathcal{R}$ . The Green function with pole  $x = 0$  can be written as  $G(0, y) = K(0, y) + w(y)$  where  $K(x, y) = H(\|x - y\|)$  is the fundamental solution to the Laplace equation and  $w(y)$  is a harmonic function depending on the domain. Then

$$f_\epsilon(0) = \int_{\mathcal{R}} G(0, y)v_\epsilon(y)dy = \int_{\mathcal{R}} H(\|y\|)v_\epsilon(y)dy + \int_{\mathcal{R}} w(y)v_\epsilon(y)dy$$

and we notice that the last integral approaches zero as  $\epsilon \rightarrow \infty$ . The integral containing the fundamental solution contains the terms from (4.2) which shows that there exist constants  $C'' > C' > 0$  and a choice of  $C_0(\epsilon)$  and  $C_1(\epsilon)$  such that  $C' \leq C_j(\epsilon) \leq C''$  for both  $j = 0, 1$  and all  $\epsilon > 0$  such that  $f_\epsilon(0) = 0$ . By construction, since the functions  $q_0$  and  $q_1$  are smooth,  $f_\epsilon(x) \in C^2(\overline{\mathcal{R}})$  and  $f_\epsilon(x) = 0$  on the boundary  $\partial\mathcal{R}$ . This shows that  $f_\epsilon \in \mathcal{D}_0$  and has the properties

$$(4.4) \quad \liminf_{\epsilon \rightarrow 0} \Delta f_\epsilon(0) > 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \int_{\mathcal{R}} \rho(x)\Delta f_\epsilon(x)dx = 0.$$

At this point we integrate  $f_\epsilon(x)$  against the solution  $\nu(dx)$  and obtain

$$0 = \int_{\mathcal{R}} \Delta f_\epsilon(x) \nu(dx) = \int_{\mathcal{R}} \Delta f_\epsilon(x) \rho(x) dx + \Delta f_\epsilon(0) \nu(\{0\}).$$

If we let  $\epsilon \rightarrow 0$  the weight  $\nu(\{0\})$  is forced to be zero.

*The solution  $\rho$  is a fundamental solution for the Laplacian.* We have already shown that  $\Delta \rho = 0$  in the punctured domain  $\mathcal{R} \setminus \{0\}$ . Let  $\nu(dx) = \rho(x) dx$  denote the solution to (4.1). We apply the second Green's identity to the functions  $\rho$  and  $f \in C_0^2(\mathcal{R})$  with the property that  $f(0) = 0$  on the open domain  $\mathcal{R}_\epsilon = \mathcal{R} \setminus B(0, \epsilon)$  with a sufficiently small  $\epsilon$  such that the ball be a subset of  $\mathcal{R}$ . It is clear that  $f \in \mathcal{D}_0$ . Then

$$(4.5) \quad \int_{\mathcal{R}_\epsilon} \rho(x) \Delta f(x) dx - \int_{\mathcal{R}_\epsilon} \Delta \rho(x) f(x) dx = \int_{\partial \mathcal{R}_\epsilon} \rho(x) \frac{\partial f}{\partial n}(x) dS_x - \int_{\partial \mathcal{R}_\epsilon} \frac{\partial \rho}{\partial n}(x) f(x) dS_x$$

and all integrals make sense because we keep away from the boundary  $\partial \mathcal{R}_\epsilon$ . It follows that, for all  $f \in C_0^2(\mathcal{R})$  with  $f(0) = 0$ ,

$$(4.6) \quad \lim_{\epsilon \rightarrow 0} \left( \int_{\partial B(0, \epsilon)} \rho(x) \frac{\partial f}{\partial n}(x) dS_x - \int_{\partial B(0, \epsilon)} \frac{\partial \rho}{\partial n}(x) f(x) dS_x \right) = 0.$$

The condition (4.6) is local about  $x = 0$  henceforth we can simply write for any  $f \in C^2(\mathcal{R})$ ,

$$(4.7) \quad \lim_{\epsilon \rightarrow 0} \left( \int_{\partial B(0, \epsilon)} \rho(x) \frac{\partial f}{\partial n}(x) dS_x - \int_{\partial B(0, \epsilon)} \frac{\partial \rho}{\partial n}(x) (f(x) - f(0)) dS_x \right) = 0.$$

Let  $f \in C_0^2(\mathcal{R})$ . We can apply (4.5) once again and obtain

$$\begin{aligned} \int_{\mathcal{R}_\epsilon} \rho(x) \Delta f(x) dx &= \int_{\partial B(0, \epsilon)} \rho(x) \frac{\partial f}{\partial n}(x) dS_x - \\ &- \int_{\partial B(0, \epsilon)} \frac{\partial \rho}{\partial n}(x) (f(x) - f(0)) dS_x - f(0) \int_{\partial B(0, \epsilon)} \frac{\partial \rho}{\partial n}(x) dS_x \end{aligned}$$

The last integral  $-\int_{\partial B(0, \epsilon)} \frac{\partial \rho}{\partial n}(x) dS_x$  is constant because  $\rho$  is harmonic in the punctured domain and we can always equate the surface integrals over two concentric shells included

in  $\mathcal{R} \setminus \{0\}$ . Let  $C$  be its value. The function  $\rho(x)$  is a probability density, hence integrable.

By dominated convergence

$$\int_{\mathcal{R}} \rho(x) \Delta f(x) dx = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{R}_\epsilon} \rho(x) \Delta f(x) dx = Cf(0),$$

implying that, if  $K(\xi, x)$  is the  $d$ -dimensional fundamental solution for the Laplacian, then  $\rho - CK(0, x)$  is harmonic in  $\mathcal{R}$ . The same fact is true then for the Green function with pole  $\xi = 0$ .

*The solution  $\rho$  is continuous and has zero boundary conditions.* Let  $\phi \in C^2(\overline{\mathcal{R}})$  and  $h_\phi(x)$  be the solution to

$$(4.8) \quad \begin{cases} \Delta h = 0 \\ h|_{\partial\mathcal{R}} = \phi|_{\partial\mathcal{R}} \end{cases}$$

and  $g \in C_0^2(\overline{\mathcal{R}})$  such that  $g(0) = \phi(0) - h_\phi(0)$ . Then, the function  $f(x) = \phi(x) - h_\phi(x) - g(x) \in \mathcal{D}_0$ . We look at

$$\int_{\mathcal{R}} \rho(x) \Delta f(x) dx = \int_{\mathcal{R}} \rho(x) \Delta \phi(x) dx - \int_{\mathcal{R}} \rho(x) \Delta h_\phi(x) dx - \int_{\mathcal{R}} \rho(x) \Delta g(x) dx.$$

The left-hand side of the equality is zero by definition; the function  $h_\phi$  is harmonic. Let  $x_0 \in \mathcal{R}$ , which will typically be a point close to the boundary. If we denote by  $\delta_\epsilon(x)$  a smooth approximation of  $\delta(x - x_0)$  with support in the ball  $B(x_0, \epsilon)$  and we take  $\phi(x) = \phi_\epsilon(x)$  such that  $\Delta \phi_\epsilon(x) = \delta_\epsilon$  and  $g(x) = g_\epsilon(x)$ , recalling that  $\rho$  is smooth in  $\mathcal{R} \setminus \{0\}$ , we have

$$\rho(x_0) = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{R}} \rho(x) \Delta \phi(x) dx = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{R}} \rho(x) \Delta g(x) dx = C \lim_{\epsilon \rightarrow 0} (\phi(0) - h_\phi(0)).$$

Define the function

$$(4.9) \quad \phi(x) = \int_{\mathcal{R}} G(x, y) \delta_{\epsilon}(y) dy \quad \Delta \phi(x) = \delta_{\epsilon}(x)$$

for which  $h_{\phi}(x) = 0$ . For a fixed interior point  $x_0$  all functions under the integral are smooth, henceforth bounded. We can pass to the limit as  $\epsilon \rightarrow 0$  and we get  $G(0, x_0)$ , which approaches zero as  $d(x_0, \partial\mathcal{R}) \rightarrow 0$ .

*Conclusion.* We derived that  $v = \rho - CG(0, x)$  is harmonic in  $\mathcal{R}$  and both  $\rho(x)$  and  $G(0, x)$  vanish on the boundary and are continuous on  $\overline{\mathcal{R}}$ . The Dirichlet problem for the Laplacian has a unique solution. We have shown that  $\rho(x) = CG(0, x)$ .  $\square$

*Remark 1:* The Green function does not change sign in the open region  $\mathcal{R}$ . It has a limit at  $x = 0$  which is infinite in  $d > 1$ , since  $x = 0$  is a pole. By carving out a small ball  $B(0, \epsilon) \subset \mathcal{R}$  and applying the maximum principle we make sure that  $CG(0, x)$  is an appropriate density function.

*Remark 2:* In  $d = 1$  (see [3]) this equation has a (unique) solution, linear on each connected component of  $\mathcal{R} \setminus \{0\}$ , continuous at  $x = 0$  and equal to zero on the boundary, uniquely determined by the condition that it integrates to one.

## 5. Appendix.

**Proposition 3.** *Assume the bounded open region  $\mathcal{R}$  has a piecewise smooth boundary and satisfies the exterior cone condition. Then, the Green function for the Dirichlet problem for the Laplacian on  $\mathcal{R}$  exists and is smooth up to the boundary.*

*Proof.* The Dirichlet problem for  $f \in C(\mathcal{R})$

$$(5.1) \quad \begin{cases} \Delta u = 0 \\ u|_{\partial\mathcal{R}} = f \end{cases}$$

can always be solved uniquely if the domain  $\mathcal{R}$  has the *exterior cone condition* (see [2]). We want to know when we can extend smoothly the solution up to the boundary in order to be able to apply Green's formula (4.5). If the bounded open region  $\mathcal{R}$  has a smooth boundary, say  $\partial\mathcal{R} \in C^2$ , it is known (for example [1] and an application of theorem 6, page 326) that, if  $f \in C^2$  the solution to the Dirichlet problem (5.1) will be smooth up to the boundary, that is  $u \in C^2(\overline{\mathcal{R}})$ . In the case of a piecewise smooth boundary, let's choose  $x_0 \in \partial\mathcal{R}$  such that there exists  $r > 0$  and a ball  $B(x_0, r)$  with the property  $\partial\mathcal{R} \cap B(x_0, r)$  is smooth. There exists an open region  $\mathcal{R}' \subset \mathcal{R}$  with smooth boundary  $\partial\mathcal{R}'$  which coincides with  $\partial\mathcal{R}$  on a neighborhood of  $x_0$ , say on  $B(x_0, r/2)$ . The solution  $u'$  of (5.1) for  $\mathcal{R}'$  with  $f \rightarrow f_1 = u|_{\partial\mathcal{R}'}$  is smooth up to the boundary and has to coincide with  $u$  on  $\mathcal{R}'$ . We conclude that  $u$  is smooth up to  $\partial\mathcal{R} \cap B(x_0, r/2)$ . On the other hand, any partial derivative  $\partial_{x_i} u$  is harmonic in  $\mathcal{R}$  and we can pick its boundary values exactly  $\partial_{x_i} f$  at all points from the interior of the smooth pieces of the boundary. However, once again (5.1) has unique continuous solutions up to the boundary henceforth  $\partial_{x_i} u$  will be continuous up to the boundary *everywhere*, given that the set of discontinuities of  $\partial\mathcal{R}$  has codimension 2 in  $R^2$  and the Dirichlet problem has a unique weak solution. Determining the Green function is equivalent to solving (5.1) for particular boundary conditions  $f(x) = -K(\xi, x)$ , for a fixed  $\xi \in \mathcal{R}$ , where  $K(\cdot, \cdot)$  is the fundamental solution, which is obviously of class  $C^\infty(\partial\mathcal{R})$ . □

**Proposition 4.** *Let  $F(t)$  be a continuous function defined for  $t > 0$  such that there exists an  $x_0 \in \mathbb{R}$  with the property that*

$$\int_0^{\infty} e^{-x_0 t} |F(t)| dt < \infty.$$

*Then, the Laplace transform  $\hat{F}(\alpha)$  is analytic in the half-plane  $\operatorname{Re}(\alpha) > x_0$  and the following inversion formula is valid*

$$(5.2) \quad F(t) = P.V \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\alpha t} \hat{F}(\alpha) d\alpha$$

*where  $x \geq x_0$  is arbitrary.*

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