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Brownian Motion on the Figure Eight

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In an interval containing the origin we study a Brownian motion which returns to zero as soon as it reaches the boundary. We determine explicitly its transition probability, prove it is ergodic and calculate the decay rate to equilibrium. It is shown that the process solves the martingale problem for certain asymmetric boundary conditions and can be regarded as a diffusion on an eight shaped domain. In the case the origin is situated at a rationally commensurable distance from the two endpoints of the interval we give the complete characterization of the possibility of collapse of distinct paths.

KEY WORDS: Absorbing Brownian motion; Laplace transform; decay rate; ergodicity.

1. INTRODUCTION

Let $W_x = (w_x(t, \omega), \{\mathscr{F}_t\}_{t \geqslant 0})$ be a Brownian motion on R such that $P(w_x(0, \omega) = x) = 1$ and two real numbers such that $a < 0 < b$. We wish to study the behavior of a family of particles, indexed by their starting points x in the interval (a, b). Individually, they evolve in the following manner. For a random duration lasting until the boundary is reached for the first time, the particle coincides with the Brownian motion $w_x(t, \omega)$. At the moment when the boundary is reached, the particle moves *instantaneously* to zero, situated inside the interval, and starts over again its Brownian motion. Even though the particles, with different initial location, keep a constant distance between themselves for *random* durations between

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boundary hits by *either* of them, the pattern of these distances changes in a complex fashion and opens the question whether there is mixing, in any sense, of the moving configuration. Any initial profile of particles (assume we follow a family of particles such that their empirical distribution at time $t=0$ equals a probalility measure $\mu(dx)$) generates a measure-valued random process $\mu(t, dx, \omega)$. The law of this random configuration can be known from its integration against smooth test functions. Assume we would be able to describe the time evolution of each individual particle, starting at any $x \in (a, b)$ and denote this family of processes by $\{z_x(t, \omega)\}_{x \in (a,b)}$. Then, the mean value of the moving profile at any time $t \geqslant 0$ is exactly the probability distribution of the time marginal of the process $\{\int_a^b z_x(\cdot, \omega)\, \mu(dx)\}$. In this paper, we focus on the description of the processes $\{z_x(t, \omega)\}_{x \in (a,b)}$, with emphasis on the investigation of their ergodic properties.

The particles will move in Markovian fashion, since the first exit times from (a, b) are stopping times. Naturally the paths will be discontinuous, because there are jumps to zero and the stopping times are almost surely finite. Based on this view of the process we can derive explicitly precise estimates concerning the convergence of any configuration to the *unique* probability measure $\rho(y)\, dy$ (defined in (2.9)). This is the object of Theorem 1. We would like to understand better the analytical properties of the model. Theorem 2 identifies a martingale involving the number of hits to the boundary, in fact, a Doob–Meyer decomposition of the Itô semimartingale (2.14). The derivation is explicit and constructive. This lays out the boundary conditions (2.15) which finally enable us to see the process as a diffusion. If we would regard the motion on the compact state space $[a, b]$, the domain (2.15) would not be dense in $C[a, b]$ and we wouldn't be able to construct the infinitesimal generator of the process. The condition that continuous functions on the state space not differentiate between the points a, b and zero says only that the topology should paste together the three points. The state space emerges naturally as the eight shaped domain X known in topology as the *figure eight* (according to Ref. 5). Theorem 4 proves that our process is a diffusion on X. Naturally, there is more than one kind of boundary conditions on this domain. It is interesting that (2.15) seems to require a symmetric condition on a, b and zero. However, a particle moving *inside* the interval (a, b) will go back and forth about the origin while once at a or b there is no immediate return. The asymmetry is contained in that, even though the functions in the domain are smooth up to any of the three points, it is only at zero that the first and second derivatives are matching, hence only one passage way through zero (from $0-$ to $0+$ and back) is open, whereas the other passages are one sided like in an oriented graph or an electrical circuit. Finally, Theorem 4 sheds light on the issue whether paths starting at distinct points can (and eventually would) meet.

2. THE RESULTS

We shall define inductively the increasing sequence of stopping times $\{\tau_n\}_{n\geqslant 0}$, together with the pair of adapted nondecreasing processes $\{N_x^a(t,\omega)\}_{t\geqslant 0}$ and $\{N_x^b(t,\omega)\}_{t\geqslant 0}$ and the process $\{z_x(t,\omega)\}_{t\geqslant 0}$, starting at $x\in(a,b)$. Let $\tau_0=T_x=\inf\{t\colon w_x(t,\omega)\notin(a,b)\}$, while for $t\leqslant\tau_0$ we set $N_x^a(t,\omega)=1_{\{a\}}(w_x(t,\omega))$, $N_x^b(t,\omega)=1_{\{b\}}(w_x(t,\omega))$ and $z_x(t,\omega)=w_x(t,\omega)-aN_x^a(\tau_0,\omega)-bN_x^b(\tau_0,\omega)$. By induction for $n\in Z_+$

$$\tau_{n+1}=\inf\{t>\tau_n : w_x(t,\omega)-aN_x^a(\tau_n,\omega)-bN_x^b(\tau_n,\omega)\notin(a,b)\} \tag{2.1}$$

which enables us to define

$$\begin{aligned} N_x^a(t,\omega)&=N_x^a(\tau_n,\omega)+1_{\{a\}}(z_x(t,\omega)),\\ N_x^b(t,\omega)&=N_x^b(\tau_n,\omega)+1_{\{b\}}(z_x(t,\omega)), \end{aligned} \tag{2.2}$$

as well as

$$z_x(t,\omega)=w_x(t,\omega)-aN_x^a(t,\omega)-bN_x^b(t,\omega) \tag{2.3}$$

for $\tau_n<t\leqslant\tau_{n+1}$. We notice that $z_x(t,\omega)=0$ for all $t=\tau_n$. The construction is well defined due to the following result.

Proposition 1. The sequence of stopping times $\tau_0<\tau_1<\cdots<\tau_n<\cdots$ are finite for all n and $\lim_{n\to\infty}\tau_n=\infty$, both almost surely. Also, the processes $N_x^a(t,\omega)$ and $N_x^b(t,\omega)$ defined for $t\geqslant 0$ have the properties

(i) they are nondecreasing, piecewise constant, predictable and right-continuous

(ii) $P(N_x^a(t,\omega)<\infty)=P(N_x^b(t,\omega)<\infty)=1$.

Proof. The formula (6.7) from Lemma 4 gives the Laplace transform of the density of the first exit time from (a,b). The time intervals between τ_n and τ_{n+1} (we include $\tau_{-1}=0$), for any $n\geqslant -1$ are either T_x for the first exit time or independently identically distributed as T_0 for all the rest. Since $P(T_x=0)=0$ for any $x\in(a,b)$ the sequence is strictly increasing. Moreover, $E[\tau_n]<\infty$ from the Laplace transform, which implies $P(\tau_n<\infty)=1$. In the same time, if $N>0$ is fixed, $P(\lim_{n\to\infty}\tau_n\leqslant N)\leqslant P(T_0^1+T_0^2+\cdots<N)$ for a sequence of i.i.d. $T_0^i\sim T_0$. If the sum $T_0^1+T_0^2+\cdots$ is finite we must have elements in the summation arbitrarily small, for instance $T_0^k<\epsilon$, for an infinite sequence of increasing ranks k. We can find a value ϵ such that $P(T_0 < \epsilon) < 1$. From the independence condition, we derive that $P(\lim_{n\to\infty} \tau_n \leqslant N) = 0$. But $\{\lim_{n\to\infty} \tau_n < \infty\}$ is the union of these events when $N \to \infty$.

The processes $N^a_x(t, \omega)$ and $N^b_x(t, \omega)$ are clearly nondecreasing and piecewise constant. They are right-continuous by construction (2.2) preserving the same value until the next boundary hit. Predictability is a consequence of the fact that the first exit times T_x are stopping times. □

The law of the process $\{z_x(t, \omega)\}_{t\geqslant 0}$, adapted to $\{\mathscr{F}_t\}_{t\geqslant 0}$ will be denoted by Q_x and the family of processes $\{Q_x\}_{x\in(a,b)}$ will be denoted simply by $\{Q\}$. The construction described by Eqs. (2.1) through (2.3) can be made deterministically for each path $w(\cdot) \in C([0, \infty), R)$, resulting in a predictable mapping

$$\Phi(w(\cdot)) = w(t) - bN^b_x(t) - aN^a_x(t). \tag{2.4}$$

With this notation Φ: $C([0, \infty), R) \to D([0, \infty), (a, b))$ and $Q_x = W_x \circ \Phi^{-1}$ is the law of the process $\{z_x(t, \omega)\}_{t\geqslant 0}$ with values in the interval (a, b), a measure on the Skorohod space $D([0, \infty), (a, b))$.

The Brownian motion with absorbing boundary conditions at a and b, denoted by $(w^{abs}_x(t, \omega))_{t\geqslant 0}$, has infinitesimal generator (see Ref. 4)

$$\left(\frac{1}{2}\frac{d^2}{dx^2}, \mathscr{D}_{abs}\right) \qquad \mathscr{D}_{abs} = \{f \in C[a, b] : f'' \in C[a, b], f''(a) = f''(b) = 0\}. \tag{2.5}$$

For $t > 0$, $x \in [a, b]$, we denote by $P_{abs}(t, x, dy) = P(w_x(t, \omega) \in dy, T_x > t)$. This transition probability function is defective (it doesn't sum up to one) since it does not account for the endpoints a and b. However, this is sufficient for our purposes. If $x \in (a, b)$ the transition probability has density $p_{abs}(t, x, y)$ with respect to the Lebesgue measure

$$p_{abs}(t, x, y) = \frac{2}{b-a} \sum_{k=1}^{\infty} \exp\left(-\frac{\lambda_k^2}{2} t\right) \sin \lambda_k(x-a) \sin \lambda_k(y-a) \tag{2.6}$$

where $\lambda_k = k\pi/(b-a)$ and $\{-\lambda_k^2 : 1 \leqslant k \leqslant \infty\}$ are the eigenvalues of (2.5). Let $T_x = \inf\{t: w_x(t, \omega) \notin (a, b)\}$ which is the same as τ_0. Then

$$\begin{aligned} P(T^x > t) &= P(w^{abs}_x(t, \omega) \in (a, b)) = \int_a^b p_{abs}(t, x, y)\, dy \\ &= \frac{4}{\pi} \sum_{k=1,\,\text{odd}}^{\infty} \frac{1}{k} \exp\left(-\frac{\lambda_k^2}{2} t\right) \sin \lambda_k(x-a). \end{aligned} \tag{2.7}$$

The probability density function of T^x is

$$h^x(t) = \frac{2\pi}{(b-a)^2} \sum_{k=1,\,\mathrm{odd}}^{\infty} k \exp\left(-\frac{\lambda_k^2}{2} t\right) \sin \lambda_k (x-a). \tag{2.8}$$

Let $\rho(y)$ be the probability density function on $[a, b]$ defined as

$$\rho(y) = \begin{cases} -\dfrac{2}{(b-a)\,b}(y-b) & \text{if } \; y \in [0, b] \\[2ex] -\dfrac{2}{(b-a)\,a}(y-a) & \text{if } \; y \in [a, 0] \end{cases} \tag{2.9}$$

The function is continuous on $[a, b]$ and $v(dy) = \rho(y)\,dy$ is a probability measure on (a, b). Since $\rho(a) = \rho(b) = 0$ the measure can be regarded as a probability measure on a compact space which would not differentiate between a and b. This fact will be studied in Theorem 3. It is easy to verify that the Fourier sine series of $\rho(y)$ is

$$\frac{4(b-a)}{\pi^2 ab} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin\left(\frac{k\pi a}{b-a}\right) \sin\left(\frac{k\pi(y-a)}{b-a}\right) \tag{2.10}$$

which implies, from the continuity of $\rho(y)$, that $\rho(y)$ is *equal* to the series (2.10).

We are ready to state the results.

Theorem 1. Let $P(t, x, dy)$ be the transition probability for the process $\{Q_x\}_{x \in (a, b)}$. For any $t > 0$ the measure $P(t, x, dy)$ is absolutely continuous with respect to the Lebesgue measure on (a, b) and, if $N_x(t) = N_x^a(t) + N_x^a(t)$ is the total number of visits to the boundary up to time $t > 0$, its probability density function $p(t, x, y)$ is given by

$$p(t, x, y) = p_{abs}(t, x, y) + \int_0^t p_{abs}(t-s, 0, y)\, dE[N_x(s)] \tag{2.11}$$

and satisfies the properties:

(i) $p(t, x, y)$ has a time-variable Laplace transform equal to

$$\hat{p}(\alpha, x, y) = \widehat{p_{abs}}(\alpha, x, y) + \widehat{p_{abs}}(\alpha, 0, y)\, \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} \tag{2.12}$$

which is a meromorphic function with simple poles at

$$\{0\} \cup \left\{ \alpha_0 k^2, 4\alpha_0 k^2 \left(1 + \frac{|a|}{b}\right)^2, 4\alpha_0 k^2 \left(1 + \frac{b}{|a|}\right)^2 : k \in Z_+ \right\},$$

where $\alpha_0 = -\pi^2(\sqrt{2}(b-a))^{-2}$,

(ii) the residue at $\alpha = 0$ is $\rho(y)$

$$\lim_{t \to \infty} \sup_{x, y \in (a, b)} |p(t, x, y) - \rho(y)| = 0 \tag{2.13}$$

and, moreover, the decay rate to the invariant measure $\nu(dy) = \rho(y)\, dy$ is given by

(iii)

$$\limsup_{t \to \infty} \frac{1}{t} \log(\sup_{x, y \in (a, b)} |p(t, x, y) - \rho(y)|) = \alpha_0.$$

Corollary 1. The process $\{Q\}$ is ergodic.

The next result characterizes Q_x as the solution to a martingale problem.

Theorem 2. If $f \in \{f \in C[a, b] : f'' \in C[a, b]\}$, then

$$f(z_x(t, \omega)) - f(x) - \int_0^t \tfrac{1}{2} f''(z_x(s, \omega))\, ds - (f(0) - f(b))\, N_x^b(t, \omega)$$
$$- (f(0) - f(a))\, N_x^a(t, \omega) \tag{2.14}$$

is a $\mathscr{F}_t$-martingale with respect to Q_x.

Let

$$\mathscr{D}_0 = \{f \in C[a, b] : f'' \in C[a, b], f(a) = f(0) = f(b)\}. \tag{2.15}$$

Corollary 2. If $f \in \mathscr{D}_0$ then

$$f(z_x(t, \omega)) - f(x) - \int_0^t \tfrac{1}{2} f''(z_x(s, \omega))\, ds \tag{2.16}$$

is a $\mathscr{F}_t$-martingale with respect to Q_x.

The next theorem allows us to regard $\{z_x(t, \omega)\}_{t \geqslant 0}$ as a process with continuous paths on the compact state space "figure eight."

Let $X = (0, b) \cup (a, 0) \cup \{0\}$ with the topology $\mathscr{T}$ generated by the neighborhood basis

$$V_{x,\epsilon} = \{(x-\epsilon, x+\epsilon) : \forall \epsilon > 0 \text{ such that } (x-\epsilon, x+\epsilon) \subset (0, b) \cup (a, 0)\}$$
$$\text{if} \quad x \neq 0$$
$$V_{0,\epsilon} = \{(-\epsilon, 0) \cup (0, +\epsilon) \cup (b-\epsilon, b) \cup (a, a+\epsilon) \cup \{0\} : \forall \epsilon < \min(|a|, b)\}$$
$$\text{if} \quad x = 0.$$

The space $(X, \mathscr{T})$ is a compact subspace of R^2 with the usual topology.

We define the class of functions

$$\mathscr{D}(X) = \{f \in C(X \setminus \{0\}) : \lim_{x \to r} f^{(j)}(x) \text{ exists and is finite,}$$
$$0 \leqslant j \leqslant 2, r = 0^+, 0^-, a, b\} \tag{2.17}$$

where the lateral limit $\lim_{x \to r} g(x)$ in the $\mathscr{T}$ topology is defined as $\lim_{x \to r} g(x)$ in the topology inherited from R of the set $(a, 0) \cap X$ for a, 0^- and $(0, b) \cap X$ for b and 0^+.

Under the inclusion mapping $\mathscr{I}: \mathscr{D}(X) \to \mathscr{D}_0$ defined as $\mathscr{D}(X) \ni f \to \mathscr{I}(f) \in \mathscr{D}_0$, where $\mathscr{I}(f)(x) = f(i(x))$ and $i(x) = x$ is the identification mapping from (a, b) to X, the domain (2.15) is equal to

$$\{f \in \mathscr{D}(X) : \lim_{x \to 0+} f^{(j)}(x) = \lim_{x \to 0-} f^{(j)}(x),$$
$$0 \leqslant j \leqslant 2 \ \lim_{x \to 0\pm} f(x) = \lim_{x \to a} f(x) = \lim_{x \to b} f(x)\} \tag{2.18}$$

and will be denoted by $\mathscr{D}_0(X)$.

Theorem 3. Let $\widehat{Q_x} = Q_x \circ i^{-1}$ be the measure induced on $C([0, \infty), X)$ by $i: (a, b) \to X$. Then, $\widehat{Q_x}$ is a diffusion process on X with infinitesimal generator

$$\mathscr{L} = \left(\frac{1}{2} \frac{d^2}{dx^2}, \mathscr{D}_0(X)\right). \tag{2.19}$$

Corollary 3. The martingale problem (2.16) is well posed.

Let's denote by $Z_{a,b} = \{ma + nb : m, n \in Z\}$ the additive subgroup of R generated by the pair (a, b). We shall say that two paths collapse if there exists a $T_c < \infty$ such that the paths will coincide for $t \geqslant T_c$.

Theorem 4. Let $\{z_x(t, \omega)\}_{t \geqslant 0}$ and $\{z_y(t, \omega)\}_{t \geqslant 0}$ be two elements of the family of processes defined by $\{Q\}$ starting from x and y, two points in (a, b).

(i) In case $x - y \notin Z_{a,b}$ the paths will never collapse.

(ii) In case a/b is rational there exist two integers a^* and b^*, mutually prime, and a real number $l > 0$ such that $a/a^* = b/b^* = l$. In this case $Z_{a,b} = \{lk : a^* \leqslant k \leqslant b^*\}$ is finite and if $x - y \in Z_{a,b}$ the paths will collapse almost surely.

Corollary 4. If a/b is rational, a finite family of paths starting at points included in $Z_{a,b}$ will collapse almost surely.

3. PROOF OF THEOREM 1

Proof. The nth boundary hitting time τ_n^x can be written as $\tau_n^x = T^x + T_1^0 + T_2^0 + \cdots + T_{n-1}^0$ with independent summands where T^x is the first hitting time of the boundary for the standard Brownian motion starting at x and T_i^0, $1 \leqslant i \leqslant n-1$ are independent identically distributed hitting times when $x = 0$. The probability density functions $h_n^x(t)$ of τ_n^x can be written as $h_n^x(t) = (h^x * (h^0)^{*, n-1})(t)$.

For $B \in \mathscr{B}([a, b])$

$$
\begin{aligned}
P(t, x, B)& \\
&= P_{abs}(t, x, B) + \sum_{n=1}^{\infty} P(\tau_n^x \leqslant t < \tau_{n+1}^x, w_0^{abs}(t - \tau_n^x) \in B) \\
&= P_{abs}(t, x, B) + \sum_{n=1}^{\infty} \int_0^t P(0 \leqslant u < T_n^0, w_0^{abs}(u) \in B) \frac{d}{du} P(t - \tau_n^x \leqslant u)\, du
\end{aligned}
\tag{3.1}
$$

which leads to

$$
p(t, x, y) = p_{abs}(t, x, y) + \sum_{n=1}^{\infty} \int_0^t p_{abs}(u, 0, y)\, h_n^x(t - u)\, du. \tag{3.2}
$$

The total number of visits to the boundary $N_x(s) = N_x^a(s) + N_x^b(s)$ up to time $s > 0$ has the property

$$P(N_x(s) \geqslant n) = P(\tau_n^x \leqslant s) = \int_0^s (h^x * (h^0)^{*, n-1})(r)\, dr$$

and

$$E[N_x(s)] = \sum_{n=1}^{\infty} P(N_x(s) \geqslant n) = \int_0^s \sum_{n=1}^{\infty} (h^x * (h^0)^{*, n-1})(r)\, dr$$

which implies that

$$\frac{d}{ds} E[N_x(s)] = \sum_{n=1}^{\infty} (h^x * (h^0)^{*, n-1})(s).$$

We were able to pass to the limit in the sum $P(\tau_n^x \leqslant s) = \int_0^s (h^x * (h^0)^{*, n-1})(r)\, dr$ due to the monotone convergence theorem. The increasing function $s \to E[N_x(s)]$ is continuous as a time integral. We can calculate the transition probability (3.2) as (2.11).

The Laplace transform of a function $g(t)$ is equal to $\hat{g}(\alpha) = \int_0^{\infty} e^{-\alpha t} g(t)\, dt$ whenever the integral converges. In the case of the transition probability function (3.2) the transform is

$$\begin{aligned}\hat{p}(\alpha, x, y) &= \widehat{p_{abs}}(\alpha, x, y) + \widehat{p_{abs}}(\alpha, 0, y) \left(\widehat{\sum_{n=1}^{\infty} (h^x * (h^0)^{*, n-1})} \right)(\alpha) \\ &= \widehat{p_{abs}}(\alpha, x, y) + \widehat{p_{abs}}(\alpha, 0, y) \left(\sum_{n=1}^{\infty} \widehat{h^x}(\alpha) (\widehat{h^0}(\alpha))^{n-1} \right).\end{aligned}$$

The value of $\widehat{h^0}(\alpha)$ belongs to $(0, 1)$ for $\alpha > 0$, as shown by (3.5), which implies that $\hat{p}(\alpha, x, y)$ is equal to (2.12). Proposition 2 and Lemma 1 show that all functions in (2.12) are analytic in the complex plane with the exception of simple poles on the negative real axis and possibly at $\alpha = 0$. In the fraction $\widehat{h^x}(\alpha)/(1 - \widehat{h^0}(\alpha))$ the denominator $\cosh \sqrt{2\alpha}(\frac{b-a}{2})$ from Eq. (3.5) simplifies. The remaining poles of the function are where $\cosh \sqrt{2\alpha}(\frac{b-a}{2}) - \cosh \sqrt{2\alpha}(-\frac{b+a}{2}) = 0$ which gives us

$$\{0\} \cup \left\{ 4\alpha_0 k^2 \left(1 + \frac{|a|}{b}\right)^2, 4\alpha_0 k^2 \left(1 + \frac{b}{|a|}\right)^2 : k \in Z_+ \right\},$$

where $\alpha_0 = -\pi^2(\sqrt{2}(b-a))^{-2}$. Since $\widehat{p_{abs}}(\alpha, x, y)$ and $\widehat{p_{abs}}(\alpha, 0, y)$ had simple poles $\{\alpha_0 k^2 : k \in Z_+\}$ we obtain (i) from Theorem 1.

Proposition 2 and Lemma 1 allow us to apply the inverse Laplace transform to $\hat{p}(\alpha, x, y)$ on the domain U as defined in Proposition 2 instead of the simple vertical line of Theorem 5. This fact together with the uniform bounds at infinity obtained once again in Proposition 2 make Proposition 4 applicable and conclude the proof of (ii) and (iii) of Theorem 1. $\square$

Proposition 2. Let $\zeta_0 > -\frac{\lambda_1^2}{2}$ and $\phi \in (\frac{\pi}{2}, \pi)$ defining the domain $U = U_{\zeta_0} \subset C$ containing the positive axis above ζ_0, bounded by the half-lines starting from ζ_0 with slopes $\pm \tan \phi$ whose union will be denoted by $L = L(\zeta_0)$.

(i) If $\zeta_0 \geqslant 0$, the time variable Laplace transform of the transition probability function $p(t, x, y)$ is analytic in U and for any $r \in (0, 1/2)$

$$\lim_{|\alpha| \to \infty} \sup_{x, y \in (a, b)} |\alpha^r \hat{p}(\alpha, x, y)| = 0.$$

(ii) If $\zeta_0 \in (-\frac{\lambda_1^2}{2}, 0)$, the time variable Laplace transform of the transition probability function $p(t, x, y)$ has a simple pole at $\alpha = 0$.

Proof. The function $\widehat{p_{abs}}(\alpha, x, y)$ is analytic in U.

$$\widehat{p_{abs}}(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p_{abs}(t, x, y)\, dt$$

$$= \frac{2}{b-a} \sum_{k=1}^\infty \frac{2}{2\alpha + \lambda_k^2} \sin \lambda_k (x-a) \sin \lambda_k (y-a) \tag{3.3}$$

and

$$\widehat{p_{abs}}(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p_{abs}(t, x, y)\, dt$$

$$= \frac{2}{b-a} \sum_{k=1}^\infty \frac{2}{2\alpha + \lambda_k^2} \sin \lambda_k (x-a) \sin \lambda_k (y-a).$$

The series is absolutely convergent and has a uniform upper bound

$$\frac{2}{b-a} \frac{1}{\sin \phi} \sum_{k \geqslant 1} \frac{1}{\frac{\lambda_k^2}{2} + \zeta_0}$$

sufficient to guarantee uniform convergence on any compact subset K of U.

Let $|\alpha-\zeta_0|\geqslant M>0$.

$$\frac{b-a}{2}|\alpha^r\widehat{p_{abs}}(\alpha,x,y)|\leqslant\sum_{k\geqslant 1}\frac{|\alpha|^r}{|\alpha+\frac{\lambda_k^2}{2}|}\leqslant\sum_{k,\,\lambda_k\leqslant\sqrt{2|\alpha|}}\frac{|\alpha|^r}{|\alpha+\frac{\lambda_k^2}{2}|}+\sum_{k,\,\lambda_k>\sqrt{2|\alpha|}}\frac{|\alpha|^r}{\frac{\lambda_k^2}{2}-|\alpha|}.$$

We notice that the estimates are uniform in $x, y\in(a,b)$ immediately after the first inequality. If we write $k(\alpha)=[\frac{b-a}{\pi}\sqrt{2|\alpha|}]$, then the first term is bounded above by

$$\frac{|\alpha|^r}{|\alpha|\sin\phi}\left(\frac{b-a}{\pi}\sqrt{2|\alpha|}\right)\sim O(|\alpha|^{r-\frac{1}{2}})$$

while the second is less than

$$\begin{aligned}\frac{2(b-a)^2}{\pi^2}&|\alpha|^r\sum_{k=k(\alpha)+1}^{\infty}\frac{1}{k^2-k(\alpha)^2}\\ &=\frac{2(b-a)^2}{\pi^2}|\alpha|^r\sum_{l=1}^{\infty}\frac{1}{l(l+2k(\alpha))}\\ &=\frac{2(b-a)^2}{\pi^2}|\alpha|^r\frac{1}{2k(\alpha)}\sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+2k(\alpha)}\right)\\ &\leqslant\frac{(b-a)^2}{\pi^2}\frac{|\alpha|^r}{k(\alpha)}(\ln(2k(\alpha))+c_0)\end{aligned}$$

where c_0 is Euler's constant. The second term is of order $|\alpha|^{r-\frac{1}{2}}\ln(|\alpha|)$. As $M\to\infty$ we obtain $\lim_{|\alpha|\to\infty}|\alpha^r\widehat{p_{abs}}(\alpha,x,y)|=0$.

To complete the proof it is sufficient to show that $\widehat{h^x}(\alpha)/(1-\widehat{h^0}(\alpha))$ has a simple pole at $\alpha=0$ and is uniformly bounded away from zero. Lemma 1 shows that $\widehat{h^x}(\alpha)$ is analytic and bounded on U uniformly in x and ζ_0 if $\zeta_0>-\lambda_1^2/2$. The function $1-\widehat{h^0}(\alpha)$ does not depend on x, is analytic on U and has a simple zero at $\alpha=0$. To see that it is bounded below away from zero, assume we look at $U'=U\setminus\{|\alpha|<\lambda_1^2/4\}$. In this domain, $1-\widehat{h^0}(\alpha)$ is analytic and has no zeros, while it is positive on the real axis, implying the result. □

Lemma 1. The Laplace transform $\widehat{h^x}(\alpha)$ of the probability density function of the first boundary hitting time (2.8) is analytic on the complex plane with the exception of the simple poles $\{-\frac{\lambda_k^2}{2}: k \text{ odd}\}$ and if $\zeta_0>-\lambda_1^2/2$, then it is uniformly bounded with respect to $x\in(a,b)$ and its bound does not depend on ζ_0.

Proof. We fix $x \in (a, b)$ and prove that $\widehat{h^x}(\alpha)$ is analytic in U and there exists a constant $0 < c_1 < 1$, depending on x, such that $\widehat{h^x}(\alpha) < c_1$. The Laplace transform (2.8) is

$$\widehat{h^x}(\alpha) = \frac{2\pi}{(b-a)^2} \sum_{k=1,\,\mathrm{odd}}^{\infty} \frac{k}{\alpha + \frac{\lambda_k^2}{2}} \sin \lambda_k (x-a). \tag{3.4}$$

The series is convergent (due to Abel's convergence criterion for series) but not absolutely convergent. Let γ be a closed contour in U. Since the partial sums are analytic in U the sequence of contour integrals of the partial sums is zero. To prove analyticity, it is sufficient to show that the real and imaginary parts series are uniformly bounded for any $\alpha \in \gamma$. The real part of the series is

$$\frac{2\pi}{(b-a)^2} \sum_{k=1,\,\mathrm{odd}}^{\infty} \frac{k(Re(\alpha) + \frac{\lambda_k^2}{2})}{(Re(\alpha) + \frac{\lambda_k^2}{2})^2 + (Im(\alpha))^2} \sin \lambda_k (x-a)$$

which can be bounded above by a series of the form $P(k) \sin \lambda_k (x-a)$ with the property that $P(k)$ becomes decreasing for $k \geqslant k(\gamma)$, a rank depending only on the contour and not on any particular $\alpha \in \gamma$. An analogous bound is obtained for the imaginary part. Abel's criterion and dominated convergence concludes the proof that the function is analytic.

The Uniform Bound. Due to the independent alternative derivation of (3.4) given in (6.7) and the analyticity of the two functions (which coincide on the positive real axis)

$$\widehat{h^x}(\alpha) = \frac{2\pi}{(b-a)^2} \sum_{k=1,\,\mathrm{odd}}^{\infty} \frac{k}{\alpha + \frac{\lambda_k^2}{2}} \sin \lambda_k (x-a) = \frac{\cosh \sqrt{2\alpha}\,(x - \frac{b+a}{2})}{\cosh \sqrt{2\alpha}\,(\frac{b-a}{2})}. \tag{3.5}$$

The square root is well defined on U. The function $\cosh(z) = \sum_n \frac{1}{(2n)!} z^{2n}$ remains analytic on U when we plug in the value $\sqrt{2\alpha}\, C$, with C any real constant. This shows that the singularities of $\widehat{h^x}(\alpha)$ are poles, and one can verify they are exactly (with multiplicity one) $\{-\frac{\lambda_k^2}{2}\}$, where k is odd ($\widehat{p_{abs}}(\alpha, x, y)$ has the same poles, but for all k). A simple way to verify (3.5) is to calculate the Fourier sine series of the right term as a continuous function of x.

The square of the complex norm of $\cosh(z)$ is the half of $\cosh(2\,|Re(z)|) + \cos(2\,|Im(z)|)$. The ratio

$$\left| \frac{\cosh \sqrt{2\alpha}\,(x - \frac{b+a}{2})}{\cosh \sqrt{2\alpha}\,(\frac{b-a}{2})} \right|^2 \leqslant \frac{\cosh(2\,|Re(z)|) + \cos(2\,|Im(z)|)}{\cosh(2\,|Re(z')|) + \cos(2\,|Im(z')|)}$$

where $z=\sqrt{2\alpha}(x-\frac{b+a}{2})$ and $z'=\sqrt{2\alpha}(\frac{b-a}{2})$. We already know that the denominator is bounded away from zero. On a vertical strip about the origin $|Re(z)|\leqslant 1$ the $\cosh(2\,|Re(z)|)/\cosh(2\,|Re(z')|)$ is bounded independently of x because the whole argument $|Re(z)|=|\sqrt{2\alpha}(x-\frac{b+a}{2})|$ is uniformly bounded. On the other hand, while as $|Re(\sqrt{2\alpha})|\to\infty$ the ratio is of order $O(\exp(2\,|Re(\sqrt{2\alpha})|\,(x-\frac{b+a}{2}-\frac{b-a}{2}))$. The proof is complete because $(x-\frac{b+a}{2}-\frac{b-a}{2})<0$. If we need the function to vanish at infinity we can only obtain a bound depending on x. However, if we are only interested in a uniform bound, this is given by $(x-\frac{b+a}{2}-\frac{b-a}{2})\leqslant 0$, which makes the numerator never larger than two. □

Lemma 2. For the function $F(t)=p(t,x,y)$ the vertical line of integration from the inversion formula (3.10) can be replaced by the contour L defined in Proposition 2 with $\zeta_0>0$.

In this case

$$p(t,x,y)=\frac{1}{2\pi i}\int_L e^{\alpha t}\hat{p}(\alpha,x,y)\,d\alpha \tag{3.6}$$

and, for any $t>0$,

$$\frac{d^n}{dt^n}p(t,x,y)=\frac{1}{2\pi i}\int_L \alpha^n e^{\alpha t}\hat{p}(\alpha,x,y)\,d\alpha. \tag{3.7}$$

Proof. We consider $x_0>\zeta_0>0$. For $R>0$ we denote by L_R the union of the line segments with one endpoint at $C=(\zeta_0,0)$ and the other at $A_\pm=(R\cot\phi+\zeta_0,\pm R)$. The horizontal lines through $A_\pm$ intersect $Re(\alpha)=x_0$ at $B_\pm=(x_0,\pm R)$. We have to show that

$$\lim_{R\to\infty}\left[\frac{1}{2\pi i}\int_{A_-CA_+} e^{\alpha t}\hat{p}(\alpha,x,y)\,d\alpha-\frac{1}{2\pi i}\int_{B_-B_+} e^{\alpha t}\hat{p}(\alpha,x,y)\,d\alpha\right]=0.$$

Since the integrand contains no singularities inside the contour $A_-B_-B_+A_+C$ we only have to show that

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{A_+B_+} e^{\alpha t}\hat{p}(\alpha,x,y)\,d\alpha=0$$

and the analogous limit for the lower segment A_-B_-. If we denote by u the running real parameter from $R\cot\phi+\zeta_0$ to x_0 and we recall that

$\hat{p}(\alpha, x, y)\,|\alpha|^{+r}$ approaches zero as $\alpha \to \infty$, hence is bounded by a constant $M > 0$, the integrand

$$|e^{\alpha t}\hat{p}(\alpha, x, y)| \leqslant e^{ut}\,|\alpha|^{-r}\,(\hat{p}(\alpha, x, y)\,|\alpha|^{+r}) \leqslant Me^{ut}\,\frac{1}{R^r}$$

therefore

$$\left|\int_{A_+B_+} e^{\alpha t}\hat{p}(\alpha, x, y)\,d\alpha\right| \leqslant \frac{M}{R^r}\int_{R\cot\phi+\zeta_0}^{x_0} e^{ut}\,du \leqslant \frac{M}{R^r}\frac{1}{t}\,(e^{x_0 t} - e^{(R\cot\phi+\zeta_0)\,t}).$$

For any $t > 0$ we let $R \to \infty$ and obtain the result.

The Differentiation. We have shown that

$$p(t, x, y) = \frac{1}{2\pi i}\int_L e^{\alpha t}\hat{p}(\alpha, x, y)\,d\alpha.$$

Our goal is to differentiate the left side function with respect to t an arbitrary number of times $n \in Z_+$.

$$\int_L e^{\alpha t}\hat{p}(\alpha, x, y)\,d\alpha = \lim_{R\to\infty}\left\{\int_{-R}^{\zeta_0} e^{(u+i(\tan\phi)(u-\zeta_0))\,t}\hat{p}(u+i(\tan\phi)(u-\zeta_0), x, y)\,du\right.$$
$$\left. + \int_{\zeta_0}^{-R} e^{(u-i(\tan\phi)(u-\zeta_0))\,t}\hat{p}(u-i(\tan\phi)(u-\zeta_0), x, y)\,du\right\}$$

by parametrizing the contour with u running from $-R$ to ζ_0. Let $t \in [t_0, t_1]$ away from zero ($t_1 > 0$). It is sufficient to have

$$\limsup_{R\to\infty}\ \sup_{t_0\leqslant t\leqslant t_1}\int_{-R}^{\zeta_0} |u+i(\tan\phi)(u-\zeta_0)|^{n+2}\,|e^{(u+i(\tan\phi)(u-\zeta_0))\,t}|\,du < \infty. \qquad (3.8)$$

The variable $|u+i(\tan\phi)(u-\zeta_0)|$ is bounded on $u \in [-1, \zeta_0]$ so the integral will stay finite if and only if it is finite on $u \in [-R, -1]$ when $R \to \infty$. $|u+i(\tan\phi)(u-\zeta_0)|^{n+2} = O(|u|^{n+2})$ and the exponential is bounded by e^{ut_0} which brings us to

$$\limsup_{R\to\infty}\int_{-R}^{-1} |u|^{n+2}\,e^{ut_0}\,du \leqslant \frac{\Gamma(n+3)}{t_0^{n+3}}.$$

The similar estimate for the other arm of the contour concludes the proof. □

Proposition 3. The residue of the Laplace transform of the transition probability function $\hat{p}(\alpha, x, y)$ at $\alpha = 0$ is

$$\lim_{|\alpha| \to 0} \alpha \hat{p}(\alpha, x, y) = \lim_{|\alpha| \to 0} \left[\alpha \widehat{p_{abs}}(\alpha, x, y) + \left(\frac{\alpha \widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} \right) \widehat{p_{abs}}(\alpha, 0, y) \right] = \rho(y). \tag{3.9}$$

where $\rho(y)$ is defined in (2.9).

Proof. We recall the Fourier sine series of $\rho(y)$ given in Eq. (2.10).

$$\widehat{p_{abs}}(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p_{abs}(t, x, y)\, dt$$

$$= \frac{2}{b-a} \sum_{k=1}^{\infty} \frac{2}{2\alpha + \lambda_k^2} \sin \lambda_k (x-a) \sin \lambda_k (y-a).$$

The series is uniformly convergent in $\alpha \geqslant 0$ due to the absolute convergence of the harmonic series $\sum k^{-2}$. The uniform convergence theorem for continuous functions applied to the partial sums implies that we can commute the limit in α and n.

$$\lim_{|\alpha| \to 0} \widehat{p_{abs}}(\alpha, x, y) = \frac{2}{b-a} \sum_{k=1}^{\infty} \frac{2}{\lambda_k^2} \sin \lambda_k (x-a) \sin \lambda_k (y-a) \neq \infty.$$

We only have to show that

$$\lim_{|\alpha| \to 0} \frac{\alpha \widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} = -\frac{1}{ab}.$$

From Eq. (3.5) we see that, for any $x \in (a, b)$ $\widehat{h^x}(\alpha)$ is analytic in a neighborhood of $\alpha = 0$ and

$$\lim_{|\alpha| \to 0} \frac{\alpha \widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} = \lim_{|\alpha| \to 0} \frac{\alpha \cosh \sqrt{2\alpha}(x - \frac{b+a}{2})}{\cosh \sqrt{2\alpha}(\frac{b-a}{2}) - \cosh \sqrt{2\alpha}(-\frac{b+a}{2})}$$

$$= \lim_{|\alpha| \to 0} \frac{\alpha}{\cosh \sqrt{2\alpha}(\frac{b-a}{2}) - \cosh \sqrt{2\alpha}(-\frac{b+a}{2})}$$

$$= \lim_{|\alpha| \to 0} \frac{\alpha}{\frac{1}{2}(2\alpha)[(\frac{b-a}{2})^2 - (-\frac{b+a}{2})^2] + \alpha^2 C(\alpha)} = -\frac{1}{ab}$$

where $C(\alpha) \sim O(1)$ in a neighborhood of $\alpha = 0$. □

We recall a result concerning the existence of the inverse Laplace transform referring to Ref. 1 for the proof.

Theorem 5. Let $F(t)$ be a continuous function defined for $t>0$ such that there exists an $x_0 \in R$ with the property that

$$\int_0^\infty e^{-x_0 t}\,|F(t)|\,dt<\infty.$$

Then, the Laplace transform $\hat{F}(\alpha)$ is analytic in the half-plane $Re(\alpha)>x_0$ and the following inversion formula is valid

$$F(t)=P.V\,\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{\alpha t}\hat{F}(\alpha)\,d\alpha \tag{3.10}$$

where $x \geqslant x_0$ is arbitrary.

We recall that for $\zeta \in R$ we define a domain U_ζ with boundary $L(\zeta)$ as in Proposition 2.

Proposition 4. Let $\zeta_1' < \zeta_1 < \zeta_0 < \zeta_2 < \zeta_2'$ and let $f(\alpha)$ be analytic in the domain $V=U_{\zeta_1'}\backslash \bar{U}_{\zeta_2'}$ with the exception of $\alpha=\zeta_0$ which is a pole of order $m \in Z_+$ with the principal part of the Laurent expansion about ζ_0 equal to

$$\frac{c_1}{(\alpha-\zeta_0)}+\cdots+\frac{c_m}{(\alpha-\zeta_0)^m}.$$

Assume that there exist $R_0>0$ and $M>0$ with the property $|f(\alpha)| \leqslant M$ if $|\alpha-\zeta_0| \geqslant R_0$. Then there exists a $T>0$ such that the integral (in principal value sense)

$$F(t)=\frac{1}{2\pi i}\int_{L(\zeta_2)} e^{\alpha t} f(\alpha)\,d\alpha$$

is uniformly convergent for $t \geqslant T$ and for $t \to \infty$ we have the asymptotic expansion

$$F(t)=e^{\zeta_0 t}\left(c_1+\frac{c_2}{1!}\,t+\cdots+\frac{c_m}{(m-1)!}\,t^{m-1}\right)+o(e^{\zeta_1 t}). \tag{3.11}$$

Proof. For $R>0$ we denote by $A_\pm$ the points $\zeta_1+\cot\phi R+\pm iR$ on the boundary of U_{ζ_1} and by $B_\pm$ the points $\zeta_2+\cot\phi R+\pm iR$ on the boundary of U_{ζ_2} as well as $C=\zeta_1$ and $D=\zeta_2$.

$$\frac{1}{2\pi i}\int_{A_-B_-DB_+A_+CA_-} e^{\alpha t} f(\alpha)\,d\alpha = Res(e^{\alpha t} f(\alpha))(\zeta_0)$$

$$= e^{\zeta_0 t}\left(c_1+\frac{c_2}{1!}t+\cdots+\frac{c_m}{(m-1)!}t^{m-1}\right)$$

and the integrals along the horizontal line segments A_-B_- and B_+A_+ go to zero as $R\to\infty$ because the integration is carried out on a segment of finite length $|\zeta_2-\zeta_1|$ and the real part of the exponent tends to $-\infty$. By definition,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{B_-DB_+} e^{\alpha t} f(\alpha)\,d\alpha$$

converges to $F(t)$ for $t\geqslant T$. We first want to show that

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{A_-CA_+} e^{\alpha t} f(\alpha)\,d\alpha\sim O(e^{\zeta_1 t}).$$

We can bound

$$\left|\frac{1}{2\pi i}\int_{A_-CA_+} e^{\alpha t} f(\alpha)\,d\alpha\right|$$

$$\leqslant\frac{M}{2\pi}\int_{\zeta_1+\cot\phi R}^{\zeta_1} |e^{(u-i\tan\phi(u-\zeta_1))\,t}|\,du+\frac{M}{2\pi}\int_{\zeta_1+\cot\phi R}^{\zeta_1} |e^{(u+i\tan\phi(u-\zeta_1))\,t}|\,du$$

$$\leqslant\frac{M}{\pi}\int_{\zeta_1+\cot\phi R}^{\zeta_1} e^{ut}\,du=\frac{M}{\pi T}e^{\zeta_1 t}(1-e^{\cot\phi R})$$

and notice that $\cot\phi<0$ implies the asymptotic expansion as $R\to\infty$. Since $\zeta_1\in(\zeta_1',\zeta_0)$ is arbitrary we can see that the error bound can be improved to $o(e^{\zeta_1 t})$. □

4. PROOF OF THEOREM 2

Proof. We shall prove an equivalent formulation of (2.14), that is, for any $f\in C^2[a,b]$ and any $0\leqslant s<t$,

$$f(z_x(t,\omega)) = f(z_x(s,\omega)) + \int_s^t \tfrac{1}{2} f''(z_x(u,\omega))\,du + \int_s^t f'(z_x(u,\omega))\,dw_0(u,\omega)$$
$$+ (f(0)-f(b))(N_x^b(t,\omega)-N_x^b(s,\omega))$$
$$+ (f(0)-f(a))(N_x^a(t,\omega)-N_x^a(s,\omega)). \tag{4.1}$$

If we denote by $N_x(t,\omega) = N_x^a(t,\omega)+N_x^b(t,\omega)$ the total number of boundary hits up to time $t>0$, the sequence of times when the process hits the boundary will be denoted by $\{\tau_l\}_{l'\leqslant l\leqslant l''}$, where $l' = \max\{l\colon \tau_l \leqslant s\}$ and $l'' = \min\{l\colon \tau_l > t\}-1$.

$$f(z_x(t,\omega)) - f(z_x(s,\omega)) - \int_s^t \tfrac{1}{2} f''(z_x(u,\omega))\,du - [(f(0)-f(b))(N_x^b(t,\omega)$$
$$-N_x^b(s,\omega)) + (f(0)-f(a))(N_x^a(t,\omega)-N_x^a(s,\omega))]$$
$$= f(z_x(t,\omega)) - f(z_x(\tau_{l''},\omega)) - \int_{\tau_{l''}}^t \tfrac{1}{2} f''(z_x(u,\omega))\,du \tag{4.2}$$
$$+ \sum_{l=l'+1}^{l''-1} \left\{ \left[f(z_x(\tau_{l+1},\omega)) - f(z_x(\tau_l,\omega)) - \int_{\tau_l}^{\tau_{l+1}} \tfrac{1}{2} f''(z_x(u,\omega))\,du \right] \right.$$
$$-[(f(0)-f(b))(N_x^b(\tau_{l+1},\omega)-N_x^b(\tau_l,\omega))$$
$$\left. + (f(0)-f(a))(N_x^a(\tau_{l+1},\omega)-N_x^a(\tau_l,\omega))] \right\} \tag{4.3}$$
$$+ \left\{ \left[f(z_x(\tau_{l'+1},\omega)) - f(z_x(s,\omega)) - \int_s^{\tau_{l'+1}} \tfrac{1}{2} f''(z_x(u,\omega))\,du \right] \right.$$
$$-[(f(0)-f(b))(N_x^b(\tau_{l'+1},\omega)-N_x^b(s,\omega))$$
$$\left. + (f(0)-f(a))(N_x^a(\tau_{l'+1},\omega)-N_x^a(s,\omega))] \right\}. \tag{4.4}$$

We want to show that the conditional expectation of all the terms listed above is zero. We start with the middle terms (4.3). From construction, seen in (2.2) and (2.3), the process $z_x(t,\omega)$ coincides with a Brownian motion between successive stopping times τ_l, since $aN_x^a(t,\omega)+bN_x^b(t,\omega)$ is constant on $[\tau_l, \tau_{l+1})$ for any $l \in Z_+$. The paths are broken at τ_{l+1} and the correction needed is exactly the quantity $f(0)-f(b)$ or $f(0)-f(a)$ respectively, which compensates the jump of $-b$ or $-a$ (we recall that $a<0$). More precisely, $z_x(\tau_{l+1},\omega)=0$ while the path $w_x(t,\omega)-aN_x^a(t,\omega)-bN_x^b(t,\omega)$, $t\in[\tau_l,\tau_{l+1})$ would end with the value a or b according to the place on the boundary where the boundary hit occurs. On the interval $[\tau_l, \tau_{l+1})$ we can substitute $z_x(t, \omega)$ with the actual Brownian motion $\hat{w}_x(t, \omega) = w_x(t, \omega) - aN_x^a(t, \omega) - bN_x^b(t, \omega)$ all the way to τ_{l+1} by adding the value $(f(a) - f(0))\, N_x^a(\tau_{l+1}, \omega) + (f(b) - f(0))\, N_x^b(\tau_{l+1}, \omega)$ lost by the jump to zero. Then we can re-write the conditional expectation of (4.3) as

$$E\left[f(\hat{w}_x(\tau_{l+1}, \omega)) - f(\hat{w}_x(\tau_l, \omega)) - \int_{\tau_l}^{\tau_{l+1}} \tfrac{1}{2} f''(\hat{w}_x(u, \omega))\, du \,\middle|\, \mathscr{F}_s \right]$$

$$= E\left[E\left[f(\hat{w}_x(\tau_{l+1}, \omega)) - f(\hat{w}_x(\tau_l, \omega)) - \int_{\tau_l}^{\tau_{l+1}} \tfrac{1}{2} f''(\hat{w}_x(u, \omega))\, du \,\middle|\, \mathscr{F}_{\tau_l} \right] \middle|\, \mathscr{F}_s \right]$$

due to the towering property of the filtration σ-fields.

$$E\left[f(\hat{w}_x(\tau_{l+1}, \omega)) - f(\hat{w}_x(\tau_l, \omega)) - \int_{\tau_l}^{\tau_{l+1}} \tfrac{1}{2} f''(\hat{w}_x(u, \omega))\, du \,\middle|\, \mathscr{F}_{\tau_l} \right]$$

$$= f(\hat{w}_x(\tau_{l+1} - \tau_l, \omega)) - f(\hat{w}_x(0, \omega)) - \int_0^{\tau_{l+1} - \tau_l} \tfrac{1}{2} f''(\hat{w}_x(u, \omega))\, du$$

by the strong Markov property. The optional sampling theorem (f and its derivatives are smooth and bounded and we stop the Itô martingale at the first exit time $T_0 = \tau_{l+1} - \tau_l$) shows that the expected value is zero. The same reasoning applies to the terms (4.2) and (4.4). If we want to go further and show (4.1) we simply write down the martingale term

$$\int_{\tau_l}^{\tau_{l+1}} f'(z_x(u, \omega))\, d\hat{w}(u, \omega)$$

which is not affected by a change of the function f' at one point; this, together with the fact that $aN_x^a(t, \omega) + bN_x^b(t, \omega)$ is constant on $[\tau_l, \tau_{l+1})$ proves that

$$\int_{\tau_l}^{\tau_{l+1}} f'(z_x(u, \omega))\, d\hat{w}(u, \omega) = \int_{\tau_l}^{\tau_{l+1}} f'(z_x(u, \omega))\, dw_0(u, \omega) \quad \text{a.s.} \qquad \square$$

5. PROOF OF THEOREM 3

Proof. We follow Ref. 4 and first prove that $(\frac{1}{2} \frac{d^2}{dx^2}, \mathscr{D}_0)$ is a closed Markov pregenerator.

(a) $1 \in \mathscr{D}_0$ and $\mathscr{L}1 = 0$ are obvious.

(b) $\mathscr{D}_0$ is dense in $C(X)$. This is the first point where the particular topology of X comes into play. The domain $\mathscr{D}_0$ is *not* dense in $C[a, b]$.

(c) We have to prove that if $f \in \mathscr{D}_0$ has a global maximum at $x \in X$, then $f''(x) \leqslant 0$ (the maximum principle). For interior points of X there is nothing to prove, we are essentially on a subspace of the real line. For $0 \in X$, $f(x) \leqslant f(0)$ for any $x \in X$ implies that $f'(0+) \leqslant 0$ and $f'(0-) \geqslant 0$. However, $f \in \mathscr{D}_0$ implies $f'(0+) = f'(0-)$ hence both are zero. Since $f \in \mathscr{D}_0$ implies the existence and continuity of $f''(x)$ we can write the Taylor expansion about $x = 0$ and conclude that $f''(0) \leqslant 0$. The other two limits at a and b are superfluous in this reasoning; we do not need to prove anything about the nonexistent points a and b. This is the second place where the structure of $(X, \mathscr{T})$ is essential.

(d) The operator $\mathscr{L}$ is closed by a standard argument. We can easily see this because a sequence of functions $\{f_n\}$ defined on a compact X, converging at at least one point $x_0 \in X$ with the property that their derivatives converge uniformly (that is, in the supremum norm) to a given function $g(x)$ will necessarily converge uniformly to a function $f(x)$ and $f(x)$ will be differentiable with $f'(x) = g(x)$. This can be applied twice and we obtain the result.

(e) The next step is to show that $\mathscr{L}$ is a Markov generator. We will be done as soon as $\mathrm{R}(\alpha I - \mathscr{L}) = C(X)$ for sufficiently large positive α. This will be true if for $f \in C(X)$ we can show that, for sufficiently large α,

$$[(\alpha I - \mathscr{L})^{-1} f](x) = \int_X \hat{p}(\alpha, x, y)\, f(y)\, dy \in \mathscr{D}_0.$$

We first prove that

$$\int_X \hat{p}(\alpha, x, y)\, f(y)\, dy \in \mathscr{D}_0.$$

and later on we identify the resolvent as stated above. We can translate this requirement into the conditions on the closed interval $[a, b]$. What is needed is that for any $f \in C[a, b]$ with $f(a) = f(b) = f(0)$, the integral

$$g(x) = \int_X \hat{p}(\alpha, x, y)\, f(y)\, dy$$

be a function of class $C^2[a, b]$ with $g(a) = g(b) = g(0)$.

Using (2.12) we see that

$$g(x) = \int_a^b \widehat{p_{abs}}(\alpha, x, y)\, f(y)\, dy + \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} \int_a^b \widehat{p_{abs}}(\alpha, 0, y)\, f(y)\, dy.$$

We shall prove separately that the integral

$$\int_a^b \widehat{p_{abs}}(\alpha, x, y)\, f(y)\, dy$$

and the function $x \to \widehat{h^x}(\alpha)$ are in $C^2(a, b)$.

Recall (5.3) and (3.5). The condition needed to differentiate under the integral are met due to the exponential factor with negative exponent. It is easy to verify that all the functions in the summation (5.3) as well as $x \to \widehat{h^x}(\alpha)$ satisfy the equation $v'' = -2\alpha v$ (or $\alpha v - \mathscr{L} v = 0$), with the possible exception of

$$\int_a^b \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha}\,|y-x|} f(y)\, dy$$

presenting a singularity at $x = y$. A way to look at it is to notice it is the Laplace transform of the heat kernel on the real line, and consider a function $f(x)$ identically equal to zero outside the interval $[a, b]$. To avoid any technical difficulty, we can simply re-write it as

$$\tilde{f}(x) = \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha}\,x} \int_a^x e^{\sqrt{2\alpha}\,y} f(y)\, dy + \frac{1}{\sqrt{2\alpha}}\, e^{\sqrt{2\alpha}\,x} \int_x^b e^{-\sqrt{2\alpha}\,y} f(y)\, dy$$

and differentiate with respect to x to obtain

$$-e^{-\sqrt{2\alpha}\,x} \int_a^x e^{\sqrt{2\alpha}\,y} f(y)\, dy + \frac{1}{\sqrt{2\alpha}} f(x) - \frac{1}{\sqrt{2\alpha}} f(x) + e^{\sqrt{2\alpha}\,x} \int_x^b e^{-\sqrt{2\alpha}\,y} f(y)\, dy$$

$$= -e^{-\sqrt{2\alpha}\,x} \int_a^x e^{\sqrt{2\alpha}\,y} f(y)\, dy + e^{\sqrt{2\alpha}\,x} \int_x^b e^{-\sqrt{2\alpha}\,y} f(y)\, dy.$$

Due to the cancellation of the middle terms the derivative is differentiable without further regularity conditions on $f(x)$. We differentiate once more

$$\tilde{f}''(x) = \sqrt{2\alpha}\, e^{-\sqrt{2\alpha}\,x} \int_a^x e^{\sqrt{2\alpha}\,y} f(y)\, dy - f(x)$$

$$+ \sqrt{2\alpha}\, e^{\sqrt{2\alpha}\,x} \int_x^b e^{-\sqrt{2\alpha}\,y} f(y)\, dy - f(x)$$

showing that $\tilde{f}''(x) = 2\alpha\tilde{f}(x) - 2f(x)$ or $\alpha\tilde{f} - \mathscr{L}\tilde{f} = f(x)$. Summing up the results and verifying directly the limits as $x \to a$ and $x \to b$ to check that $g(a) = g(b) = g(0)$, part (e) of the proof is concluded. □

Lemma 3. Let $q(t, x, y) = (2\pi t)^{-1/2} \exp(\frac{(y-x)^2}{2t})$ be the heat kernel on the real line. The heat kernel $q_L(t, x, y)$ for the interval $[0, L]$, $L > 0$, with periodic boundary conditions, can be written as

$$\begin{aligned} q_L(t, x, y) &= \sum_{n \in Z} q(t, x, y+nL) \\ &= \frac{1}{L} + \frac{2}{L} \sum_{k=1}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{2\pi k}{L}\right)^2 t\right) \cos\frac{2\pi k}{L}(x-y). \end{aligned} \tag{5.1}$$

The transition probability $p_{abs}(t, x, y)$ of the Brownian motion with absorbing boundary conditions at a and b, for the interval $[a, b]$ with length $2(b-a) = L$ is

$$\begin{aligned} p_{abs}(t, x, y) &= \sum_{k \in Z} (q(t, x-a, y-a+2(b-a)k) \\ &\quad - q(t, x-a, -y+a-2(b-a)k)) \\ &= \frac{1}{\sqrt{2\pi t}} \sum_{k \in Z} \left[\exp\left(-\frac{1}{2t}(y-x+2(b-a)k)^2\right)\right. \\ &\quad \left. - \exp\left(-\frac{1}{2t}(y+x-2a+2(b-a)k)^2\right)\right] \end{aligned} \tag{5.2}$$

and the time variable Laplace transform of (5.2) is

$$\begin{aligned} \widehat{p_{abs}}(\alpha, x, y) &= \frac{1}{\sqrt{2\alpha}} (\exp(-\sqrt{2\alpha}\,|y-x|) - \exp(-\sqrt{2\alpha}(y+x-2a))) \\ &\quad + \frac{1}{\sqrt{2\alpha}} \frac{\exp(-\sqrt{2\alpha}(b-a))}{\sinh\sqrt{2\alpha}(b-a)} \\ &\quad \times (\cosh\sqrt{2\alpha}(y-x) - \cosh\sqrt{2\alpha}(y+x-2a)). \end{aligned} \tag{5.3}$$

Proof. Formula (5.1) can be obtained by identifying the two forms of the solution, either as a Fourier series or as the convolution with the heat kernel on the line with the periodic extension of an arbitrary test function $\phi \in C[0, L]$. The relation (5.2) can be obtained by various methods (see Ref. 3) or simply by computation and comparing with (2.6). The series

(5.2) is absolutely convergent uniformly in x, y for $t>0$. The Laplace transform of the heat kernel on the real line

$$\int_0^{\infty} \exp(-\alpha t) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(y-x)^2\right) dt = \frac{1}{\sqrt{2\alpha}} \exp(-\sqrt{2\alpha}\,|y-x|)$$

and the summation of a geometric series yields the formula (5.3). □

6. PROOF OF THEOREM 4

Proof. (i) The difference between the two paths $z_x(t, \omega)$ and $z_y(t, \omega)$ will stay piecewise constant between successive hits to the boundary by either of them. If the two were to collapse, this could only happen at zero. From (2.4) we can see that

$$\begin{aligned} z_x(t, \omega) - z_y(t, \omega) &= x - y - bN_x^b(t) - aN_x^a(t) + bN_y^b(t) + aN_y^a(t) \\ &= x - y + a(N_y^a(t) - N_x^a(t)) + b(N_y^b(t) - N_x^b(t)) \end{aligned}$$

which proves (i).

(ii) Without loss of generality we shall assume that $l=1$ and $b \geqslant |a|$. We denote by T_c the time of collapse of two paths $z_x(t, \omega)$ and $z_y(t, \omega)$

$$T_c = \inf\{t \colon z_x(t, \omega) = 0 \text{ and } z_y(t, \omega) = 0\} \tag{6.1}$$

with the convention that $T_c = \infty$ if the paths never collapse.

The union of increasing sequences of a.s. finite times of hitting the boundary $\{\tau_n^x\}$ and $\{\tau_n^y\}$ corresponding to x and y from the interval (a, b) can be rearranged in increasing order; the new increasing sequence of stopping times will be simply denoted by $\{\tau_n\}$. Since the initial distance between the piecewise parallel paths is an integer $x-y<b-a$ we notice that after each time the boundary is hit the distance will change into a new value from Z. The boundary is hit by one of the paths at a time, otherwise their distance would have already been $b-a$ which is impossible. This implies that all hitting times can be re-arranged in increasing order as desired. At each such hitting time, one of the paths will fall back to zero, while the other one will be in the set $Z \cap (a, b)$. It is important to recall that the two paths can only collapse at 0, since they evolve in parallel fashion between the times τ_n. If $|x-y| = |a|$ or $|x-y| = b$ the paths will have only two possibilities at τ_1: (1) the paths collapse, and there is nothing to prove and (2) the distance changes into $b-|a| = b+a \in [a+1, b-1] \cap Z$. Let's denote $S = \{k \colon a+1 \leqslant k \leqslant b-1\}$. We can verify that $k \in S \setminus \{0\}$ then both $b-|k| \in S$ and $a+|k| \in S$.

These considerations allow us to define a Markov chain $Y_n(\omega) = z_r(\tau_n, \omega)$, where r is either x or y in such a way that $z_r(\tau_n, \omega)$ is the point which is *not* situated at zero at time τ_n. The chain has transition probability

$$P_{k,j} = \begin{cases} \dfrac{|a|-|k|}{b+|a|-|k|} & \text{if} \quad j=b-|k| \quad \text{and} \quad k<0 \\ \dfrac{b}{b+|a|-|k|} & \text{if} \quad j=a+|k| \quad \text{and} \quad k<0 \\ \dfrac{|a|}{b+|a|-|k|} & \text{if} \quad j=b-|k| \quad \text{and} \quad k>0 \\ \dfrac{b-|k|}{b+|a|-|k|} & \text{if} \quad j=a+|k| \quad \text{and} \quad k>0 \\ 1 & \text{if} \quad j=k=0 \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

based on the formulas (6.5) applied to the intervals associated to the strip determined by the two paths. We only have to prove that $P_{k_0}(Y_n = 0 : n < \infty) = 1$ for any initial state $k_0 \in S$ of the Markov chain, since we know that the hitting times $\{\tau_n\}$ are finite almost surely.

Let T_k be the first hitting time of the state k and $N(k)$ be the number of visits to the state k with the convention that $N(k) = \infty$ if a state is visited infinitely many times. We have to prove that $P_{k_0}(T_0 < \infty) = 1$ for any k_0.

All States Except $k = 0$ Are Transient. The state $k \in S$ is *transient* if and only if $P_k(T_k = \infty) > 0$. The transition probabilities P_{kj} which are not equal to zero have a common positive lower bound

$$0 < P_{a,b} = \frac{1}{b+|a|} \leqslant P_{kj}$$

(they are also bounded above strictly below one). Since $k = 0$ is an absorbing state we have $P_k(T_k = \infty) \geqslant P(\{k \text{ reaches } 0 \text{ in finite time}\})$.

Algorithm to Reach the Absorbing State $k = 0$ in a Finite Number of Steps from any $k \in S \setminus \{0\}$. We have assumed that $0 < |a| \leqslant b$. Let $q = [b/|a|] > 0$.

Step 1. If $k = 0$ there is nothing to prove and the algorithm stops. If $k \in (a, 0)$ we go to Step 2. If $k > 0$ we can subtract $|a|$ from k for a number $q_1 > 0$ of times until $k - q_1|a| = k + q_1 a \in (a, 0]$. This procedure can take at most q steps. To subtract $|a|$ from $k > 0$, the Markov chain $\{Y_n\}$ must go to

$j = a+k$, according to (6.2), which occurs with probability $\frac{b-|k|}{b+|a|-|k|} \geqslant P_{a,b} > 0$. In terms of the actual process, the lower boundary is hit q_1 times in a row.

Step 2. If $k \in (a, 0]$ we move to $b-|k| = b+k$ which occurs with probability $\frac{|a|-|k|}{b+|a|-|k|} \geqslant P_{a,b} > 0$ according to (6.2); in terms of the actual process, the upper boundary is hit. Since $b-|k| \in (0, b-1]$ unless $k = 0$. In both cases we go to Step 1.

We want to check if the algorithm produces any repetitions of the current value k. The outcomes are of the form $mb+na+k_{\text{initial}}$, with $m, n \in Z_+$. The integer m represents the number of times we have to add b to the current value k while $n = \sum q_i$, where i is the number of times we subtract $|a|$ (that is, we hit the lower boundary) from Step 1. If two outcomes of the algorithm are equal, let's say $mb+na+k_{\text{initial}} = m'b+n'a+k_{\text{initial}}$, with the second outcome being obtained at a later iteration than the first, we should have $(m'-m)\, b = (n'-n)\, |a|$. Since the two numbers a and b are mutually prime, it follows that $m'-m$ must be a multiple of $|a|$, henceforth $m'-m \geqslant |a|$. This proves that the current value k must run through all integer values in the interval $(a, 0]$ before any repetition occurs, which forces k to achieve the value zero in at most $|a| \cdot (q+1)$ iterations. This quantity is easily bounded by $2b = 2\max\{|a|, b\}$. This, together with the fact that all transition probabilities in the iterations involved in the algorithm are bounded below by $P_{a,b}$ and the fact that $k = 0$ is absorbing imply that $P_{k_0}(N(k) < \infty) > 0$ for any $k \neq 0$. We have shown that all nonzero states are transient.

Let $k \neq 0$. We denote by $A_k = \{N(k) = \infty\}$ and by $A = \{T_0 < \infty\}$. Then

$$A \supseteq \bigcap_{k \neq 0} A_k^c = \left(\bigcup_{k \neq 0} A_k\right)^c$$

which implies that

$$P_{k_0}(A) \geqslant 1 - P_{k_0}\left(\bigcup_{k \neq 0} A_k\right) \geqslant 1 - \sum_{k \neq 0} P_{k_0}(A_k) = 1$$

due to the fact that $P_{k_0}(A_k) = 0$ for all transient states (see Ref. 2), in this case all $k \in S \setminus \{0\}$. We conclude that $P_{k_0}(A) = 1$. □

The following lemma calculates the probability that a standard Brownian motion starting at $x \in (a, b)$ will reach one of the endpoints of the interval before reaching the other.

Lemma 4. Let $T(a) = \inf\{t > 0 : w_x(t, \omega) \leqslant a\}$, $T(b) = \inf\{t > 0 : w_x(t, \omega) \geqslant b\}$ and $T_x = \min\{T(a), T(b)\}$. Then,

(i)

$$E_x[e^{-\alpha T(a)} 1_{T(a)<T(b)}] = \frac{\sinh\sqrt{2\alpha}(b-x)}{\sinh\sqrt{2\alpha}(b-a)}, \tag{6.3}$$

$$E_x[e^{-\alpha T(b)} 1_{T(b)<T(a)}] = \frac{\sinh\sqrt{2\alpha}(x-a)}{\sinh\sqrt{2\alpha}(b-a)} \tag{6.4}$$

and

(ii)

$$P_x(T(a) < T(b)) = \frac{b-x}{b-a}, \qquad P_x(T(b) < T(a)) = \frac{x-a}{b-a}. \tag{6.5}$$

Proof. We shall basically follow Ref. 6 (the result exists in Ref. 3 as an exercise on p. 100). The proof of exactly what we need is not given in either book.

We define $\Phi(x) = E_x[e^{-\alpha T(a)} 1_{T(a)<T(b)}]$ and prove that it solves

$$\begin{aligned} &\tfrac{1}{2}\Phi''(x) - \alpha\Phi(x) = 0, \qquad x \in (a, b) \\ &\Phi(a) = 1 \qquad \Phi(b) = 0. \end{aligned} \tag{6.6}$$

Naturally, $E_x[e^{-\alpha T_x} 1_{T(a)<T(b)}] = E_x[e^{-\alpha T(a)} 1_{T(a)<T(b)}]$. Assume the function $\Phi(x)$ is the solution to (6.6). Then

$$e^{-\alpha t}\Phi(w_x(t,\omega)) - \Phi(w_x(0,\omega)) - \int_0^t (-\alpha\Phi(w_x(s,\omega)) + \tfrac{1}{2}\Phi''(w_x(s,\omega)))\, e^{-\alpha s}\, ds$$

is a $\{\mathscr{F}_t\}$ martingale according to Itô's formula. Since T_x is a stopping time, we apply the optional sampling theorem and get that

$$\begin{aligned} &e^{-\alpha(t\wedge T_x)}\Phi(w_x(t\wedge T_x,\omega)) - \Phi(w_x(0\wedge T_x,\omega)) \\ &\quad - \int_0^{t\wedge T_x} (-\alpha\Phi(w_x(s,\omega)) + \tfrac{1}{2}\Phi''(w_x(s,\omega)))\, e^{-\alpha s}\, ds \end{aligned}$$

is a martingale. The integrand is zero as long as $s < t\wedge T_x$, that is, when the process stays in the interval (a, b). This implies that $e^{-\alpha(t\wedge T_x)}\Phi(w_x(t\wedge T_x,\omega))$ is a martingale. We notice that $\alpha > 0$ and $\Phi(x)$ is bounded (it is continuous) hence the martingale is uniformly integrable. If we take the expected value

$$E_x[e^{-\alpha(t\wedge T_x)}\Phi(w_x(t\wedge T_x,\omega))]|_{t=0}$$
$$= E_x[e^{-\alpha(t\wedge T_x)}\Phi(w_x(t\wedge T_x,\omega))\, 1_{T(a)<T(b)}]|_{t\to\infty}$$
$$+ E_x[e^{-\alpha(t\wedge T_x)}\Phi(w_x(t\wedge T_x,\omega))\, 1_{T(b)<T(a)}]|_{t\to\infty}$$

which means that

$$\Phi(x) = E_x[e^{-\alpha T(a)} 1_{T(a)<T(b)}]$$

since the second term is zero. However, (6.6) has a unique solution given by (6.3). The analogue computation for (6.4) will prove the result. To show (6.5) we can reproduce the proof from above with $\alpha = 0$ or calculate

$$\lim_{\alpha\to 0} \frac{\sinh\sqrt{2\alpha}(b-x)}{\sinh\sqrt{2\alpha}(b-a)} = \frac{b-x}{b-a}. \qquad \square$$

Corollary 5. The Laplace transform $\widehat{h^x}(\alpha)$ of the first exit time T_x from the interval (a, b) is

$$E_x[e^{-\alpha T_x}] = \frac{\cosh\sqrt{2\alpha}(x-\frac{b+a}{2})}{\cosh\sqrt{2\alpha}(\frac{b-a}{2})}. \tag{6.7}$$

Proof. We can reproduce the proof of the preceding lemma with $\Phi(x)$ satisfying the boundary conditions $\Phi(a) = \Phi(b) = 1$ and notice that (6.7) satisfies the equation. The uniqueness of the PDE concludes the proof. Alternatively, we could look at the sum of the two solutions (6.3) and (6.4). $\square$

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