Name: Answer

MA407 — Thursday, November 19, 2009

TEST 3

Department of Mathematics
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SHOW YOUR WORK. NO WORK = NO CREDIT.

1. (20%) Let $G = \mathbb{Z}_{20}$ and $H =< 4 >$ subgroup of G.

   (a) List the elements of the factor group G/H.

   $$G/H = \{H,\ 1+H,\ 2+H,\ 3+H\}$$

   since $G = \mathbb{Z}_{20} = \{0, 1, 2, \cdots, 19\}$ and
   $H = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$

   (b) Is G/H isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$? Justify your answer.

   $$(1+H) + (1+H) = 2+H \neq H$$
   $$\Rightarrow o(1+H) \neq 2 \Rightarrow o(1+H) = 4$$
   $$\Rightarrow \langle 1+H \rangle = G/H \approx \mathbb{Z}_4$$

2. (20%) Show whether the following maps are homomorphisms of groups or not. If it is a homomorphism find its kernel.

   (a) $\varphi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $\varphi(a, b) = a - b$ for all $a, b \in \mathbb{Z}$.

   Let $(a,b), (c,d) \in \mathbb{Z} \oplus \mathbb{Z}$
   $$\varphi((a,b)+(c,d)) = \varphi(a+c, b+d) = (a+c)-(b+d)$$
   $$\varphi(a,b) + \varphi(c,d) = (a-b)+(c-d) = (a+c)-(b+d) = \varphi((a,b)+(c,d))$$
   $\Rightarrow \varphi$ is a homomorphism
   $$\ker\varphi = \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid \varphi(a,b) = 0\} = \{(a,a) \mid a \in \mathbb{Z}\}$$

   (b) $\varphi : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{10}$ given by $\varphi(a) = 3a$ for all $a \in \mathbb{Z}_{12}$.

   Take $4, 8 \in \mathbb{Z}_{12}$, $\varphi(4+8) = \varphi(12) = \varphi(\underset{\mathbb{Z}_{12}}{0}) = 0 \in \mathbb{Z}_{10}$
   $\varphi(4) = 12 = 2 \in \mathbb{Z}_{10}$, $\varphi(8) = 24 = 4 \in \mathbb{Z}_{10}$
   So $\varphi(4) + \varphi(8) = 6 \neq \varphi(4+8)$
   $\therefore \varphi$ is not a homomorphism.

3. (20%) Consider the group $G = \mathbb{Z}_9 \oplus \mathbb{Z}_3$.

(a) List all elements of order 3 in G. $(a, b) \in \mathbb{Z}_9 \oplus \mathbb{Z}_3$

$o(a)$	$o(b)$	elements
3	1	$(3,0), (6,0)$
1	3	$(0,1), (0,2)$
3	3	$(3,1), (6,1), (3,2), (6,2)$

(Total 8 elements)

(b) How many subgroups of order 3 does G have? Justify your answer.

Any subgroup of order 3 is cyclic. Also any cyclic subgroup of order 3 contain two distinct elements of order 3. So # of subgroups $= \frac{8}{2} = 4$

4. (a) (12%) Find all Abelian groups (up to isomorphism) of order $108 = 2^2 3^3$.

$\mathbb{Z}_4 \oplus \mathbb{Z}_{27}$, $\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$, $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27}$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

are the isomorphism classes abelian groups of order 108.

(b) (8%) Is $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_9$ isomorphic to $\mathbb{Z}_6 \oplus \mathbb{Z}_{18}$? Justify your answer.

$\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_9 \approx \mathbb{Z}_6 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \approx \mathbb{Z}_6 \oplus \mathbb{Z}_{18}$

since $\gcd(2,9) = 1$.

5. (a) (12%) Show that $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$ is a subring of the ring of complex numbers $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$. (Here $i = \sqrt{-1}$).

Let $a+ib, c+id \in \mathbb{Z}[i]$. Then

$(a+ib)-(c+id) = (a-c)+i(b-d) \in \mathbb{Z}[i]$

$(a+ib)(c+id) = (ac-bd)+i(bc+ad) \in \mathbb{Z}[i]$

$\Rightarrow \mathbb{Z}[i]$ subring by subring test.

(b) (8%) Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of the ring of integers $\mathbb{Z}$.

$3 \in 3\mathbb{Z}$, $2 \in 2\mathbb{Z} \Rightarrow 3, 2 \in 2\mathbb{Z} \cup 3\mathbb{Z}$

But $3-2 = 1 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$

$\therefore 2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring.