SHOW YOUR WORK. NO WORK = NO CREDIT.

1. (15%) Find all cyclic subgroups of $D_4$.

\[
D_4 = \{ I, R, R^2, R^3, H, HR, HR^2, HR^3 \}
\]

\[
\langle I \rangle = \{ I \}, \quad \langle R \rangle = \{ I, R, R^2, R^3 \} = \langle R^3 \rangle
\]

\[
\langle R^2 \rangle = \{ I, R^2 \}, \quad \langle H \rangle = \{ I, H \}, \quad \langle HR \rangle = \{ I, HR \}
\]

\[
\langle HR^2 \rangle = \{ I, HR^2 \}, \quad \langle HR^3 \rangle = \{ I, HR^3 \}
\]

2. (10%) List all elements of order 10 in $\mathbb{Z}_{80}$. Justify your answer.

\[
\frac{80}{10} = 8, \quad |\langle 8 \rangle| = 10, \quad U(10) = \{ 1, 3, 7, 9 \}
\]

Elements in $\mathbb{Z}_{80}$ of order 10 are

\[
\{ 8^1, 8^2, 8^3, 8^4 = 24, 8^5 = 56, 8^6 = 72 \}
\]

3. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 6 & 8 & 7 & 4 & 9 & 3 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 1 & 8 & 9 & 7 \end{pmatrix}$.

(a) (15%) Write $\alpha$, $\beta$, and $\alpha \beta$ as product of disjoint cycles and determine their order.

\[
\alpha = (12579)(3648), \quad o(\alpha) = \text{lcm}\{5,4\} = 20
\]

\[
\beta = (123456)(789), \quad o(\beta) = \text{lcm}\{6,3\} = 6
\]

\[
\alpha \beta = (123456789)(2319)(56874231) = (154738)(26), \quad o(\alpha \beta) = \text{lcm}\{6,2\} = 6
\]

(b) (10%) Write $\alpha$, $\beta$, and $\alpha \beta$ as product of transpositions. State whether they are even or odd permutations?

\[
\alpha = (19)(17)(15)(12)(38)(34)(36), \quad \text{odd}
\]

\[
\beta = (16)(15)(14)(13)(12)(79)(78), \quad \text{odd}
\]

\[
\alpha \beta = (18)(13)(17)(14)(15)(26), \quad \text{even}
\]
4. (15%) Is $U(20)$ isomorphic to $U(24)$? Justify your answer.

\[ U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\} \quad U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\} \]

\[
\begin{align*}
0(1) &= 1 \\
0(3) &= 4 = 0(7) = 0(13) = 0(17) \\
0(9) &= 2 = 0(11) = 0(19)
\end{align*}
\]

Hence, $U(20)$ has 4 elements of order 4 and $U(24)$ contains no element of order 4. Therefore, $U(20) \not\cong U(24)$.

5. (10%) Let $\mathbb{R}^+$ be the group of positive real numbers under multiplication. Prove that the mapping $f(x) = \sqrt{x}$ is an automorphism of $\mathbb{R}^+$.

\[ f(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = f(x)f(y) \Rightarrow f \text{ is a homomorphism.} \]

\[ f(x) = f(y) \Rightarrow \sqrt{x} = \sqrt{y} \Rightarrow x = y, \quad \therefore f \text{ is 1-1.} \]

For $y \in \mathbb{R}^+$, we have $y^2 \in \mathbb{R}^+$ and $f(y^2) = \sqrt{y^2} = y$

Therefore, $f$ is onto.

$\Rightarrow f$ is an automorphism on $\mathbb{R}^+$.

6. (10%) Consider the group $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$ under multiplication modulo 15. List the distinct left cosets of the subgroup $H = \{1, 11\}$ in the group $U(15)$.

\[
H = \{1, 11\} = 1H \\
2H = \{2, 7\} \\
4H = \{4, 14\} \\
8H = \{8, 13\}
\]

7. (15%) Suppose $K$ is a subgroup of $H$ and $H$ is a subgroup of the group $G$. Assume $K \neq H \neq G$. If $|K| = 10$ and $|G| = 100$. What are the possible orders of $H$? Justify your answer.

By Lagrange Theorem

\[ 10 \mid |1H| \text{ and } |1H| \mid 100 \text{ since } |K| = 10, \quad |G| = 100 \]

Since $K \neq H$ and $H \neq G$, we know $|1H| \neq 10 \text{ or } 100$

Hence $|1H| = 20 \text{ or } 50$