MA407 TEST 2 Department of Mathematics
Thursday, Oct. 22, 2009 NCSU

SHOW YOUR WORK. NO WORK = NO CREDIT.

1. (15%) Find all cyclic subgroups of D_4.

$D_4 = \{I, R, R^2, R^3, H, HR, HR^2, HR^3\}$

$\langle I \rangle = \{I\}$, $\langle R \rangle = \{I, R, R^2, R^3\} = \langle R^3 \rangle$

$\langle R^2 \rangle = \{I, R^2\}$, $\langle H \rangle = \{I, H\}$, $\langle HR \rangle = \{I, HR\}$

$\langle HR^2 \rangle = \{I, HR^2\}$, $\langle HR^3 \rangle = \{I, HR^3\}$

2. (10%) List all elements of order 10 in $\mathbb{Z}_{80}$. Justify your answer.

$80/10 = 8$, $|\langle 8 \rangle| = 10$, $U(10) = \{1, 3, 7, 9\}$

Elements in $\mathbb{Z}_{80}$ of order 10 are

$\{8^1 = 8, 8^3 = 24, 8^7 = 56, 8^9 = 72\}$

3. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 6 & 8 & 7 & 4 & 9 & 3 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 1 & 8 & 9 & 7 \end{pmatrix}$.

(a) (15%) Write α, β, and $\alpha\beta$ as product of disjoint cycles and determine their order.

$\alpha = (1\,2\,5\,7\,9)(3\,6\,4\,8)$, $o(\alpha) = lcm\{5, 4\} = 20$

$\beta = (1\,2\,3\,4\,5\,6)(7\,8\,9)$, $o(\beta) = lcm\{6, 3\} = 6$

$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 8 & 7 & 4 & 2 & 3 & 1 & 9 \end{pmatrix} = (1\,5\,4\,7\,3\,8)(2\,6)$

$o(\alpha\beta) = lcm\{6, 2\} = 6$

(b) (10%) Write α, β, and $\alpha\beta$ as product of transpositions. State whether they are even or odd permutations?

$\alpha = (1\,9)(1\,7)(1\,5)(1\,2)(3\,8)(3\,4)(3\,6)$, odd

$\beta = (1\,6)(1\,5)(1\,4)(1\,3)(1\,2)(7\,9)(7\,8)$, odd

$\alpha\beta = (1\,8)(1\,3)(1\,7)(1\,4)(1\,5)(2\,6)$, even

4. (15%) Is $U(20)$ isomorphic to $U(24)$? Justify your answer.

$U(20)=\{1,3,7,9,11,13,17,19\}$
$o(1)=1$
$o(3)=4=o(7)=o(13)=o(17)$
$o(9)=2=o(11)=o(19)$

$U(24)=\{1,5,7,11,13,17,19,23\}$
$o(1)=1$
$o(5)=2=o(7)=o(11)=o(13)$
$o(17)=2=o(19)=o(23)$

Hence $U(20)$ has 4 elements of order 4 and $U(24)$ contains no element of order 4. Therefore, $U(20) \not\cong U(24)$.

5. (10%) Let $\mathbb{R}^+$ be the group of positive real numbers under multiplication. Prove that the mapping $f(x) = \sqrt{x}$ is an automorphism of $\mathbb{R}^+$.

$f(xy)=\sqrt{xy}=\sqrt{x}\sqrt{y}=f(x)f(y) \Rightarrow f$ is a homomorphism.
$f(x)=f(y) \Rightarrow \sqrt{x}=\sqrt{y} \Rightarrow x=y$. $\therefore f$ is 1-1.
For $y \in \mathbb{R}^+$, we have $y^2 \in \mathbb{R}^+$ and $f(y^2)=\sqrt{y^2}=y$
Therefore, f is onto.
$\Rightarrow f$ is an automorphism on $\mathbb{R}^+$.

6. (10%) Consider the group $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$ under multiplication modulo 15. List the distinct left cosets of the subgroup $H = \{1, 11\}$ in the group $U(15)$.

$H=\{1,11\}=1H$
$2H=\{2,7\}$
$4H=\{4,14\}$
$8H=\{8,13\}$

7. (15%) Suppose K is a subgroup of H and H is a subgroup of the group G. Assume $K \neq H \neq G$. If $|K| = 10$ and $|G| = 100$. What are the possible orders of H? Justify your answer.

By Lagrange Theorem
$10 \mid |H|$ and $|H| \mid 100$ since $|K|=10$, $|G|=100$
Since $K \neq H$ and $H \neq G$, we know $|H| \neq 10$ or 100
Hence $|H| = 20$ or 50