Tori Invariant under an Involutorial Automorphism I

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Introduction

Let G be a connected reductive linear algebraic group defined over an field k of characteristic not 2, $\theta \in \mathrm{Aut}(G)$ an involutional k-automorphism of G and $K = G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of θ. Denote the set of k-rational points of G by G_k.

In this paper we shall classify the K^0-conjugacy classes of θ-stable maximal tori of G. This is shown to be independent of the characteristic of k and can be applied to describe all the orbits of affine symmetric spaces under the action of a minimal parabolic subgroup in the case that the affine symmetric space is of type (G, K^0) (see Helminck [8]).

This paper is part of a series of papers leading towards a classification of all orbits of affine symmetric spaces under the action of a minimal parabolic subgroup (see also [10]). The results and techniques in this paper will be used for the classification in the remaining cases.

The K^0-conjugacy classes of θ-stable maximal tori of G occur (in a slightly different form) in the representation theory of real semisimple Lie groups. Namely if θ is a Cartan involution of G as in [10, 11.7], then the K^0-conjugacy classes of θ-stable maximal tori of G correspond one to one with the G_k-conjugacy classes of maximal k-tori of G. So for $k = \mathbb{R}$ this leads to a one to one correspondence with the conjugacy classes of Cartan subalgebras of a real simisimple Lie algebra. We give two characterizations of these conjugacy classes. The first characterization relates the K^0-conjugacy classes with conjugacy classes of involutions

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of a maximal θ-split torus and the second characterization relates the K^0-conjugacy classes with conjugacy classes of involutions of a maximal torus of K. The first characterization not only generalizes the classification of the conjugacy classes of Cartan subalgebras in a real semisimple Lie algebra as in Kostant [13] and Sugiura [31], but also gives a considerable simplification. Similarly the second characterization generalizes some results of Schmidt [24] on Cartan subalgebras. We also derive the diagrams of the adjacent conjugacy classes, what generalizes some results of Hirai [12] on Cartan subalgebras.

For an involution $\theta \in \mathrm{Aut}(G)$ we write $\mathcal{T}$ for the set of θ-stable maximal tori of G and $\mathcal{T}_K$ for the set of K^0-conjugacy classes of θ-stable maximal tori of G. If T is a θ-stable torus of G, then we write:

$$T^+ = (T \cap K)^0 \text{ and } T^- = \{t \in T \mid \theta(t) = t^{-1}\}^0.$$

The second torus is called a θ-*split* torus of G. Denote the set of characters, the set of roots and the Weyl group of T with respect to G by respectively, $X^*(T)$, $\Phi(T)$ and $W(T)$.

If we fix a θ-stable maximal torus T of G, then any other θ-stable maximal torus of G is of the form gTg^{-1} with $n = g^{-1}\theta(g) \in N_G(T)$. Denote the image of n in $W(T)$ by w_g. Then $\theta(w_g)w_g = id$, i.e., w_g is a θ-twisted involution in $W(T)$. The representative g of the K^0-conjugacy class of gTg^{-1} is unique up to left translations from K^0 and right translations from $N_G(T)$. If $h = kgtw$ is another representative ($k \in K$, $t \in T$, $w \in W(T)$), then $w_h = w^{-1}w_g\theta(w)$. So this leads to θ-twisted conjugacy classes of θ-twisted involutions. In the case that T^+ (or T^-) is maximal it is easy to show that we can choose g in such a way that w_g becomes an involution (see 3.4). These involutions in $W(T)$ are called θ-singular. From the above remarks it follows now that there is a one to one correspondence between the K^0-conjugacy classes of maximal tori of G and the θ-twisted conjugacy classes of θ-singular involutions in $W(T)$. We can restrict the above correspondence to conjugacy classes of θ-singular involutions in $W(T)$:

Theorem 4.5 *Let T be a θ-stable maximal torus of G with maximal T^- (resp. T^+). Then there is a one to one correspondence between the K^0-conjugacy classes of $\mathcal{T}$ and the $W(T)$-conjugacy classes of θ-singular involutions in $W(T)$.*

It remains to classify the conjugacy classes of θ-singular involutions of $W(T)$. Unfortunately these are difficult to classify for an arbitrary θ-stable maximal torus T. However in the case that T^- is maximal this appears to be easy:

Theorem 4.6 *Let T be a θ-stable maximal torus of G with maximal T^- and $w \in W(T)$, $w^2 = e$. Then the following are equivalent:*

(*i*) *w is θ-singular*

(*ii*) $T_w^- \subset T^-$

A classification of the conjugacy classes of θ-singular involutions of $W(T)$ in the case that T^- is a maximal θ-split torus of G, can be derived from the classification of involutorial automorphisms of G in Helminck [8].

There exists a natural ordering on $\mathcal{T}_K$. For $[T_1], [T_2] \in \mathcal{T}_K$ define $[T_1] < [T_2]$ if $T_1^- \subset T_2^-$ for some representatives T_i of $[T_i]$ $(i = 1, 2)$. The corresponding diagram easily follows from the classification of the conjugacy classes of θ-singular involutions in $W(T)$.

A brief summary of the contents is as follows. After some preliminaries in section 1, we derive all the properties needed about involutions in section 2. Section 3 deals with the characterization of the K^0-conjugacy classes of θ-stable maximal tori of G in terms of conjugacy classes of θ-singular involutions in $W(T)$ in the case that either T^+ is a maximal torus of K or T^- is a maximal θ-split torus of G. In section 4 we characterize the conjugacy classes of θ-singular involutions and in section 5 we show that, in the case that θ is a Cartan involution of G as in [10], the K^0-conjugacy classes of θ-stable maximal tori correspond bijectively with the G_k-conjugacy classes of maximal k-tori of G. The ordering on $\mathcal{T}_K$ is introduced in section 6. Finally in section 7 we give a classification of involutions in Weyl groups and use this to classify the conjugacy classes of θ-singular involutions together with their diagrams.

Some of the above results were announced in [9].

1 Preliminaries and recollections

1.1 We use as our basic references for algebraic groups the books of Humphreys [13] and Springer [25] and we shall follow their notations and terminology. All algebraic groups considered are linear algebraic groups.

Let k denote a field of characteristic not 2 and G an algebraic k-group (i.e., an algebraic group defined over k). The set of k-rational points of G is denoted by G_k or $G(k)$. For most of this paper (except in chapter 5), we will assume that k is an algebraiccally closed field.

For any closed subgroup H of G, denote its Lie algebra by the corresponding (lower case) German letter $\mathfrak{h}$ and write H^o for the identity component. The center of H will be denoted by $Z(H)$.

For a subtorus T of H let $X^*(T)$ denote the additively written group of rational characters of T and $X_*(T)$ the group of rational one-parameter multiplicative subgroups of T, i.e. the

group of homomorphisms (of algebraic groups): $GL_1 \to T$. The group $X^*(T)$ can be put in duality with $X_*(T)$ by a pairing $< \cdot\,,\,\cdot >$ defined as follows: if $\chi \in X^*(T),\ \lambda \in X_*(T)$, then $\chi(\lambda(t)) = t^{<\chi,\lambda>}$ for all $t \in F^*$. The torus T acts on the Lie algebra $\mathfrak{h}$ of H by the adjoint representation. For $\alpha \in X^*(T)$ let $\mathfrak{h}_\alpha$ denote the weight space for the character α on $\mathfrak{h}$ and let $\Phi(T,H)$ denote *the set of roots of H with respect to T*, i.e. $\Phi(T,H)$ is the set of non-trivial characters $\alpha \in X^*(T)$ such that $\mathfrak{h}_\alpha \neq 0$. Set $W(T,H) = N_H(T)/Z_H(T)$, where

$$N_H(T) = \{\, x \in H \mid xTx_1 \subset T \,\},$$

$$Z_H(T) = \{\, x \in H \mid xt = tx \ \text{ forall t} \in \text{T} \,\}.$$

If H is connected, $W(T,H)$ is called the *Weyl group of H relative to T*. It is a finite group, which acts on $T,\ X^*(T)$ and $X_*(T)$. Moreover the set of roots $\Phi(T,H)$ is stable under the action of $W(T,H)$ on $X^*(T)$. In the case $H = G$ we shall write $\Phi(T)$ for $\Phi(T,G)$ and $W(T)$ for $W(T,G)$.

If T is a torus of G such that $\Phi(T)$ is a root system in the subspace $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(T)$ and if $W(T)$ is the corresponding Weyl group, then for each $\alpha \in \Phi(T)$ the subgroup $G_\alpha = Z_G((Ker\,\alpha)^0)$ is nonsolvable. If we choose now $n_\alpha \in N_{G_\alpha}(T) - Z_{G_\alpha}(T)$ and let s_α be the element of $W(T)$ defined by n_α, then there exists a unique one parameter subgroup $\alpha^\vee \in X_*(T)$ such that $< \alpha, \alpha^\vee >= 2$ and $s_\alpha(\chi) = \chi - < \chi, \alpha^\vee > \alpha$ $(\chi \in X^*(T))$. We call $\alpha^\vee$ the *coroot* of α and denote the set of these $\alpha^\vee$ in $X_*(T)$ by $\Phi^\vee(T)$. We have a bijection of $\Phi^\vee(T)$ onto $\Phi(T)$.

For $x, y \in G$ denote the commutator $xyx^{-1}y^{-1}$ by (x,y). If $A,\ B$ are subgroups of G, the subgroup of G generated by all $(x,y),\ x \in A,\ y \in B$ will be donoted by $[A,B]$.

1.2 Involutorial automorphisms of G

Let $\theta \in$ Aut(G) be a k-involution of G, i.e., θ is defined over k and $\theta^2 = id$. We denote the automorphism of $\mathfrak{g}$, induced by θ also by θ and write $K = G_\theta = \{x \in G \mid \theta(x) = x\}$ for the group of fixed points of θ. This is a closed reductive k-subgroup of G (see Vust [33, §1] and [10]). If $k = \mathbb{C}$, then G/K is the complexification of a space $G(\mathbb{R})/K(\mathbb{R})$ with $G(\mathbb{R})$-invariant Riemannian structure.

1.3 θ-split tori

Let T be a θ-stable torus of G. (Recall that according to a result of Steinberg [27, 7.5], there exists a θ-stable torus T of G.) If we write $T_\theta^+ = (T \cap K)^0$ and $T_\theta^- = \{x \in T \mid \theta(x) = x^{-1}\}^0$,

then it is easy to verify that the product map

$$\mu : T_\theta^+ \times T_\theta^- \to T, \;\; \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. So in particular $T = T_\theta^+ T_\theta^-$ and $T = T_\theta^+ \cap T_\theta^-$ is a finite group. (In fact it is an elementary abelian 2-group.) If T is a torus in a θ-stable subgroup H of G, then the automorphisms of $\Phi(T, H)$ and $W(T, H)$ induced by $\theta|H$ will also be denoted by θ.

A torus A of G is called *θ-split* if $\theta(a) = a^{-1}$ for every $a \in A$. If $\theta \neq id$, then non-trivial θ-split tori exist (see Vust [33, §1]), so in particular there are maximal ones. The following result can be found in Vust [33, §1]:

Proposition 1.4 *Let A be a maximal θ-split torus of G. Then:*

(1) *A is the unique θ-split torus of of $Z_K(A)$;*

(2) *$(Z_G(A),\ Z_G(A)) \subset K^0$ and $Z_G(A)$ is the almost direct product of $Z_K(A)^o$ and A;*

(3) *If T is a maximal torus of G, containing A, then T is θ-stable.*

Moreover all maximal θ-split tori of G are conjugate under K^0 and so are all maximal tori of G containing a maximal θ-split torus of G.

Let A be a maximal θ-split torus of G and $T \supset A$ a maximal torus. Take $W_1 = \{w \in W(T) \mid w(A) \subset A\}$ and $W_0 = \{w \in W(T) \mid w|A = id_A\}$. The restriction map $W_1 \to W_0,\ w \to w|A$ induces an isomorphism of W_1/W_0 onto $W(A, G)$ (see Helminck [8] or Richardson [21]). The group $W(A, G)$ is even the Weyl group of a root system.

Proposition 1.5 *Let A be a maximal θ-split torus of G and let E_0 denote the vector subspace of $X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(A)$. Then $\Phi(A)$ is a root system in E_0 and the corresponding Weyl group is given by the restriction of $W(A)$ to E_0. Moreover every element of $W(A)$ has a representative in $N_{K^0}(A)$.*

For a proof, see Richardson [21, 4.7].

Note that if T is a maximal torus of G containing A, then $\Phi(A)$ coincides with the set of restrictions of the elements of $\Phi(T)$ to A.

Another important K^0-conjugacy class of θ-stable maximal tori of G is the one containing a maximal torus of K. They appear as θ-stable maximal tori in a θ-stable Borel subgroup of G.

Proposition 1.6 *If B is a θ-stable Borel subgroup of G and T a θ-stable maximal torus of B, then $B \cap K^0$ is a Borel Subgroup of K^0 and $K^0 \cap T$ is a maximal torus of K^0.*

All these maximal tori are conjugate under K^0:

Proposition 1.7 *If S is a maximal torus of K, then $Z_G(S)$ is a maximal torus of G. All maximal tori of G, containing a maximal torus of K^0, are conjugate under K^0.*

For a proof of these results, see Richardson [21].

2 Some properties of involutions of root systems

In the sequel we shall need various properties of involutions in Weyl groups, which commute with a fixed involution θ of the corresponding root system. We shall review these properties here. Our basic reference for root systems is Bourbaki [5].

2.1 Let E be a finite dimensional real vector space, $\Phi \subset E$ a (reduced) root system, $(\,\cdot\,,\,\cdot\,)$ an $\mathrm{Aut}(\Phi)$-invariant inner product of E and $X \supset \Phi$ a free abelian group contained in the weight lattice of Φ. For a closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite subgroup of $\mathrm{Aut}(\Phi)$ generated by the reflections s_α, $\alpha \in \Phi_1$. We will also write W for the Weyl group $W(\Phi)$.

2.2 θ-order on Φ

Let $\theta \in \mathrm{Aut}(\Phi)$ be an involution, i.e. $\theta^2 = id$. Denote the eigenspace of θ for the eigenvalue ξ, by $E(\theta, \xi)$. Similarly as in [8] we define a θ-order on Φ related to the projection of E on $E(\theta, -1)$.

Let $X_0(\theta) = \{\chi \in X \mid \theta(\chi) = \chi\}$ and $\Phi_0(\theta) = \Phi \cap X_0(\theta)$. Clearly $X_0(\theta)$ and $\Phi_0(\theta)$ are θ-stable and $\Phi_0(\theta)$ is a closed subsystem of Φ. We denote the Weyl group of $\Phi_0(\theta)$ by $W_0(\theta)$ and identify it with the subgroup $W(\Phi_0(\theta))$ of W. Let $W_1(\theta) = \{w \in W \mid w(X_0(\theta)) = X_0(\theta)\}$, $\overline{X}_\theta = X/X_0(\theta)$ and let π be the natural projection from X to $\overline{X}_\theta$. Every $w \in W_1(\theta)$ induces an automorphism $\pi(w)$ of $\overline{X}_\theta$ and $\pi(w(\chi)) = \pi(w)(\pi(\chi)) \quad (\chi \in X)$. If $\overline{W}_\theta = \{\pi(\theta) \mid w \in W_1(\theta)\}$, then $\overline{W}_\theta \cong W_1(\theta)/W_0(\theta)$. (See Satake [23, 2.1.3]). This is called the *restricted Weyl group* with respect to the action of θ on X. It is not necessarily a Weyl group in the sense of Bourbaki [5, Ch. V, no. 1].

Let $\overline{\Phi}_\theta = \pi(\Phi - \Phi_0(\theta))$ denote the set of *restricted roots of* Φ *relative to* θ. Similarly as in [8] we define a θ-order on Φ by choosing orders on $X_0(\theta)$ and $\overline{X}_\theta$. To be more precise:

Definition 2.3 *An order $\succ$ on X is called a θ-order if it has the following property:*

$$\text{if } \chi \in \mathrm{X},\ \chi \succ 0, \text{ and } \chi \notin \mathrm{X}_0(\theta), \text{ then } \theta(\chi) \prec 0.$$

A basis Δ of Φ with respect to a θ-order on X will be called a θ-basis of Φ. We then write $\Delta_0(\theta) = \Delta \cap \Phi_0(\theta)$ and $\overline{\Delta}_\theta = \pi(\Delta - \Delta_0(\theta))$. Clearly $\Delta_0(\theta)$ is a basis of $\Phi_0(\theta)$ and a similar property holds for $\overline{\Delta}_\theta$ (see [8, 2.4]).

2.4 A characterization of θ on a θ-basis of Φ

Let Δ be a θ-basis of Φ. Similarly as in [8, 2.8] we can write $\theta = -id \cdot \theta^* \cdot w_0(\theta)$, where $w_0(\theta) \in W_0(\theta)$ is the longest element of $W_0(\theta)$ with respect to $\Delta_0(\theta)$ and $\theta^* \in \mathrm{Aut}(\mathrm{X}, \Phi, \Delta, \Delta_0(\theta)) = \{\phi \in \mathrm{Aut}(\mathrm{X}, \Phi) \mid \phi(\Delta) = \Delta,\ \phi(\Delta_0(\theta)) = \Delta_0(\theta)\},\ (\theta^*)^2 = \mathrm{id}$. For more details see [8, §2]. This will be called a characterization of θ on its $(+1)$-eigenspace (because $W_0(\theta)$ is the Weyl group of $\Phi_0(\theta) \subset E(\theta, +1)$).

2.5. Remark: The above characterization of involutions in $\mathrm{Aut}(\Phi)$ is useful, if one needs properties of both the involution and its restricted root system, like in [8]. To characterize only the involution (without the restricted root system) it is a lot easier to work in the opposite setting and use the (-1)-eigenspace to characterize the involution, i.e. start with a basis of $\Phi \cap E(\theta, -1)$ and extend it to a $(-\theta)$-basis of Φ. To be more precise: consider the involution $-\theta = -id \cdot \theta \in \mathrm{Aut}(\Phi)$. Then $\Phi_0(-\theta) = \{\alpha \in \Phi \mid \theta(\alpha) = -\alpha\}$ and $W_0(-\theta) = W(\Phi_0(-\theta))$. Choose a $(-\theta)$-basis Δ of Φ corresponding to a $(-\theta)$-order on Φ. Then

$$\theta = (-\theta)^* w_0(-\theta) \tag{1}$$

where $w_0(-\theta)$ is the longest element of $W_0(-\theta)$ with respect to $\Delta_0(-\theta)$.

Especially for dealing with involutions in W this characterization is a lot easier. But unfortunately we will also need the restricted root system. Therefore both characterizations will be used and to avoid confusion we shall use a different notation for the characterizations of involutions on their (-1)-eigenspace (so related to $-\theta$).

So if Δ is a $(-\theta)$-basis of Φ, then write $\Phi(\theta)$ for $\Phi_0(-\theta)$, $\Delta(\theta)$ for $\Delta_0(-\theta)$, $W(\theta)$ for $W_0(-\theta)$ and $w(\theta)$ for $w_0(-\theta)$, the longest element of $W_0(-\theta)$ with respect to $\Delta(\theta)$. Note that $W(\theta) \subset W_1(\theta)$.

Remark: Instead of working with the θ-order resp. $(-\theta)$-order on Φ one can also use regular vectors in $E(\theta, -1)$ resp. $E(\theta, +1)$, like in Springer [27].

Definition 2.6 *Two roots $\alpha, \beta \in \Phi$ are called strongly orthogonal if $(\alpha, \beta) = 0$ and $\alpha \pm \beta \notin \Phi$.*

We have the following characterization of involutions in W:

Proposition 2.7 *Let $\theta \in \mathrm{Aut}(\Phi)$, $\theta^2 = id$ and let Δ be a $(-\theta)$-basis of Φ. Write $\theta = (-\theta)^* \, w(\theta)$ as in (1). Then the following are equivalent.*

(*i*) $\theta \in W$

(*ii*) $\theta = s_{\alpha_1} \cdots s_{\alpha_n}$ *with* $\alpha_1, \cdots, \alpha_n \in \Phi(\theta)$ *strongly orthogonal*

(*iii*) $(-\theta)^* = id$

(*iv*) $-id \in W(\Phi(\theta)) = W(\theta)$

(*v*) *The irreducible components of $\Phi(\theta)$ are of type A_1, B_n, C_n, D_{2n} $(n \geq 2)$, E_7, E_8, F_4 or G_2*

To prove this we will need the following wellknown result (see Carter [6, 2.5.5]).

Lemma 2.8 *Let $v \in E$ and $w \in W$ such that $w(v) = v$. Then w is the product of reflections corresponding to roots orthogonal to v.*

2.9 Proof of Proposition 2.7: $(i) \Leftrightarrow (iii)$ is immediate from (1) and $(ii) \Rightarrow (i)$ is also clear.

$(i) \Rightarrow (ii)$. We use induction on $\mathrm{rank}(\Phi)$. If θ has an eigenvalue 1, then use induction and (2.8). If $\theta = -id$, choose a root $\alpha \in \Phi$, which is a long root of some irreducible subsystem (if different root lengths occur). Then $s_\alpha \theta$ has an eigenvalue 1 (with eigenvector α) and is a product of reflections commuting with α, in strongly orthogonal roots. Since α is a long root, the result follows.

$(iv) \Rightarrow (ii)$ follows with a similar argument. Since $\theta|\Phi(\theta, -) = -id$, $(ii) \Rightarrow (iv)$ is also clear. Finally $(iv) \Leftrightarrow (v)$ follows from [8, 2.9.2], which proves the result.

2.10 Remark: If $\alpha_1, ..., \alpha_r \in \Phi$ are pairwise orthogonal roots of Φ, then $w = s_{\alpha_1}....s_{\alpha_r} \in W$ is an involution with $E(w, -1)$ spanned by $\alpha_1, ..., \alpha_r$. From the above result it follows, that there are $\beta_1, ..., \beta_r \in \Phi$ strongly orthogonal, such that $w = s_{\beta_1}....s_{\beta_r}$.

In the remainder of this section we study involutions in W, which commute with θ. An important invariant for the conjugacy class of an involution $w \in W$ is the dimension of its (-1)-eigenspace. If it is maximal, then it is the opposition involution with respect to some basis of Φ.

Proposition 2.11 *Let $w \in W$ such that $w^2 = id$ and $\dim \mathrm{E(w, -1)}$ is maximal, (i.e. if $r \in W$, $r^2 = id$, $E(r, -1) \supset E(w, -1)$ then $r = w$). We have the following:*

(1) *There is a basis Δ of Φ such that w is the opposition involution of W with respect to Δ.*

(2) *All involutions $w \in W$ with $E(w, -1)$ maximal are conjugate.*

Proof: Consider the involution $-w = -id \cdot w \in \mathrm{Aut}(\Phi)$. Let Δ be a basis of Φ such that $\Delta \cap E(-w, -1)$ is a basis for $\Phi(-w) = \{\alpha \in \Phi \mid w(\alpha) = \alpha\}$.

If $\alpha \in \Phi(-w)$, then $\dim E(s_\alpha w, -1) = \dim \mathrm{E(w, -1)} + 1$, which contradicts the fact that $E(w, -)$ is maximal. So $\Phi(-w) = \emptyset$ and hence $-w$ is a diagram automorphism of Δ. This proves (1).

(2) follows from the fact that all opposition involutions in W are conjugate (see Bourbaki [5]).

Definition 2.12 *An involution $w \in W$ is called a θ-maximal involution of W if $E(w, -1) \subset E(\theta, -1)$ and w is a maximal involution of $W(\theta)$.*

Corollary 2.13 *Let $\theta \in \mathrm{Aut}(\Phi)$, $\theta^2 = id$. All θ-maximal involutions of W are conjugate under $W(\theta) \subset W_1(\theta)$.*

Proof: This result follows immediately from (2.11) applied to the root system $\Phi(\theta)$.

Corollary 2.14 *Let $\theta \in \mathrm{Aut}(\Phi)$, $\theta^2 = id$ and w_0 a θ-maximal involution of W. If $w \in W$ with $E(w, -) \subset E(\theta, -)$ then there exists $w_1 \in W(\theta)$ such that $E(w_1 w w_1^{-1}, -) \subset E(w_0, -) \subset E(\theta, -)$.*

Proof: Let w' be a θ-maximal involution of $W(\theta)$ with $E(w',-) \supset E(w,-)$. The result follows now from (2.13).

We will need the following result:

Lemma 2.15 *Let $\theta \in \mathrm{Aut}(\Phi)$, $\theta^2 = id$ and assume $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$. If Φ is irreducible, then $\overline{\Phi}_\theta$ is irreducible.*

Proof: It suffices to prove that for any two orthogonal roots $\alpha, \beta \in \Phi$, there exists a $\gamma \in \Phi$ such that $(\alpha, \gamma) \neq 0$ and $(\beta, \gamma) \neq 0$.

If α and β are contained in a subsystem Φ_0 of type B_2 or G_2, then the result is clear. So assume $\pm(\alpha \pm \beta)$ and $\pm 1/2(\alpha \pm \beta) \notin \Phi$. Let $e_1, ..., e_n$ be a basis of E with $e_1 = \alpha,\ e_2 = \beta$. Define a total ordering on E as follows: If $x = x_1 e_1 + ... + x_j e_j$ is a vector with $x_j \neq 0$, then $x > 0$ if and only if $x_j > 0$, and $y > z$ if $y - z > 0$ $(y, z \in E)$.

It is easy to verify that $\Phi^+ = \{\lambda \in \Phi \mid \lambda > 0\}$ is a system of positive roots and that α, β are elements of the corresponding basis Δ of Φ.

Let $\alpha_1, ..., \alpha_r \in \Delta$, with $\alpha_1 = \alpha,\ \alpha_r = \beta$, be the shortest chain in the Dynkin diagram connecting α and β. Then $\gamma = \sum_{k=2}^{r-1} \alpha_k \in \Phi$ and $(\alpha, \gamma) \neq 0$ and $(\beta, \gamma) \neq 0$. This proves the result.

Conjugacy classes of involutions in $W \cap \overline{W}_\theta$ can be characterized as follows.

Proposition 2.16 *Let θ be an involution of Φ such that $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$ and let $w_1, w_2 \in W$ be involutions with $E(w_i, -1) \subset E(\theta, -1)$ $(i = 1, 2)$. Then w_1 and w_2 are conjugate under W if and only if they are conjugate under $W_1(\theta)$.*

Proof: The "only if" part being obvious, we assume $w \in W$ such that $w w_1 w^{-1} = w_2$. If $w_1 = id$, then the statement is clear. So we assume that $w_i \neq id$ $(i = 1, 2)$. We will use induction on the rank of Φ.

If $\mathrm{rank}\Phi = 1$, then, since $E(w_i, -1) \neq 0 \quad (i = 1, 2)$, we have $\Phi = \overline{\Phi}_\theta$ and the result is clear.

Assume now that the result is proved for root systems of rank $\leq n$ and assume that Φ has rank $n + 1$. Since $w\ w_1 w^{-1} = w_2$, we have $w(\Phi(w_1)) = \Phi(w_2)$. Let $\alpha \in \Phi(w_1)$ (exists by (2.7)) and $\beta = w(\alpha) \in \Phi(w_2)$. Since $E(w_i, -1) \subset E(\theta, -1)$ $(i = 1, 2)$, it follows that both $\alpha, \beta \in \overline{\Phi}_\theta$. Moreover, since $\beta = w(\alpha)$, both α and β are contained in one irreducible component of Φ. Hence by (2.15) in one irreducible component of $\overline{\Phi}_\theta$. Since $W(\overline{\Phi}_\theta) \cong W_1(\theta)/W_0(\theta)$, it follows that there exists $\tilde{w} \in W_1(\theta)$ such that $\tilde{w}(\alpha) = \beta$.

Let $r_1 = s_\beta \tilde{w} w_1 \tilde{w}^{-1}$, $r_2 = s_\beta w_2$, $w_0 = w\tilde{w}^{-1}$ and $\Phi_\beta = \Phi_0(s_\beta) = \{\gamma \in \Phi \mid (\beta, \gamma) = 0\}$. Now $r_1(\beta) = r_2(\beta) = w_0(\beta) = \beta$, so by (2.8) $r_1, r_2, w_0 \in W(\Phi_\beta) = W_0(s_\beta)$. Since $w_0 r_1 w_0^{-1} = r_2$ it follows that there exists $\tilde{r} \in W_1(\theta) \cap W_0(s_\beta)$ such that $\tilde{r} r_1 \tilde{r}^{-1} = r_2$. Now $r = \tilde{r}\tilde{w} \in W_1(\theta)$ and $r w_1 r^{-1} = w_2$, which proves the result.

Corollary 2.17 *Let θ, $\overline{\Phi}_\theta$ be as in 2.16 and let w_0 be a θ-maximal involution of W. If $w_1, w_2 \in W$ are involutions such that $E(w_i, -1) \subset E(w_0, -1) \subset E(\theta, -1) \quad (i = 1, 2)$, then w_1 and w_2 are conjugate under $W_1(\theta)$ if and only if $w_0 w_1$ and $w_0 w_2$ are conjugate under $W_1(\theta)$.*

Proof: It suffices to prove one side of the implication. Assume $w \in W$ such that $w w_1 w^{-1} = w_2$.

By (2.16) we may assume that $w \in W_1(w_0)$, i.e. $w_0 w = w w_0$. But then $w\ w_0 w_1 w^{-1} = w_0 w_2$, which proves the result.

The following result is immediate from (2.16)

Corollary 2.18 *Let $\theta, \overline{\Phi}_\theta$ as in (2.16) and let w_0 be a θ-maximal involution of W. If $w_1, w_2 \in W$ such that $E(w_i, -1) \subset E(w_0, -1) \subset E(\theta, -1) \quad (i = 1, 2)$ then the following are equivalent.*

(1) *w_1 and w_2 are conjugate under W.*

(2) *w_1 and w_2 are conjugate under $W_1(\theta)$.*

(3) *w_1 and w_2 are conjugate under $W_1(w_0)$.*

2.19 Remark: Combining (2.14) and (2.18) it follows that we can study conjugacy classes (under W) of involutions in $W(w_0)$ instead of in $W(\theta)$. Here w_0 is a θ-maximal involution of W.

The following result will be needed in the sequel:

Lemma 2.20 *Let $\sigma, \theta \in \mathrm{Aut}(\Phi)$ be involutions with $E(\sigma, -) \subset E(\theta, -)$. Then we have the following:*

(1) *$\sigma\theta = \theta\sigma$.*

(2) *There exists a basis Δ of Φ such that $\Delta(\theta, -)$ and $\Delta(\sigma, -)$ are bases of $\Phi(\theta, -)$ and $\Phi(\sigma, -)$ respectively.*

Proof: (1) Consider $\phi = \theta\sigma\theta \in \mathrm{Aut}(\Phi)$. Clearly ϕ is an involution and $E(\sigma, -) \subset E(\phi, -)$. On the other hand, if $x \in E(\phi, -)$, then $\sigma(\theta(x)) = -\theta(x)$. Since $E(\sigma, -) \subset E(\theta, -)$ it follows that $x \in E(\sigma, -)$, hence $E(\sigma, -) = E(\phi, -)$. But then $\phi = \sigma$ and $\sigma\theta = \theta\sigma$.

(2) follows by choosing a $(-\sigma)$-bases for $\Phi(\theta, -)$ and extending this to a $(-\theta)$-bases for Φ.

3 Characterization of the conjugacy classes

In this section we show that the K^0-conjugacy classes of θ-stable maximal tori can be characterized by conjugacy classes of involutions in a Weyl group. These results are similar to those for groups with a Cartan involution (see [10]). However in this case much stronger results can be proved.

3.1 For a θ-stable torus T, we reserve the notation T^+ and T^- for T_θ^- and T_θ^+ respectively. For other involutions of T we shall keep the subscript. Let $\mathcal{T}$ denote the set of θ-stable maximal tori of G.

Definition 3.2 *For $T_1, T_2 \in \mathcal{T}$ the pair (T_1, T_2) is called standard if $T_1^- \subset T_2^-$ and $T_1^+ \supset T_2^+$. In this case, we also say that T_1 is standard with respect to T_2.*

The θ-stable maximal tori of G can be put in standard position.

Lemma 3.3 *Let $T_1, T_2 \in \mathcal{T}$ such that $T_1^+ \supset T_2^+$ (resp. $T_1^- \subset T_2^-$). Then there exists $x \in Z_{K^0}(T_2^+)$ (resp $Z_{K^0}(T_1^-)$) such that $(T_1,\ xT_2x^{-1})$ is standard. In particular if T_1^+ and T_2^+ (resp. T_1^- and T_2^-) are K^0-conjugate, so are T_1 and T_2.*

Proof: Let $M = Z_G(T_2^+)$. Then T_1^- and T_2^- are θ-split tori of M. Let $A \subset M$ be a maximal θ-split torus with $A \supset T_1^-$. Since T_2^- is a maximal θ-split torus of M, there exists by (1.4) a $x \in (M \cap K)^0$ such that $xAx^{-1} = T_2^-$ and hence $xT_1^-x^{-1} \subset T_2^-$.

The second assertion follows with a similar argument.

A standard pair (T_1, T_2) of θ-stable maximal tori of G gives rise to an involution in $W(T_1)$ (resp. $W(T_2)$).

Lemma 3.4 *Let (T_1, T_2) be a standard pair of θ-stable maximal tori of G. Then we have the following conditions:*

(*i*) *There exists* $g \in Z_G(T_1^- T_2^+)$ *such that* $gT_1g^{-1} = T_2$.

(*ii*) *If* $n_1 = \theta(g)^{-1}g$ *and* $n_2 = \theta(g)g^{-1}$, *then* $n_1 \in N_G(T_1)$ *and* $n_2 \in N_G(T_2)$.

(*iii*) *Let* w_1 *and* w_2 *be the images of* n_1 *and* n_2 *in* $W(T_1)$ *and* $W(T_2)$ *respectively. Then* $w_1^2 = e$, $w_2^2 = e$ *and* $(T_1)^+_{w_1} = (T_2)^+_{w_2} = T_1^- T_2^+$, *which characterizes* w_1 *and* w_2.

Proof: Consider the group $M = Z_G(T_1^- T_2^+)$. Since T_1 and T_2 are maximal tori of M, the first statement is clear.

(*ii*) and (*iii*). Consider first n_1. Since $g \in M$ and $T_1 = (T_1^- T_2^+)(T_1^+ \cap g^{-1}T_2^- g)$ it suffices to look at $T_1^+ \cap g^{-1}T_2^- g$. So let $x \in T_1^+ \cap g^{-1}T_2^- g$ and write $x = g^{-1}tg$ with $t \in T_2^-$. Then $n_1 x n_1^{-1} = \theta(g^{-1}t^{-1}g) = \theta(x)^{-1} = x^{-1}$. It follows that

$$Int(n_1)|T_1^- T_2^+ = 1,\ Int(n_1)|T_1^+ \cap g^{-1}T_2^- g = -id$$

which implies that $n_1 \in N_G(T_1)$, $w_1^2 = e$ and $(T_1)^+_{w_1} = T_1^- T_2^+$.

The assertion for n_2 and w_2 follows with a similar argument.

Remark: By (*iii*) of Lemma 3.4, w_1 and w_2 are independent of the choice of $g \in G$ with $gT_1g^{-1} = T_2$.

Definition 3.5 *Let* T_1, T_2, $w_1 \in W(T_1)$ *and* $w_2 \in W(T_2)$ *be as in Lemma* 3.4. *We call* w_1 (*resp.* w_2) *the* T_2*-standard involution* (*resp.* T_1*-standard involution*) *of* $W(T_1)$ (*resp.* $W(T_2)$).

For a θ-stable torus T of G we write $W(T,K)$ for $N_{K^0}(T)/Z_{K^0}(T)$. Note that $W(T,K) \subset \{w \in W(T) \mid w(T^-) \subset T^-\}$.

In the following we fix an element $T_0 \in \mathcal{T}$ (resp. $S \in \mathcal{T}$) such that T_0^- (resp. S^+) is a maximal θ-split torus of G. (resp. a maximal torus of K). Then by [21] $W(T_0,K) = \{w \in W(T_0) \mid w(T_0^-) \subset T_0^-\}$.

The following result is stronger than the similar result in the case of groups with a Cartan involution (see [8]).

Theorem 3.6 *Assume that* $T_1, T_2 \in \mathcal{T}$ *such that they are standard with respect to* T_0. *Let* w_1 *and* w_2 *be the* T_1*-standard and* T_2*-standard involutions in* $W(T_0)$ *respectively. Then the following are equivalent:*

(*i*) T_1 *and* T_2 *are* K^0*-conjugate.*

(ii) w_1 *and* w_2 *are conjugate under* $W(T_0, K)$.

(iii) w_1 *and* w_2 *are conjugate under* $W(T_0)$.

Proof: $(i) \Rightarrow (iii)$. Let $k \in K^0$ such that $kT_1k^{-1} = T_2$. Then $kT_1^+k^{-1} = T_2^+$, $kT_1^-k^{-1} = T_2^-$. We will first show that T_1^- and T_2^- are conjugate under $N_{K^0}(T_0)$.

Let $M = Z_G(T_2^-)$. Then T_0^- and $kT_0^-k^{-1}$ are maximal θ-split tori of M, so by (1.4) there exists $k_1 \in (M \cap K)^0$ such that $k_1kT_0^-k^-k_1^{-1} = T_0^-$. Since $k_1kT_0^+kk_1^{-1}$ and T_0^+ are conjugate under $(Z_G(T_0^-) \cap K)^0$, it follows that we may assume that $k \in N_{K^0}(T_0)$, $kT_1k^{-1} = T_2$. But then $kT_0^-k^{-1} = T_0^-$, $kT_1^-k^{-1} = T_2^-$, so since $(T_0)^+_{w_i} = T_0^+T_i^-$ $(i = 1, 2)$, it follows that $k(T_0)^+_{w_1}k^{-1} = (T_0)^+_{w_2}$, what proves the result.

Since $(iii) \Rightarrow (ii)$ follows from (2.16) it suffices to show $(ii) \Rightarrow (i)$. Let $w \in W(T_0, K)$ such that $ww_1w^{-1} = w_2$ and let $k \in N_{K^0}(T_0^-)$ be a representative. Then $kT_0^+k^{-1} = T_0^+$, $k(T_0)^-_{w_1}k^{-1} = (T_0)^-_{w_2}$. Since $(T_0)^+_{w_i} = T_0^+T_i^-$ $(i = 1, 2)$, we have $kT_1^-k^{-1} = T_2^-$. The result follows now from (3.3).

Corollary 3.7 *Assume that* $T_1, T_2 \in \mathcal{T}$ *such that they are standard with respect to* S. *Let* w_1 *and* w_2 *be the* T_1*-standard and* T_2*-standard involutions in* $W(S)$ *respectively. Then the following are equivalent:*

(i) T_1 *and* T_2 *are* K^0*-conjugate.*

(ii) w_1 *and* w_2 *are conjugate under* $W(S, K)$.

(iii) w_1 *and* w_2 *are conjugate under* $W(S)$.

Proof: $(i) \Rightarrow (ii)$ Let $k \in K^0$ such that $kT_1k^{-1} = T_2$. As before $kT_1^+k^{-1} = T_2^+$, $kT_1^-k^{-1} = T_2^-$. Let $M = Z_G(T_2^+)$ and $\tilde{S} = kSk^{-1}$. Now S^+ and $\tilde{S}^+$ are maximal tori of $(M \cap K)^0$, so by (1.7) there is a $k_1 \in (M \cap K)^0$ such that $k_1\tilde{S}^+k_1^{-1} = S^+$. Since $S = Z_G(S^+)$ (see 1.7) it follows that $k_1k_1 \in N_{K^0}(S)$. So $k_1kS^-k^{-1}k_1^{-1} = S^-$ and $k_1kT_1^+k^{-1}k_1^{-1} = T_2^+$. Since $S^+_{w_i} = S^-T_i^+$ $(i = 1, 2)$ it follows that $k_1kS^+_{w_1}k^{-1}k_1^{-1} = S^+_{w_2}$ and $k_1kS^-_{w_1}k^{-1}k_1^{-1} = S^-_{w_2}$. This proves the result.

$(ii) \Rightarrow (iii)$ being clear, we are left to prove $(iii) \Rightarrow (i)$. For this we will transfer the involutions in S to involutions in a maximal θ-split torus and then use Theorem 3.6.

Let $w \in W(S)$ be such that $ww_1w^{-1} = w_2$ and let $g \in N_G(S)$ be a representative. For $i = 1, 2$ let $g_i \in Z_G(T_i^+S^-)$ be such that $T_i = g_iSg_i^{-1}$. Let $\tilde{T}_i \subset Z_G(T_i^-)$ be maximal tori such that $\tilde{T}_i^-$ is a maximal θ-split torus of $Z_G(T_i^-)$ and $\tilde{T}_i^+ \subset T_i^+ \subset S^+$ $(i = 1, 2)$.

Then T_i is standard with respect to $\tilde{T}_i$ $(i = 1, 2)$. Let $\tilde{g}_i \in Z_G(T_i^+ T_i^-)$ be such that $\tilde{g}_i T_i \tilde{g}_i^{-1} = \tilde{T}_i$ $(i = 1, 2)$. Since $\tilde{T}_1^-$ and $\tilde{T}_2^-$ are maximal θ-split tori of $Z_G(S^-)$, there exists $k \in (Z_G(S^-) \cap K)^0$ such that $k\tilde{T}_1 k^{-1} = \tilde{T}_2$.

Write $T = \tilde{T}_2$ and put $h_1 = k\tilde{g}_1 g_1,\ h_2 = \tilde{g}_2 g_2$. Then $h_i S h_i^{-1} = T$ $(i = 1, 2)$.

Let ϕ_1 and $\phi_2 : X^*(S) \tilde{\to} X^*(T)$ be the isomorphisms induced by h_1 and h_2. Denote the corresponding isomorphisms of $W(S)$ and $W(T)$ also by ϕ_1 and ϕ_2. Set $\tilde{w}_1 = \phi_1(w_1)$ and $\tilde{w}_2 = \phi_2(w_2) \in W(T)$. Then:

$$T^-_{\tilde{w}_1} = h_1 S^-_{w_1} h_1^{-1} \subset k\tilde{g}_1 T_1^- \tilde{g}_1^{-1} k^{-1} \subset T^- \text{and} \quad \mathrm{T}^-_{\tilde{\mathrm{w}}_2} = \mathrm{h}_2 \mathrm{S}^-_{\mathrm{w}_2} \mathrm{h}_2^{-1} \subset \tilde{\mathrm{g}}_2 \mathrm{T}_2^- \tilde{\mathrm{g}}_2^{-1} = \mathrm{T}_2^- \subset \mathrm{T}^-$$

In particular, since $T_i^- = S^- \cdot g_i S^-_{w_i} g_i^{-1}$ $(i = 1, 2)$ we have:

$$kT_1^- k^{-1} = S^- k g_1 S^-_{w_1} g_1^{-1} k^{-1} = S^- h_1 S^-_{w_1} h_1^{-1} = S^- T^-_{\tilde{w}_1} \text{and} \qquad \mathrm{T}_2^- = \mathrm{S}^- \mathrm{h}_2 \mathrm{S}^-_{\mathrm{w}_2} \mathrm{h}_2^- = \mathrm{S}^- \mathrm{T}^-_{\tilde{\mathrm{w}}_2}.$$

Let $g_0 = h_2 g h_1^{-1} \in N_G(T)$. Then $g_0 T^-_{\tilde{w}_1} g_0^{-1} = T^-_{\tilde{w}_2}$, hence $\tilde{w}_1$ and $\tilde{w}_2$ are conjugate under $W(T)$.

Since S is standard with respect to T, there exists a S-standard involution $w_0 \in W(T)$ such that $T^+_{w_0} = T^+ \cdot S^-$. Moreover $w_0 \tilde{w}_i = \tilde{w}_i w_0$ $(i = 1, 2)$ and $T^-_{\tilde{w}_i} \subset T^-_{w_0}$.

By (2.17) $w_0 \tilde{w}_1$ and $w_0 \tilde{w}_2$ are conjugate under $W(T)$ so by (3.6), there exists a $r \in W(T, K)$ such that $r w_0 \tilde{w}_1 r^{-1} = w_0 \tilde{w}_2$. Let $h \in N_{K^0}(T)$ be a representative of r.

Then $hT^+_{w_0 \tilde{w}_1} h^{-1} = T^+_{w_0 \tilde{w}_2}$, but since $T^+_{w_0 \tilde{w}_i} = T^+ T_i^-$ $(i = 1, 2)$ and $hT^+h^{-1} = T^+$, it follows that $hT_1^- h^{-1} = T_2^-$. The assertion now follows from (3.3).

Remark: This characterization in terms of involutions in $W(S)$ is more general than the one in Schmidt [24] for $k = \mathbb{R}$, which deals only with the case of rank G = rank K^0.

4 Characterization of the standard involutions

The results in (3.6) and (3.7) provide a sound criterion when elements in $\mathcal{T}$ are K^0-conjugate. To complete the characterization of K^0-conjugacy classes of $\mathcal{T}$, we need to determine those $w \in W(T_0)$ (resp. $W(S)$) which are the T-standard involutions for some $T \in \mathcal{T}$. This will be dealt with in this section.

4.1 Let $T \in \mathcal{T}$ and $w \in W(T)$ satisfying $w^2 = e$ and $w\theta = \theta w$. Set $G_w = Z_G(T_w^+)$. Let n be the preimage of w in $N_G(T)$. Then $n \in Z_G(T_w^+)$ and $T_w^- \cap Z(G_w)$ is finite. As a consequence T_w^- is a maximal torus of $[G_w, G_w]$.

Recall that an involution θ of a connected reductive group M is called *split* if there exists a θ-split maximal torus of M.

Definition 4.2 *Let $T \in \mathcal{T}$ and $w \in W(T)$. We say that w is θ-singular if*

(1) $w^2 = e$

(2) $\theta w = w\theta$

(3) $\theta|[G_w, G_w]$ *is split*

(4) $\mathrm{rank}\ [G_w, G_w] \cap K = \mathrm{rank}\ [\mathrm{G_w}, \mathrm{G_w}]$.

A root $\alpha \in \Phi(T)$ is called θ-singular if the corresponding reflection $s_\alpha \in W(T)$ is θ-singular.

The following result is well known:

Lemma 4.3 $\theta \in \mathrm{Int(G)}$ *if and only if* rank G = rank K.

Proof: Assume first that there exists $a \in G$ such that $\theta(x) = axa^{-1}$ for all $x \in G$. Let S be a maximal torus of K^0. Since $\theta|S = id$, it follows that $a \in Z_G(S)$. By (1.7), $Z_G(S)$ is a maximal torus of G. Hence $\theta|Z_G(S) = id$, so $S = Z_G(S)$. In other words rank G = rank K^0.

Assume next that T is a maximal torus of K^0 and G. Since $\theta|T = id \in W(T)$, there exists $t \in T$ such that $\theta(x) = txt^{-1}$ for all $x \in G$. (See Springer [25, 11.4.3].) So $\theta \in \mathrm{Int(G)}$.

Lemma 4.4 *Let T be a θ-stable maximal torus of G with maximal T^- (resp. T^+) and $w \in W(T)$ θ-singular. Then we have the following conditions:*

(i) $(T_w^-)^- = T_w^-$ *is a θ-split maximal torus of $[G_w, G_w]$ (resp. $(T_w^-)^+ = T_w^-$ is a maximal torus of $[G_w, G_w] \cap K$).*

(ii) $T_w^+ = T^+(T^-)_w^+$ *(resp. $T_w^+ = T^-(T^+)_w^+$).*

(iii) *w is the T_1-standard involution in $W(T)$ for some $T_1 \in \mathcal{T}$.*

Proof: We show the assertion for the case that T^- is maximal. The result for the case that T^+ is maximal follows with a similar argument.

(i) From $\theta w = w\theta$ it follows that $T^- = (T^-)^+_w (T^-)^-_w$. Since T^- is a maximal θ-split torus of G, also $(T^-)^-_w$ is a maximal θ-split torus of $[G_w, G_w]$. But since $\theta|[G_w, G_w]$ is split, it follows that $(T^-)^-_w$ is a maximal torus of $[G_w, G_w]$. Hence $(T^-)^-_w = T^-_w$.

(ii) is immediate from (i) and the observation that $T^+_w = (T^+_w)^+(T^+_w)^-$.

(iii) Let $\tilde{S}$ be a maximal torus of $[G_w, G_w] \cap K$. Then $T_1 = \tilde{S}T^+_w$ is a maximal torus of G and by (ii) (T_1, T) is standard. Clearly w is the T_1-standard involution in $W(T)$.

We now have the following characterization of the K^0-conjugacy classes of $\mathcal{T}$.

Theorem 4.5 *Let T be a θ-stable maximal torus of G with maximal T^- (resp. T^+). Then there is a one to one correspondence between the K^0-conjugacy classes of $\mathcal{T}$ and the $W(T)$-conjugacy classes of θ-singular involutions in $W(T)$.*

Proof: Consider first the case that T^- is maximal. By Proposition (3.6) and (iii) of Lemma 4.4, it suffices to show that the standard involutions are θ-singular. Let $T_1 \in \mathcal{T}$ such that (T_1, T) is standard and let $w \in W(T)$ denote the T_1-standard involution in $W(T)$. By (iii) of Lemma 3.4 $T^+_w = T^-_1 T^+$. Clearly T^+_w is θ-stable and so $w\theta = \theta w$. On the other hand T^-, $T^+_1 \subset G_w$ and their images in $G_w/Z(G_w)$ are maximal tori of $G_w/Z(G_w)$. It follows that $\theta|[G_w, G_w]$ is split and $\mathrm{rank}[\mathrm{G_w}, \mathrm{G_w}] = \mathrm{rank}([\mathrm{G_w}, \mathrm{G_w}] \cap \mathrm{K}.)$. So w is θ-singular.

The result for the T^+ maximal follows with a similar argument.

In the case that T^- is maximal it is easy to determine the θ-singular involutions.

Theorem 4.6 *Let T be a θ-stable maximal torus of G with maximal T^- and $w \in W(T)$, $w^2 = e$. Then the following are equivalent:*

(i) *w is θ-singular*

(ii) $T^-_w \subset T^-$

Proof: $(i) \Rightarrow (ii)$ follows immediately from Lemma 4.4 (i).
$(ii) \Rightarrow (i)$. From 2.20 it follows that w and θ commute. Since T^-_w is a θ-split maximal torus of $[G_w, G_w]$ it suffices to show that rank $[\mathrm{G_w}, \mathrm{G_w}]$ = rank $([\mathrm{G_w}, \mathrm{G_w}] \cap \mathrm{K})$. This follows from Lemma 4.3 and the observation that $\theta|T^-_w = w|T^-_w = -id \in W(T^-_w, G_w)$, hence $\theta|G_w \in \mathrm{Int}(\mathrm{G_w})$.

Remarks: (1) This result is stronger than the equivalent result in the case of groups with a Cartan involution [10, 12.10].
(2) In the case that T^+ is maximal, an involution $w \in W(T)$ with $T_w^- \subset T^+$ is not necessarily θ-singular. For example, take $\theta \in \mathrm{Int(G)}$ and G has no θ-split maximal torus.

The θ-singular involutions can be characterized in terms of a product of reflections of θ-singular strongly orthogonal roots. First we need the following:

Lemma 4.7 *Let T be a θ-stable maximal torus of G and let $\Psi = \{\alpha_1, \cdots, \alpha_r\}$ be a set of strongly orthogonal roots of $\Phi(T)$. Let G_Ψ denote the closed subgroup of G generated by the $G_{s_{\alpha_i}} (i = 1, \cdots, r)$. Then:*

$$[G_\Psi, G_\Psi] \cong \prod_{i=1}^{r} [G_{s_{\alpha_i}}, G_{s_{\alpha_i}}]$$

Moreover, if $\alpha_1, \cdots, \alpha_r$ are θ-singular, then $\theta|[G_\Psi, G_\Psi]$ is split and rank $[G_\Psi, G_\Psi] =$ rank $([G_\Psi, G_\Psi] \cap K)$.

Proof: Since $[G_{s_\alpha}, G_{s_\beta}] = e$ for $\alpha, \beta \in \Phi(T)$ strongly orthogonal, the first assertion is clear.

If $\alpha_1, \cdots, \alpha_r$ are θ-singular, then $\mathrm{rank([G_\Psi, G_\Psi] \cap K)} = \sum_{\mathrm{i=1}}^{\mathrm{r}} \mathrm{rank([G_{s_{\alpha_i}}, G_{s_{\alpha_i}}] \cap K)} = \mathrm{r} = \mathrm{rank[G_\Psi, G_\Psi]}$. The result follows now from Lemma 4.4.

Proposition 4.8 *Let T be a θ-stable maximal torus of G with* dim $\mathrm{T}^- = \mathrm{r}$, dim $\mathrm{T}^+ = \mathrm{s}$. *Then the following are equivalent:*

(*i*) $\theta \in \mathrm{Int(G)}$ *and θ is split.*

(*ii*) *there exist strongly orthogonal subsets $\{\alpha_1, \cdots, \alpha_r\}$ and $\{\beta_1, \cdots, \beta_s\}$ of θ-singular roots of $\Phi(T)$, such that $\theta(\alpha_i) = -\alpha_i$ $(i = 1, \cdots, r)$, $\theta(\beta_j) = \beta_j$ $(j = 1, \cdots, s)$ and $-id = s_{\alpha_1} \cdots s_{\alpha_r} s_{\beta_1} \cdots s_{\beta_s}$.*

Proof: We prove $(i) \Rightarrow (ii)$ by induction in several steps.
(1) If both T^+ and T^- are nontrivial, then the derived subgroup of $Z_G(T^+)$ satisfies the conditions. Hence the induction argument works. So it suffices to prove the result in the cases that $T = T^-$ or $T = T^+$.
(2) Assume first that $T = T^-$. Since $\theta \in \mathrm{Int(G)}$, we have $\theta|T = -id \in W(T)$. Let $\alpha_1, \cdots, \alpha_r \in \Phi(T)$ be strongly orthogonal such that $-id = s_{\alpha_1} \cdots s_{\alpha_r}$. By Proposition 4.6 each α_i $(i = 1, \cdots, r)$ is θ-singular.

(3) Assume next that $T = T^+$. Let T_0 be a θ-split maximal torus of G. Then $-id \in W(T_0)$ and $-id = s_{\alpha_1} \cdots s_{\alpha_r}$ with $\alpha_1, \cdots \alpha_r \in \Phi(T_0)$ strongly orthogonal θ-singular roots. Let $\Psi = \{\alpha_1, \cdots, \alpha_r\}$ and G_Ψ the subgroup of G generated by the $G_{s_{\alpha_i}} (i = 1, \cdots, r)$. From Lemma 4.7 it follows now that G_Ψ satisfies (i). Let S be a maximal torus of $G_\Psi \cap K$. Then $\Phi(S, G_\Psi) \cong \Phi(T_0, G_\Psi)$ is of type $A_1 + \cdots + A_1$ (r times) and each root of $\Phi(S, G_\Psi)$ is θ-singular, because $\theta|[G_{s_{\alpha_i}}, G_{s_{\alpha_i}}] \neq id$ for $(i = 1, \cdots, r)$. Say $\Phi(S, G_\Psi) = \{\pm\gamma_1, \cdots, \pm\gamma_r\}$. Since S and T are K^0-conjugate, $\gamma_1, \cdots, \gamma_r$ are mapped into a set of strongly orthogonal θ-singular roots of $\Phi(T)$, which proves the result.

$(ii) \Rightarrow (i)$ follows immediately from Lemma 4.4.

Corollary 4.9 *Let T be a θ-stable maximal torus of G and $w \in W(T)$ a θ-singular involution. Then there exists strongly orthogonal subsets $\{\alpha_1, \cdots, \alpha_r\}$ and $\{\beta_1, \cdots, \beta_s\}$ of θ-singular roots of $\Phi(T)$, such that $\theta(\alpha_i) = -\alpha_i$ $(i = 1, \cdots, r)$, $\theta(\beta_j) = \beta_j$ $(j = 1, \cdots, s)$ and $w = s_{\alpha_1} \cdots s_{\alpha_r} s_{\beta_1} \cdots s_{\beta_s}$.*

Proof: Consider $G_w = Z_G(T_w^+)$. Then the result follows from 4.8.

A θ-singular involution corresponds always to a K^0-conjugacy class of a θ-stable maximal torus:

Corollary 4.10 *Let T be a θ-stable maximal torus of G and $w \in W(T)$ a θ-singular involution. Then there exists $g \in G$ such that gTg^{-1} is θ-stable and $n = \theta(g)g^{-1} \in N_G(T)$ is a representative of w.*

Proof: We may assume that $G = G_w$. Then from Proposition 4.8 it follows that θ is split and rank G = rank G_θ. Let T_0 be a θ-split maximal torus of $Z_G(T^-)$ and $g_1 \in Z_G(T^-)$ such that $T_0 = g_1 T g_1^{-1}$. Let $A_0 = g_1 T^+ g_1^{-1} \subset T_0$. Then $T_0 = T^- A_0$. Consider $M = Z_G(A_0)$. By (4.8) rank $[\mathrm{M}, \mathrm{M}]$ = rank $[\mathrm{M}, \mathrm{M}] \cap \mathrm{K}$. Let S_0 be a maximal torus of $M \cap K$. Then $S = Z_M(S_0)$ is a maximal torus of M (see 1.7). Let $g_2 \in M$ such that $S = g_2 T_0 g_2^{-1}$. If we take $g = g_2 g_1$ then $S = gTg^{-1}$ and $n = \theta(g)g^{-1} \in N_G(T)$ is a representative of w.

For a θ-singular involution $w \in W(T)$ and $g \in G$ as in 4.10, we will also write ${}^wT$ for the maximal torus gTg^{-1} in $\mathcal{T}$. Note that ${}^wT$ is not necessarily standard with respect to T.

4.11 Remark: If $k = \mathbb{R}$ one uses the notion of real and imaginary roots (see [7] or [34]). This corresponds to the following: Let T be a θ-stable maximal torus of G and $\alpha \in \Phi(T)$ θ-singular. Then $\theta(\alpha) = \pm\alpha$. If $\theta(\alpha) = -\alpha$ we call α a *real* root. If $\theta(\alpha) = \alpha$ we call α

an *imaginary singular* root. If $\theta(\alpha) = \alpha$ and α is not θ-singular, then α is called a *compact imaginary root*. We have now the following:

Proposition 4.12 *Let T be a θ-stable maximal torus of G. Then we have the following:*

(i) T^+ is maximal if and only if $\Phi(T)$ has no real roots.

(ii) T^- is maximal if and only if $\Phi(T)$ has no imaginary singular roots.

(iii) If $\alpha \in \Phi(T)$ with $\theta(\alpha) = -\alpha$, then α is real.

Proof: The assertions (i) and (ii) are immediate from (4.6) and (4.8).
(iii). Let $\alpha \in \Phi(T)$ with $\theta(\alpha) = -\alpha$. Then $\theta|[G_{s_\alpha}, G_{s_\alpha}] \neq id$. If $([G_{s_\alpha}, G_{s_\alpha}] \cap K)^0 = \{e\}$, then by Richardson [21], it follows that $[G_{s_\alpha}, G_{s_\alpha}]$ is a torus, which is impossible. So $([G_{s_\alpha}, G_{s_\alpha}] \cap K)^0 \neq \{e\}$ and $[G_{s_\alpha}, G_{s_\alpha}] \cap K$ contains a non-zero torus of rank 1. So α is θ-singular.

From the classification of involutions in [8] and the above results 4.5 and 4.6 we get the following result:

Proposition 4.13 *The set of K^0-conjugacy classes in $\mathcal{T}$ is independent of the characteristic of the field k, if char $k \neq 2$.*

5 Conjugacy classes of maximal k-tori in groups with a Cartan involution

In this section we consider the conjugacy of maximal k-tori in groups with a Cartan involution, as in [10]. This includes all reductive groups defined over $\mathbb{R}$. It will be shown that these conjugacy classes are independent of the k-structure and correspond bijectively with the K^0-conjugacy classes in $\mathcal{T}$ as in (4.5)

5.1 In this section let k be a field of characteristic not two, G a connected linear reductive k-group and θ a Cartan k-involution of G, as in [10, 11.8]. The following result follows from [10, 2.4]:

Lemma 5.2 *Let T be a maximal k-torus of G. Then T is G_k-conjugate with a θ-stable maximal k-torus.*

Using the Cartan decomposition for G_k we get:

Proposition 5.3 *Let T_1, T_2 be θ-stable maximal k-tori of G. Then T_1, T_2 are conjugate under G_k if and only if T_1 and T_2 are conjugate under K_k^0.*

Proof: The "if" statement being clear, assume that $g \in G_k$ such that $gT_1g^{-1} = T_2$. Write $g = sh_1$ with $s \in \tau(G_k) = \{x\theta(x)^{-1} \mid x \in G_k\}$, $h_1 \in K_k$, as in [10, 11.7]. Let $T_3 = h_1T_1h_1^{-1}$. Since T_i^- is a maximal k-split torus of T_i $(i = 1, 2)$ and $sT_3^-s^{-1} \subset T_2$ is k-split, it follows that $sT_3^-s^{-1} = T_2^-$.

Now $s^2 \in Z_G(T_3^-)$, namely if $t \in T_3^-$, then $\theta(sts^{-1}) = s^{-1}t^{-1}s = st^{-1}s^{-1}$, hence $s^2ts^{-2} = t$. Since s is k-split semi-simple (see [10, 11.7]), it follows that $s \in Z_G(T_3^-)$. Hence $T_3^- = T_2^-$.

Let $M = [Z_G(T_3^-), Z_G(T_3^-)]$. Then $S_3 = T_3 \cap M$ and $S_2 = T_2 \cap M$ are maximal k-tori of $M \cap K$. Since $M \cap K$ is anisotropic over k, there exists $h_2 \in (M \cap K)_k^0$ such that $h_2S_3h_2^{-1} = S_2$ and hence $h_2T_3h_2^{-1} = T_2$. Let $h = h_1h_2$. Then $h \in K_k$ and $hT_1h^{-1} = T_2$. It remains to show that T_1 and T_2 are conjugate under K_k^0. Let $A \supset T_2^-$ be a maximal (θ, k)-split torus of G. By [10, 11.6(ii)] $K_k \subset A_kK_k^0$. Write $h = a \cdot h_0$, $a \in A_k$, $h_0 \in K_k^0$. With a similar argument as above it follows that there exists $h_3 \in K_k^0$ such that $h_3h_0T_1h_0k_3^{-1} = T_2$, which proves the result.

Lemma 5.4 *Let T be a θ-stable maximal torus of G. Then there exists $h \in K^0$ such that hTh^{-1} is a k-torus.*

Proof: Let A be a maximal (θ, k)-split torus of G. Then by [10, 11.4] A is maximal θ-split. Let $T_0 \supset A$ be a θ-stable maximal k-torus. By Lemma 3.3 there exists $k \in K^0$ such that kTk^{-1} is standard with respect to T_0. Now $kT^-k^{-1} \subset A$ is defined over k, so using induction on $Z_G(kT^-k^{-1})$ the result follows.

Proposition 5.5 *Let T_1, T_2 be θ-stable maximal k-tori of G. Then T_1 and T_2 are K^0-conjugate if and only if T_1 and T_2 are K_k^0-conjugate.*

Proof: The "only if" statement being clear, assume $x \in K^0$ such that $xT_1x^{-1} = T_2$. For $i = 1, 2$, let A_i be a maximal (θ, k)-split torus of G, such that $A_i \supset T_i$. Since xA_1x^{-1} and A_2 are maximal (θ, k)-split tori of $Z_G(T_2^-)$, there exists $x_1 \in (Z_G(T_2^-) \cap K)^0$ such that $x_1xA_1x^{-1}x_1^{-1} = A_2$. Let $x_0 = x_1x$. By [10, 11.4 (ii)] there exists $y \in K_k^0$ such that $yA_1y^{-1} = A_2$. Then $y^{-1}x \in N_G(A_1)$. From [10, 11.4 (iv)] it follows that there exists $n \in N_{K_k^0}(A_1)$ such that $ny^{-1}x_0 \in Z_G(A)$. Let $y_0 = yn^{-1} \in K_k^0$. Then $y_0A_1y_0^{-1} = A_2$ and $y_0T_1^-y_1^{-1} = T_2^-$. Using induction on $Z_G(T_2^-)$ the result follows.

Corollary 5.6 *Let G, θ be as above. Then there is a one to one correspondence between the G_k-conjugacy classes of maximal k-tori of G and the K^0-conjugacy classes of θ-stable maximal tori of G.*

Proof: The result is immediate from (5.2), (5.3), (5.4) and (5.5).

5.7 Remarks: (1) For $k = \mathbb{C}$ there is a one to one correspondence between the real forms of G and the involutions of G (see [8, §10]). In particular, every involution of G is a Cartan involution for some real form of G. So the K^0-conjugacy classes in $\mathcal{T}$ depend only on the involution. It is clear now that the results in sections 3 and 4 generalize the results in Kostant [13], Sugiura [31] and Schmidt [24].
(2) Instead of the action of $\mathrm{Int}(G_k)$ on the maximal k-tori one can also consider the action of $\mathrm{Aut}(G_k)$ on the maximal k-tori. Using similar arguments as above, it is not hard to prove that this results in a conjugacy of the standard involutions under $\mathrm{Aut}(T)$. We state the result, without proof:

Proposition 5.8 *Let T_0 be a θ-stable maximal k-torus with T_0^- maximal and T_1, T_2 θ-stable maximal k-tori of G, which are standard with respect to T_0. Let $w_i \in W(T_0)$ be the T_i-standard involution $(i = 1, 2)$. Then the following are equivalent:*

(*i*) *There exists $\phi \in \mathrm{Aut(G_k)}$ such that $\phi(T_1) = T_2$,*

(*ii*) *w_1 and w_2 are conjugate under $\mathrm{Aut(T_0)}$.*

6 The diagram of K^0-conjugacy classes in $\mathcal{T}$

In this section we introduce a diagram associated to the K^0-conjugacy classes in $\mathcal{T}$. For $k = \mathbb{R}$ such a diagram was used, among others, for the study of invariant eigendistributions of Laplace operators on real simple Lie groups (see Hirai [12]). We generalize the results in [12] and also provide a technique to move to different nodes in the diagram.

6.1 Denote the set of K^0-conjugacy classes in $\mathcal{T}$ by $\mathcal{T}_K$. The elements of $\mathcal{T}_K$ will be denoted by $[T]$, where $T \in \mathcal{T}$ is a representative.

Like in 3.5 fix θ-stable maximal tori T_0 and S of G with T_0^- maximal, S^+ maximal and S standard with respect to T_0. By 3.4 every class $[T]$ in $\mathcal{T}_K$ has a representative, which is standard with respect to T_0 and S simultaneously.

6.2 For a θ-stable maximal torus T of G denote the set of $W(T)$- (resp. Aut(T)-) conjugacy classes of involutions in Aut(T) by $\mathcal{A}_T$ (resp. $\mathcal{A}_T^*$) and the subset of conjugacy classes of involutions in $W(T)$ by $\mathcal{W}_T$ (resp. $\mathcal{W}_T^*$). If T is θ-stable, then we denote the subset of conjugacy classes of θ-singular involutions in $W(T)$ by $\mathcal{W}_T(\theta)$ (resp. $\mathcal{W}_T^*(\theta)$).

The elements of $\mathcal{A}_T$ (resp. $\mathcal{A}_T^*$) will be denoted by $[\sigma]$ (resp. $\langle\sigma\rangle$), where $\sigma \in \mathrm{Aut(T)}$ is an involution in the class $[\sigma]$ (resp. $\langle\sigma\rangle$).

If $T = T_0$ or S then the classes in $\mathcal{T}_K$ correspond bijectively with those in $\mathcal{W}_T(\theta)$ (see 4.5 and 4.6). The set $\mathcal{W}_T^*(\theta)$ represents the classes under outer automorphisms, as in the case of Cartan groups, (see 5.8).

An order on $\mathcal{T}_K$ can be defined as follows:

Definition 6.3 *Let $[T_1], [T_2] \in \mathcal{T}_K$. We call $[T_1] < [T_2]$ if and only if $T_1^- \subset T_2^- \subset T_0^-$ for some representatives T_i of $[T_i]$ $(i = 1, 2)$, which are standard with respect to T_0.*

Denote the corresponding diagram by $\mathcal{L}(\theta)$. Clearly $\mathcal{L}(\theta)$ is connected with $[T_0]$ as a greatest element and $[S]$ as the least element. The level function $d : \mathcal{L}(\theta) \longrightarrow \mathbb{Z}$ is defined by:

$$d([T]) = \dim(\mathrm{T}^-) - \dim(\mathrm{S}^-) \tag{2}$$

where T is a representative of $[T]$, standard with respect to S and T_0. We will say that $[T_2]$ covers $[T_1]$ if $[T_2] > [T_1]$ and $d([T_2]) = d([T_1]) + 1$.

This order $>$ can also be formulated in terms of an order on $\mathcal{W}_{T_0}$:

Definition 6.4 *Let T be a maximal torus of G and let $[\sigma_1], [\sigma_2] \in \mathcal{A}_T$. We call $[\sigma_1] < [\sigma_2]$ if and only if $(T)_{\sigma_1}^- \supset (T)_{\sigma_2}^-$, for some representatives σ_1 resp. $\sigma_2 \in \mathrm{Aut(T)}$ of $[\sigma_1]$ resp. $[\sigma_2]$. If T is θ-stable, then the restriction of $>$ to $\mathcal{W}_T(\theta)$ will also be denoted by $>$.*

6.5 Remarks: (i) An order on $\mathcal{A}_T^*$ is defined similarly as in 6.4.
(ii). In the case that $T = T_0$, Proposition 4.6 implies that if $[w_1], [w_2] \in \mathcal{W}_{T_0}(\theta)$, then

$$[w_1] < [w_2] \Leftrightarrow T_0^- \supset (T_0)_{w_1}^- \supset (T_0)_{w_2}^- \tag{3}$$

where $w_i \in W(T_0)$ is a θ-singular representative of $[w_i]$ $(i = 1, 2)$.
(iii) If we use the identification of $\mathcal{T}_K$ and $\mathcal{W}_{T_0}(\theta)$ as in 4.5, then the ordering in 6.3 and 6.4 are basicly the same. Namely if $w_1, w_2 \in W(T_0)$ are θ-singular involutions, then by 4.4 (iii) there exist maximal tori $T_1, T_2 \in \mathcal{T}$ such that w_i is the T_i-standard involution of $W(T_0)$ $(i = 1, 2)$. Then

$$T_1^- \subset T_2^- \Leftrightarrow (T_0)_{w_1}^- \supset (T_0)_{w_2}^- \tag{4}$$

Hence

$$[T_1] < [T_2] \Leftrightarrow [w_1] < [w_2].$$

(iv) For a maximal torus T and an involution $\sigma \in \mathrm{Aut(T)}$ let $\mathcal{W}(\sigma)$ (resp. $\mathcal{W}^*(\sigma)$) denote the subset of those $[w] \in \mathcal{W}_T$ (resp. $\mathcal{W}^*_T$) with $[w] > [\sigma]$. If $\sigma = \theta$ then $\mathcal{W}(\theta) \supset \mathcal{W}_T(\theta)$. The diagram of $(\mathcal{W}_T, >)$ (resp. $(\mathcal{W}^*{}_T, >)$) will be denoted by $\mathcal{L}(\Phi(T))$ (resp. $\mathcal{L}^*(\Phi(T))$) and the diagram of $(\mathcal{W}(\sigma), >)$ (resp. $(\mathcal{W}^*{}_T, >)$) by $\mathcal{L}(\sigma)$ (resp. $\mathcal{L}^*(\sigma)$). Similarly as in 6.3 we can define a level function on $\mathcal{L}(\sigma)$.

For $T = T_0$ the greatest element in $(\mathcal{W}(\theta), >)$ is $[id]$ and the least element is $[w_0]$, where w_0 is a θ-maximal involution in $W(T_0)$, (see 2.14). This involution is nothing else than the S-standard involution for some maximal torus S with S^+ maximal.

From (2.16) and (2.18) we have now the following identification.

Proposition 6.6 *Let G, T_0, θ be as above and let $w_0 \in W(T_0)$ be a θ-maximal involution as in* (2.14). *Then $\mathcal{W}_{T_0}(\theta) = \mathcal{W}(\theta) = \mathcal{W}(w_0)$.*

This result is useful for determining the diagrams.

6.7 Running through the diagram

To move in the diagram we can use the θ-singular roots as in 4.11. They enable us to go step by step through the diagram, using in each step a real root or a θ-singular imaginary root of a maximal torus corresponding to that node. The question which remains is whether we can go in one step from a given class $[T] \in \mathcal{T}_K$ to any other class in $\mathcal{T}_K$ by a θ-singular involution of $W(T)$ (see 4.10). In the remainder of this section we will show when this is the case and we will also indicate when this does not hold.

For $[T_1], [T_2] \in \mathcal{T}_K$ consider the covering set:

$$M([T_1], [T_2]) = \{[T] \in \mathcal{T}_K \mid [T] > [T_1] \text{ and } [\mathrm{T}] > [\mathrm{T}_2]\}.$$

The set $M([T_1], [T_2])$ can be ordered with the order induced by $>$ on $\mathcal{T}_K$. Let $[T_m]$ be a minimal element. One can get from $[T_m]$ to $[T_1]$ via a θ-singular involution w_1 in T_m^- and also from $[T_m]$ to $[T_2]$ via a θ-singular involution w_2 in T_m^-. In most cases w_2 can be chosen in T_1^- and w_1 in T_2^- for $\Phi(T_m)$ irreducible. The only problems arise when $\Phi(T_m)$ has two root lengths.

6.8 Example: Let G be of type B_2 and $\theta \in \mathrm{Aut(G)}$ a split involution. Then $\mathcal{T}_K$ has two classes of level 1, corresponding to the long and short root. It is not possible to move from one class to the other by a single θ-singular involution. However, if $\Phi(T)$ has only one root length, the θ-singular involutions reach the whole diagram:

Proposition 6.9 *Let G be simple and let $[T_1], [T_2] \in \mathcal{T}_K$. If $\Phi(T_1)$ is simply laced, then there exists a θ-singular involution $w \in W(T_1)$ such that ${}^wT_1 \in [T_2]$. (see 4.10).*

Proof: Let $[T] \in M([T_1], [T_2])$ be a minimal element and $T \in \mathcal{T}$ a representative of $[T]$. Choose representatives T_i of $[T_i]$ which are standard with respect to T and let $w_i \in W(T)$ be the corresponding T_i-standard involution ($i = 1, 2$). We will prove the assertion in several steps:
(1) $(T^-_{w_1} \cap T^-_{w_2})^0 = (T^-_{w_1})^-_{w_2} = \{e\}$. This is immediate from the fact that T is minimal in $M([T_1], [T_2])$.
(2) If $\alpha, \beta \in \Phi(T)$ with $\theta(\alpha) = w_1(\alpha) = -\alpha$ and $\theta(\beta) = w_2(\beta) = -\beta$, then $\alpha \perp \beta$.
Write $T^-_{w_1} = (T^-_{w_1})^+_{s_\alpha}(T^-_{w_1})^-_{s_\alpha}$ and $T^-_{w_2} = (T^-_{w_2})^+_{s_\beta}(T^-_{w_2})^-_{s_\beta}$. Let $\Phi_{\alpha,\beta}$ be the subsystem of $\Phi(T)$ spanned by α and β. Since $|\alpha| = |\beta|$, there exists $w_0 \in W(\Phi_{\alpha,\beta})$ such that $w_0(\alpha) = \beta$. Clearly w_0 stabilizes both $(T^-_{w_1})^+_{s_\alpha}$ and $(T^-_{w_2})^+_{s_\beta}$. It follows that in $\mathcal{W}_T$ we have $[s_\beta] > [w_i]$ ($i = 1, 2$). (see 6.5). But this contradicts the fact that $[T]$ is minimal in $M([T_1], [T_2])$.
(3) $T^-_{w_2} \subset T^+_{w_1}$, so in particular $T^-_{w_2} \subset T^-_1$. This follows immediately from (2) and the fact that $T^+_{w_1} = T^+ \cdot T^-_1$ (see 4.4).
(4) Let $w_3 \in W(T_1)$ be the involution induced by $w_2 \in W(T)$, i.e., $(T_1)^-_{w_3} = T^-_{w_2}$. If $w_4 \in W(T_1)$ is the T-standard involution, then $w = w_3 w_4 \in W(T_1)$ is special θ-singular and ${}^wT_1 \in [T_2]$. This proves the result.

7 Classification of $\mathcal{T}_K$

In this section we will classify the conjugacy classes of θ-singular involutions in $W(T)$, where T is a θ-stable maximal torus of G with T^- maximal. By 4.6 this gives also a classification of $\mathcal{T}_K$.

7.1 We will use the same notation as in Section 2. In particular let E be a finite dimensional real vectorspace, $\Phi \subset E$ a root system, $(\,\cdot\,,\,\cdot\,)$ an $\mathrm{Aut}(\Phi)$-invariant innerproduct on E, W the Weyl group of Φ and $\theta \in \mathrm{Aut}(\Phi)$ an involution.

We want to determine the conjugacy classes of involutions $w \in W$ with $E(w,-) \subset E(\theta,-)$. From 2.20 (ii) it follows that these involutions determine subsets of a basis Δ of Φ (i.e., parabolic subsets of (Φ,Δ)). We will first show that we can restrict to the conjugacy of these subsets.

Definition 7.2 *Let Δ be a basis of Φ.*

(1) *Two subsets $\Delta_1, \Delta_2 \subset \Delta$ are called W-conjugate (resp. $\mathrm{Aut}(\Phi)$-conjugate) if there exists $w \in W$ (resp. $\phi \in \mathrm{Aut}(\Phi)$) such that $w(\Delta_1) = \Delta_2$ (resp. $\phi(\Delta_1) = \Delta_2$).*

(2) *An involution $w \in W$ is called Δ-standard if Δ is a $(-w)$-basis of Φ (i.e., $E(w,-) \cap \Delta = \Delta(w)$ is a basis for $\Phi(w)$).*

Let Δ be a fixed basis of Φ. Clearly every involution in W is W-conjugate to a Δ-standard involution, because W acts simply transitive on the bases of Φ. For the conjugacy of Δ-standard involutions we can restrict now to the conjugacy of the corresponding parabolic subsets:

Proposition 7.3 *Let $\Delta \subset \Phi$ be a basis and w_1, w_2 Δ-standard involutions in W. Then w_1, w_2 are W-conjugate (resp. $\mathrm{Aut}(\Phi)$-conjugate) if and only if $\Delta(w_1)$, $\Delta(w_2)$ are W-conjugate (resp. $\mathrm{Aut}(\Phi)$-conjugate).*

Proof: The "if" statement follows from the fact that w_i is the longest involution of $W(w_i)$ with respect to $\Delta(w_i)$ $(i = 1, 2)$.

On the other hand, if $w \in W$ is such that $ww_1w^{-1} = w_2$, then $w(E(w_1,-)) = E(w_2,-)$, $w(\Phi(w_1)) = \Phi(w_2)$. Since $w(\Delta(w_1))$ and $\Delta(w_2)$ are bases of $\Phi(w_2)$, they are conjugate by an element of $W(w_2) \subset W$. This proves the result.

The statement for $\mathrm{Aut}\Phi$ follows with a similar argument.

Definition 7.4 *Let Δ be a $(-\theta)$-basis of Φ and w_0 a fixed θ-maximal involution of $W(\theta)$, which is Δ-standard. (w_0 exists by Lemma 2.20.(ii)). An involution $w \in W$ is called (Δ, w_0)-standard if w is Δ-standard and $\Delta(w) \subset \Delta(w_0)$.*

Corollary 7.5 *Let $\theta \in Aut(\Phi)$ be an involution, Δ a $(-\theta)$-basis of Φ, $w_0 \in W(\theta)$ a θ-maximal involution, which is Δ-standard. Then we have the following:*

(i) *If $w \in W$ is an involution with $E(w,-) \subset E(\theta,-)$, then there exists $\tilde{w} \in W_1(\theta)$ such that $\tilde{w}w\tilde{w}^{-1}$ is (Δ, w_0)-standard.*

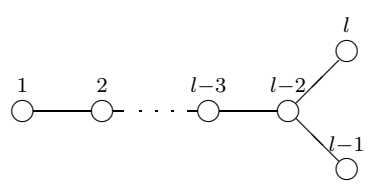
(*ii*) *If* $w_1, w_2 \in W$ *are* (Δ, w_0)*-standard involutions, then* w_1, w_2 *are* W*-conjugate* (*resp.* $\mathrm{Aut}(\Phi)$*-conjugate*) *if and only if* $\Delta(w_1)$ *and* $\Delta(w_2)$ *are* W*-conjugate* (*resp.* $\mathrm{Aut}(\Phi)$*-conjugate*).

Proof: (*i*) is immediate from Corollary 2.14 and Lemma 2.20 and (*ii*) follows from Propostion 7.4.

The proof of this result is a bit more technical. Since the result is not essential in the remainder of this paper and follows also immediately from the classification in section 7, we omit the proof. The classification of conjugacy classes in W with $E(w,-) \subset E(\theta,-)$ reduces to the study of the parabolic subsets of $\Phi(w_0)$, which correspond to an involution in W. Here w_0 denotes a θ-maximal involution in $W(\theta)$. Before we can determine these parabolic subsets and in particular the ones corresponding to the θ-maximal involutions, we will have to classify all the conjugacy classes of involutions in W. This corresponds to the case that θ is split.

7.6 Notations: Let Δ be a basis of Φ and Δ_0 a subset of Δ. If $\Phi(\Delta_0)$ is of type $A_1+\cdots+A_1$ (r times), then we will also use the notation $r \cdot A_1$.

In the case that Φ is a root system of type $D_\ell(\ell \geq 4)$ we will denote the subsystem Δ_0 corresponding to the two endnodes $\{\alpha_{\ell-1}, \alpha_\ell\}$ by D_2 (see (5))



(5)

This subsystem is not W-conjugate to any of the other subsystems of Δ of type $A_1 + A_1$. (All of the subsystems of Δ of type $A_1 + A_1$, which are not of type D_2, are W-conjugate (see 7.15)). In Kostant [13] the roots in the subsystem D_2 are called related.

To distinguish between long and short roots in B_ℓ ($\ell \geq 2$), we will write B_1 for the subsystem of type A_1 corresponding to the unique short root. Similarly we write C_1 for the long root in the bases of a root system of type C_ℓ.

The possible types of the parabolic subsets related to involutions in W follow directly from 2.7.5.

Lemma 7.7 *Let* Φ *be irreducible and* $w \in W$ *an involution. Then* $\Phi(w)$ *is of type* $r \cdot A_i + X_\ell$, *where either* $X_\ell = \emptyset$ *or one of* $B_\ell(\ell \geq 1)$, $C_\ell(\ell \geq 1)$, $D_{2\ell}(\ell \geq 1)$, E_7, E_8, F_4 *or* G_2.

In order to be able to use induction, we will have to look at orthogonal complements. For a subset R of Φ let $R^\perp = \{\alpha \in \Phi \mid (\alpha, \beta) = 0 \text{ for all } \beta \in R\}$. Clearly $R^\perp$ is a closed subsystem of Φ. If $w \in W$ is an involution and $\Delta(w)$ a basis of $\Phi(w)$, then $\Phi(w)^\perp = \Delta(w)^\perp$. Moreover $\Phi(w)^\perp = \Phi(-w)$.

Lemma 7.8 *Let Φ be irreducible, Δ a basis of Φ and $w \in W$ a Δ-standard involution. Let Δ_1 be an irreducible component of $\Delta(w)$. Then $\Phi(\Delta_1^\perp)$ is given in Table I.*

Proof: Extend the Dynkin diagram of (Φ, Δ) with the highest long and short root. (See [5]). Using this diagram and the fact that each root of Φ is conjugate to either the highest long or the highest short root, the orthogonal complements in Table I are easily derived. For E_8 note that there exists a subsystem of type D_8.

In Table I the subsystems A_{-1}, A_0 and D_1 all denote an empty subsystem of Φ.

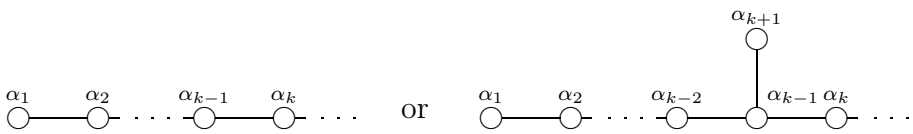
Table I

Type Φ	Type Δ_1	Type $\Delta_1^\perp$	Type Φ	Type Δ_1	Type $\Delta_1^\perp$
A_ℓ	A_1	$A_{\ell-2}$	E_8	A_1	E_7
$(\ell \geq 1)$				D_4	D_4
B_ℓ	A_1	$A_1 + B_{\ell-2}$		D_6	$2 \cdot A_1$
$(\ell \geq 2)$	B_k $(k < \ell)$	$B_{\ell-k}$		E_7	A_1
C_ℓ	A_1	$A_1 + C_{\ell-2}$		E_8	$\emptyset$
$(\ell \geq 2)$	C_k $(k < \ell)$	$C_{\ell-k}$	F_4	A_1 $(short)$	B_3
D_ℓ	A_1	$A_1 + D_{\ell-2}$		A_1 $(long)$	C_3
$(\ell \geq 4)$	D_{2k} $(2k < \ell)$	$D_{\ell-2k}$		B_2	B_2
E_6	A_1	A_5		B_3	A_1 $(short)$
	D_4	$\emptyset$		C_3	A_1 $(long)$
E_7	A_1	D_6		F_4	$\emptyset$
	D_4	$3 \cdot A_1$	G_2	A_1 $(short)$	A_1 $(long)$
	D_6	A_1		A_1 $(long)$	A_1 $(short)$
	E_7	$\emptyset$		G_2	$\emptyset$

Lemma 7.9 *Let Φ be irreducible and $w \in W$ a involution. Then $\Phi(w)^\perp$ is of type $r \cdot A_i + X_\ell$, where either $X_\ell = \emptyset$ or X_ℓ irreducible.*

Proof: This result is immediate from Table I and Lemma 7.7.

Lemma 7.10 *Let Φ be irreducible, Δ a basis of Φ and $w \in W$ a Δ-standard involution. Let $\Delta_1 = \{\alpha\}$ be an irreducible component of $\Delta(w)$ of type A_1. Assume that the Dynkin diagram of Φ contains a subdiagram of the form:*



with α_1 an endpoint, $\alpha_k = \alpha$ and $\alpha_{k+1} \notin \Delta(w)$. Then there exists $r \in W$ such that $r(\Delta(w)) \subset \Delta$ and $\mathrm{r}(\alpha) = \alpha_1$.

Proof: Let $R = \{\alpha_1, \cdots, \alpha_k\} \subset \Delta$ be the chain connecting α_1 and $\alpha = \alpha_k$. We will show that there exists $r \in W(\Phi(R))$ such that $r(\Delta(w)) \subset \Delta$ and $r(\alpha) = \alpha_1$. Since $W(\Phi(R))$ acts trivially on $\Delta(w) - (R \cap \Delta(w))$, we may assume that $\Phi = \Phi(R)$ of type A_k. We will prove the result in a few steps:
(1) If $\beta \in \Delta(w)$, $\beta = \alpha_r \neq \alpha$ and $\alpha_{r+1}, \cdots, \alpha_{k-1} \notin \Delta$, then there exists $r_1 \in W$ such that $r_1(\alpha) = \beta$, $r_1(\beta) = \alpha$ and $r_1(\alpha_i) = \alpha_i$ for $i = 1, \cdots, r-2$.

In the substystem Δ_2 spanned by $\alpha_{r+2}, \cdots, \alpha_k$ there exists $r_2 \in W(\Phi(\Delta_2))$ such that $r_2(\alpha_r) = \alpha_{k+2}$ and clearly $r_2(\alpha_i) = \alpha_i$ for $i = 1, \cdots, r$. So we may assume that $\Delta_3 = \{\alpha, \alpha_{r+1}, \beta\}$ span a subsystem of type A_3. Let $r_3 \in W(\Phi(\Delta_3))$ be such that $r_3(\alpha) = \beta$. then $r_3(\beta) = \pm\beta$. So either $r = r_3 r_2$ or $r = s_\beta r_3 r_2$ satisfies the conditions in (1).
(2) We may assume $\Delta(w) = \alpha$. Namely, using (1) we can exchange the position of α and any other root in $\Delta(w)$, left of α in the diagram. If $\alpha_2 \in \Delta(w)$, then exchange first α_1 and α_2 (in the subsystem of type A_2). Then apply (1). Finally restricting to the subsystem containing both α and α_1, the statement (2) follows.
(3) α and α_1 are conjugate under W. Since $|\alpha| = |\alpha_1|$ this statement is clear, hence the result follows.

Proposition 7.11 *Let Φ be irreducible not of type $D_\ell (\ell \geq 4)$, Δ a basis of Φ, w a Δ-standard involution of W and $\Delta_0 \subset \Delta(w)$ an irreducible component. Then $\Delta(w) - \Delta_0$ is contained in the unique irreducible component of $\Delta_0^\perp$ of the highest rank.*

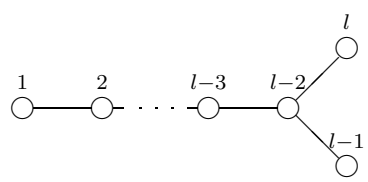
Proof: (1) Assume first that $\Delta_0 = \{\alpha\}$ of type A_1. From Lemma 7.10 it follows that we may assume that α is an endpoint of the Dynking diagram (or a highest long or short root). Since $\Delta(w) - \Delta_0$ is contained in $\Delta_0^\perp \cap \Delta$, the result follows now from Table I.

(2) If Δ_0 is not of type A_1, then, since $\Delta(w) - \Delta_0 \subset \Delta_0^\perp \cap \Delta$, the result is also immediate from Table I.

Corollary 7.12 *Let Φ be of type D_ℓ ($\ell \geq 5$) and Δ, w as in 7.11. If $\Delta(w)$ is of type $r \cdot A_1 + D_{2k}$ ($k \geq 1$), then $\Delta(w) - \Delta_0$ is contained in the unique irreducible component of $\Delta_0^\perp$ of the highest rank.*

The result follows with a similar argument as in 7.11.

7.13 Remarks: (i) Let Φ be of type D_ℓ ($\ell \geq 5$) and $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ a basis of Φ, where α_i correspond to the ith node in the Dynkin diagram.



(6)

If $\Delta_0 = \{\alpha_{\ell-1}\}$ (i.e., one of the endnodes in the subsystem of type D_2), then $\alpha_{\ell-1}^\perp \cap \Delta = \{\alpha_1, \ldots, \alpha_{\ell-3}, \alpha_\ell\}$. So the statement in 7.12 does not hold.
(ii) If Φ is of type D_4, then $\alpha^\perp$ is of type $A_1 + A_1 + A_1$ for any $\alpha \in \Phi$. However the classification of the parabolic subsets related to involutions in W is rather simple. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of Φ as in 7.12 (i). Then we have the following possibilities for $\Delta(w)$:

(1) $\Delta(w) = \Delta$ of type D_4. Then $w = -id$.

(2) $\Delta(w)$ of type A_1. Since all roots of Φ are W-conjugate there is only one conjugacy class.

(3) $\Delta(w)$ of type $3.A_1$. Then $\Delta(w) = \{\alpha_1, \alpha_3, \alpha_4\}$ and there is only one parabolic subsystem of this type.

(4) $\Delta(w)$ of type $2.A_1$. There are 3 possibilities, $\Delta_1 = \{\alpha_1, \alpha_3\}$, $\Delta_2 = \{\alpha_1, \alpha_4\}$ and $\Delta_3 = \{\alpha_3, \alpha_4\}$. Now Δ_1, Δ_2 and Δ_3 are not W-conjugate, but they are clearly $\mathrm{Aut}(\Phi)$-conjugate.

Proposition 7.14 *Let Φ be irreducible, not of type D_4, Δ a basis of Φ and let w_1, w_2 be Δ-standard involutions in W. Assume that $\Delta(w_i)$ is of type $r_i A_1 + X_{k_i}$ with X_{k_i} as in 7.7 $(i = 1, 2)$. Then we have the following.*

(1) *If* $X_{k_i} \neq \emptyset$ $(i = 1, 2)$, *then* $\Delta(w_1)$ *and* $\Delta(w_2)$ *are* W- (*resp.* $\mathrm{Aut}(\Phi)$-) *conjugate if and only if* $r_1 = r_2$ *and* $X_{k_1} = X_{k_2}$.

(2) *If* $X_{k_i} = \emptyset$ $(i = 1, 2)$ *and* Φ *not of type* $D_{2\ell}$, E_7, F_4 or G_2, *then* $\Delta(w_1)$ *and* $\Delta(w_2)$ *are* W- (*resp.* $\mathrm{Aut}(\Phi)$-) *conjugate if and only if* $r_1 = r_2$.

Proof: (1) The "if" statement being clear, assume $r_1 = r_2$ and $X_{k_1} = X_{k_2}$. Let $\Delta_0 = \Delta(w_1) \cap \Delta(w_2)$. Since $X_{k_1} \neq \emptyset$ we have $\Delta_0 \neq \emptyset$. By 7.10 and 7.11 $\Delta(w_1) - \Delta_0$ and $\Delta(w_2) - \Delta_0$ are both contained in the same irreducible component of $\Delta_0^\perp$. The result follows now with induction.
(2) Let $\Delta_1 = \{\alpha\}$ resp. $\Delta_2 = \{\beta\}$ be irreducible components of $\Delta(w_1)$ resp. $\Delta(w_2)$. Since Φ is not of type F_4 or G_2, we have $|\alpha| = |\beta|$. Moreover, since $\Delta(w_i)$ has no irreducible component of type D_2 $(i = 1, 2)$, it follows from Lemma 7.10 that we may assume that $\alpha = \beta$ is an endpoint of the Dynkin diagram. But then the result follows from 7.10 and 7.11.

Lemma 7.15 *Let* Φ *be irreducible of type* D_ℓ $(\ell > 4)$, Δ *a basis of* Φ *and let* w_1, w_2 *be* Δ*-standard involutions of* Φ. *If* $\Delta(w_1)$ *is of type* $r.A_1$ *and* $\Delta(w_2)$ *is of type* $(r-2).A_1 + D_2$ $(r \geq 2)$, *then* $\Delta(w_1)$ *and* $\Delta(w_2)$ *are not* $\mathrm{Aut}(\Phi)$*-conjugate.*

Proof: Using 7.10 and 7.12 it follows that it suffices to prove this in the case that $\Delta(w_1) = \{\alpha, \beta\}$ of type $A_1 + A_1$ and $\Delta(w_2) = \{\alpha_{\ell-1}, \alpha_\ell\}$ of type D_2. From 7.9 it follows that $\Delta(w_1)^\perp$ is of type $D_{\ell-2}$ and $\Delta(w_2)^\perp$ is of type $D_{\ell-4} + 2 \cdot A_1$. If $\phi \in \mathrm{Aut}(\Phi)$ such that $\phi(\Delta(w_1)) = \Delta(w_2)$, then also $\phi(\Delta(w_1)^\perp) = \Delta(w_2)^\perp$ which is impossible. This proves the result.

Proposition 7.16 *Let* Φ *be irreducible of type* $D_{2\ell}$ $(\ell > 2)$, $\Delta = \{\alpha_1, \cdots, \alpha_{2\ell}\}$ *a basis of* Φ *as in* 7.12(i) *and let* w_i *be a* Δ*-standard involution of* Φ *of type* $r_i.A_1$ $(i = 1, 2)$. *Then we have the following:*

(1) *If* $\ell > r_i$ $(i = 1, 2)$, *then* $\Delta(w_1)$ *and* $\Delta(w_2)$ *are* W- (*resp.* $\mathrm{Aut}(\Phi)$-) *conjugate if and only if* $r_1 = r_2$.

(2) *If* $\ell = r_1 = r_2$, *then* $\Delta_1 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2\ell-3}, \alpha_{2\ell-1}\}$ *and* $\Delta_2 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2\ell-3}, \alpha_{2\ell}\}$ *are* $\mathrm{Aut}\ \Phi$*-conjugate but not* W*-conjugate.*

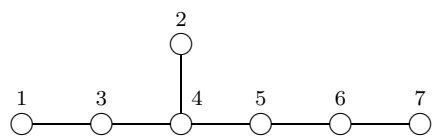
Proof: (1) follows with a similar argument as 7.14(ii).
(2) Clearly Δ_1 and Δ_2 are conjugate by a diagram automorphism in $\mathrm{Aut}(\Phi, \Delta)$. If $w \in W$ is such that $w(\Delta_1) = \Delta_2$, then using 7.10 we may assume that $w(\alpha_i) = \alpha_i$ for $i = 1, \cdots, 2r-3$. So by 7.12 we can restrict to the case that Φ is of type D_4. But then, similarly as in 7.13 (ii), Δ_1 and Δ_2 are not W-conjugate. This proves the result.

Proposition 7.17 *Let Φ be irreducible, Δ a basis of Φ and let w_1, w_2 be Δ-standard involutions of Φ with $\Phi(w_i)$ of type $r_i.A_1$ $(i = 1, 2)$. Then we have the following.*

(1) *If Φ is of type E_7 or F_4, then w_1 and w_2 are W- (resp. $\mathrm{Aut}(\Phi)$-) conjugate if and only if $r_1 = r_2$ and type $\Phi(-w_1) =$ type $\Phi(-\mathrm{w}_2)$.*

(2) *If Φ is of type G_2, then w_1 and w_2 are W- (resp. $\mathrm{Aut}(\Phi)$-) conjugate if and only if $\Delta(w_1) = \Delta(w_2)$.*

Proof: (2) being clear, it suffices to prove (1). Assume first that Φ is of type E_7, $\Delta(w_i)$ is of type $3.A_1$ $(i = 1, 2)$ and $\Delta = \{\alpha_1, \cdots, \alpha_7\}$ as below.



Using 7.10 it follows that $\Delta(w_i)$ $(i = 1, 2)$ is conjugate to $\Delta_1 = \{\alpha_1,\ \alpha_5,\ \alpha_7\}$ or $\Delta_2 = \{\alpha_2,\ \alpha_5,\ \alpha_7\}$. Since $\Delta_1^\perp$ is of type $4.A_1$ and $\Delta_2^\perp$ is of type D_4 the result follows.

If Φ is of type F_4 the result can be shown using a similar argument as above.

For Φ irreducible and $w \in W$ an involution, the types of $\Phi(w)$ and $\Phi(-w)$ are given in Table II.

7.18 Remarks: Similarly as in 6.2 let $\mathcal{W}$ denote the set of all W-conjugacy classes of involutions in W. Define an order $>$ on $\mathcal{W}$ as in 6.4. Then from 2.20 it follows that for $[w_1], [w_2] \in \mathcal{W}$ we have $[w_1] > [w_2]$ if and only if $\Delta(w_1) \subset \Delta(w_2)$ for some representatives w_i of $[w_i]$ $(i = 1, 2)$

The above results lead to the following classification of involutions in W:

Theorem 7.19 *Let Φ be irreducible, Δ a basis of Φ, $w \in W$ a Δ-standard involution of type $r \cdot A_1 + X_\ell$ with X_k as in 7.7. Then we have the following:*

(1) *The* $\mathrm{Aut}(\Phi)$*-conjugacy class of* w *is completely determined by the type of* $\Phi(w)$ *except in the cases* $E_7,\ r=3,\ \ X_k=\emptyset$*;* $F_4,\ r=1,\ X_k=\emptyset$*;* $G_2,\ r=1,\ X_k=\emptyset$ *where there are* 2 *conjugacy classes of the same type. In the cases of* E_7 *and* F_4 *the conjugacy classes are completely determined by the type of* $\Phi(w)$ *and* $\Phi(-w)$*.*

(2) *The* $\mathrm{Aut}(\Phi)$*-conjugacy class of* w *and the* W*-conjugacy class of* w *are the same, except in the cases* $D_4,\ r=2,\ X_k=\emptyset$ *and* $D_{2\ell}$ $(\ell\geq 3),\ r=\ell,\ X_k=\emptyset$*, where there are respectively* 3 *and* 2 W*-conjugacy classes of the same type.*

A list of the possible types together with the type of the diagram is given in Table II, while some of the diagrams $\mathcal{L}(\Phi)$ *are given in Table III.*

Proof: The result is immediate from 7.13, 7.14, 7.15, 7.16 and 7.17.

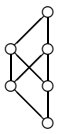
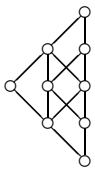
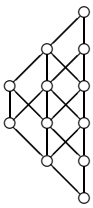
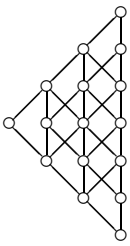
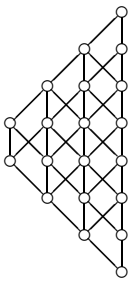
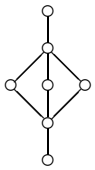
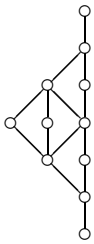
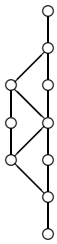
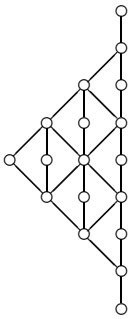
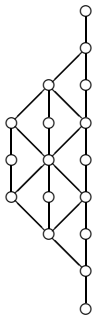
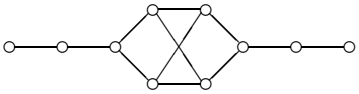
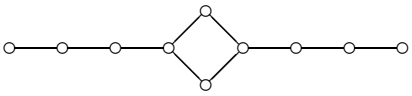
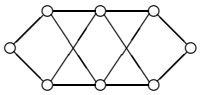
Remark: If Φ is irreducible and w_0 a maximal involution of (Φ,Δ), then $\Phi(w_0)$ is irreducible or of type $r\cdot A_1$ ($r=\mathrm{rank}\ \Phi(\mathrm{w}_0)$). Moreover $\mathcal{L}(w_0)=\mathcal{L}(-id)=\mathcal{L}(\Phi)$ and the diagrams $\mathcal{L}(\Phi)$ and $\mathcal{L}^*(\Phi)$ (see 6.5) are the same, except for $D_{2\ell}$ $(\ell\geq 2)$. In this case we will also write $D^*(2\ell)$ for $\mathcal{L}^*(\Phi)$.

If $\Phi(w_0)$ is of type $r\cdot A_1$, then we will also write $A(r)$ for the diagram $\mathcal{L}(\Phi)$. Since the diagrams for B_ℓ and C_ℓ $(\ell\geq 2)$ are identical, we will denote both of them by $B(\ell)$. Similarly the diagram $\mathcal{L}(\Phi)$ for $D_{2\ell+1}$ $(\ell\geq 2)$ is identical to the diagram $\mathcal{L}^*(\Phi)$ for $D_{2\ell}$ and will be denoted by $D^*(2\ell)$. For $E_7,\ E_8$ and F_4 we will denote the diagram $\mathcal{L}(\Phi)$ by respectively $E(7),\ E(8)$ and $F(4)$. Finally the diagram for G_2 is identical to the diagram for B_2 and will also be denoted by $B(2)$.

Table II

Type Φ	Type $\Delta(w)$		Type $\Delta(w)^{\perp}$	Type $\mathcal{L}(\Phi)$
A_ℓ $(\ell \geq 2)$	$r \cdot A_1$	$0 < 2r \leq \ell + 1$	$A_{\ell-2r}$	$A(r)$
B_ℓ $(\ell \geq 2)$	$rA_1 + B_k$	$0 < k + 2r \leq \ell$	$r \cdot A_1 + B_{\ell-k-2r}$	$B(\ell)$
C_ℓ $(\ell \geq 3)$	$r.A_1 + C_k$	$0 < k + 2r \leq \ell$	$r \cdot A_1 + C_{\ell-k-2r}$	$C(\ell)$
D_ℓ $(l \geq 4)$	$rA_1 + D_{2k}$	$0 < 2r + 2k \leq \ell$	$r \cdot A_1 + D_{\ell-2k-2r}$	$D(\ell)$ if ℓ even $D^*(\ell - 1)$ if ℓ odd
E_6	$r \cdot A_1$ D_4	$0 < r \leq 3$	$(4 - r) \cdot A_1$ $\emptyset$	$D^*(4) = A(4)$
E_7	A_1 $2 \cdot A_1$ $3 \cdot A_1$ $4 \cdot A_1$ D_4 $A_1 + D_4$ D_6 E_7		D_6 $A_1 + D_4$ D_4 $4 \cdot A_1$ $3 \cdot A_1$ $3 \cdot A_1$ $2 \cdot A_1$ A_1 $\emptyset$	$E(7)$
E_8	A_1 $2 \cdot A_1$ $3 \cdot A_1$ $4 \cdot A_1$ D_4 $A_1 + D_4$ D_6 E_7 E_8		E_7 D_6 $A_1 + D_4$ $4 \cdot A_1$ D_4 $3 \cdot A_1$ $2 \cdot A_1$ A_1 $\emptyset$	$E(8)$
F_4	A_1 $(short)$ A_1 $(long)$ $A_1 + A_1$ B_2 B_3 C_3 F_4		B_3 C_3 $A_1 + A_1$ B_2 A_1 $(short)$ A_1 $(long)$ $\emptyset$	$F(4)$
G_2	A_1 $(short)$ A_1 $(long)$ G_2		A_1 $(long)$ A_1 $(short)$ $\emptyset$	$B(2)$

Table III

A_l $(l \geq 1)$	$\mathcal{L}(A_l) = \mathcal{L}^*(A_l) = A(n)$ with $n = [\frac{l+1}{2}]$ $A(n)$:
B_l $(l \geq 2)$	$\mathcal{L}(B_l) = \mathcal{L}^*(B_l) = B(l)$ $B(2)$:  $B(3)$:  $B(4)$:  $B(5)$:  $B(6)$:  $B(7)$:  etc.
C_l $(l \geq 3)$	$\mathcal{L}(C_l) = \mathcal{L}^*(C_l) = B(l)$
D_{2l} $(l \geq 2)$	$\mathcal{L}(D_{2l}) = D(2l) \quad \mathcal{L}^*(D_{2l}) = D^*(2l)$ $D(4)$:  $D^*(4)$:  $D(6)$:  $D^*(6)$:  $D(8)$:  $D^*(8)$:  etc.
D_{2l+1} $(l \geq 2)$	$\mathcal{L}(D_{2l+1}) = \mathcal{L}^*(D_{2l+1}) = D^*(2l)$
E_6	$\mathcal{L}(E_6) = \mathcal{L}^*(E_6) = A(4)$
E_7	$\mathcal{L}(E_7) = \mathcal{L}^*(E_7) = E(7)$ $E(7)$: 
E_8	$\mathcal{L}(E_8) = \mathcal{L}^*(E_8) = E(8)$ $E(8)$: 
F_4	$\mathcal{L}(F_4) = \mathcal{L}^*(F_4) = F(4)$ $F(4)$: 
G_2	$\mathcal{L}(G_2) = \mathcal{L}^*(G_2) = B(2)$

7.20 Classification for θ not split

It is now easy to derive the θ-maximal involutions of Φ for the admissible involutions $\theta \in \mathrm{Aut}(\Phi)$ as given in [8, §4]. Without using the above classification we can prove the following characterization.

Lemma 7.21 *Let Φ be irreducible, $\theta \in \mathrm{Aut}(\Phi)$ an involution such that $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$, Δ a $(-\theta)$-basis of Φ, w_0 a θ-maximal Δ-standard involution of Φ. Then $\Delta(w_0)$ is of type $r \cdot A_1$ $(r \geq 0)$ or of type X_ℓ with X_ℓ one of $B_\ell(\ell \geq 1)$, $C_\ell(\ell \geq 1)$, $D_{2\ell}(\ell \geq 2)$, E_7, E_8, F_4, or G_2.*

The proof uses the fact that $\overline{\Phi}_\theta$ is irreducible (see 2.15) and a root connecting an irreducible component of type A_1 and the irreducible component of type X_ℓ, would project to a root in $\overline{\Phi}_\theta$, which length cannot occur. Since this result is also immediate from the classification of θ-maximal involutions, we omit the proof.

7.22 θ-maximal involutions

The types of the θ-maximal involutions are given in Table IV. For every θ-index as in [8, Table II] we give the type of $\Phi(w_0)$, with w_0 a θ-maximal involution and also the type of the diagram of $\mathcal{L}(w_0) = \mathcal{L}(\theta)$.

Note that θ is represented in the θ-index with respect to a θ-basis. So the black nodes correspond to $\Delta_0(\theta)$. On the other hand the θ-maximal involution w_0 is represented with respect to a $(-\theta)$-basis. One can easily deduce w_0 from the θ-diagram using the following observation:

$$\Phi(w_0) \subset \Delta_0(\theta)^\perp \tag{1}$$

and $\Delta_0(\theta)^\perp$ can be deduced in a similar way as in 7.8. In particular, if $\Delta_0(\theta)$ is of type $rA_1 + X_\ell$ as in 7.7 this follows immediately from Table I. We have now the following result:

Theorem 7.23 *Let Φ be irreducible, $\theta \in \mathrm{Aut}(\Phi)$ an admissible involution which is not split, Δ a $(-\theta)$-basis of Φ and $w_0 \in W$ a θ-maximal involution, which is Δ-standard. Then the W- (resp. $\mathrm{Aut}(\Phi)$-) conjugacy class of a (Δ, w_0)-standard involution $w \in W$ is completely determined by the type of $\Delta(w)$.*

This result is immediate from 7.19 and 7.20. The possible types of $\Delta(w)$ for the (Δ, w_0)-standard involutions of Φ follow from Table II and the type of the θ-maximal involution w_0 of Φ as in Table IV. The diagram of $(\mathcal{W}(\theta), \geq)$ is in these cases nothing else than $\mathcal{L}(w_0)$, which is given in Table IV.

Table IV

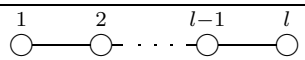
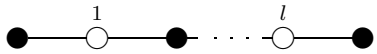
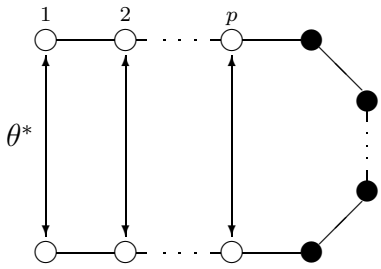
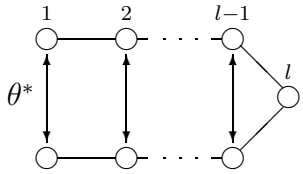
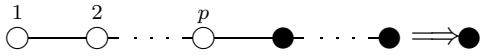
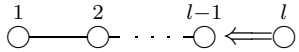
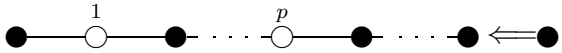
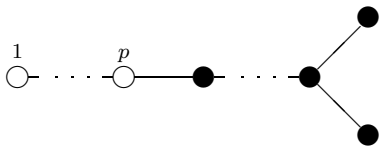
Type θ Cartan not.	θ-diagram	Type $\Phi(w_0)$	Type $\mathcal{L}(\theta)$
AI		$[\frac{l+1}{2}] \cdot A_1$	$A([\frac{l+1}{2}])$
AII		$\emptyset$	$A(0)$
$AIII_a$ (AIV $(p=1)$) $(1 \leq 2p \leq l)$		$p \cdot A_1$	$A(p)$
$AIII_b$ $(l \geq 2)$		$l \cdot A_1$	$A(l)$
BI (BII $(p=1)$) $(l \geq 2,\ 1 \leq p \leq l)$		B_p	$B(p)$
CI		C_l	$B(l)$
CII_a $(l \geq 3)$ $(1 \leq p \leq \frac{1}{2}(l-1))$		$p \cdot A_1$	$A(p)$
CII_b $(l \geq 2)$		$l \cdot A_1$	$A(l)$
DI_a (DII $(p=1)$) $(l \geq 4,\ 1 \leq p \leq l-1)$		D_p (p even) D_{p-1} (p odd)	$D^*(p)$ (p even) $D^*(p-1)$ (p odd)

Table IV

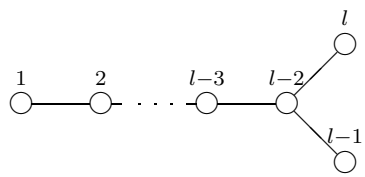
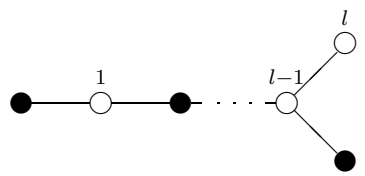
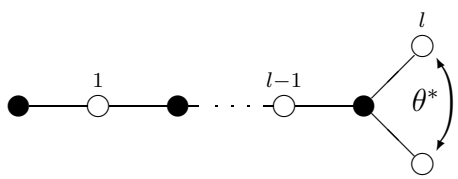
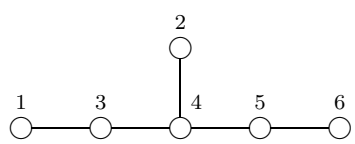
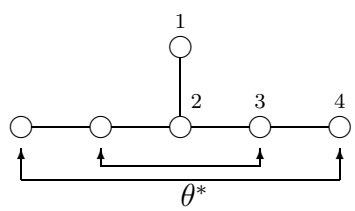
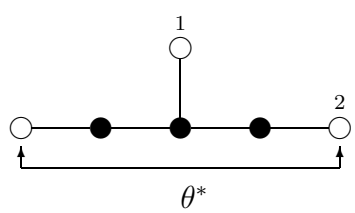
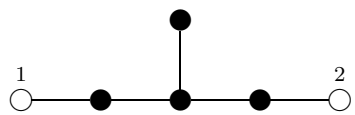
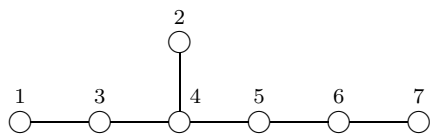
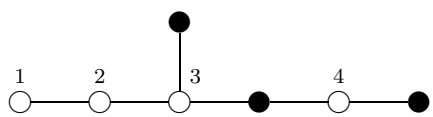
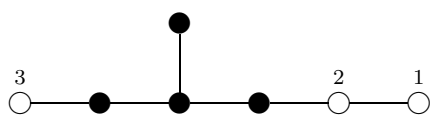
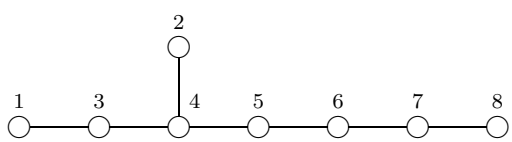
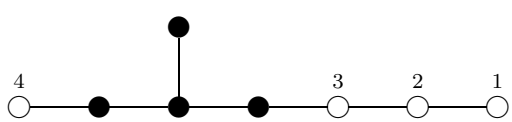
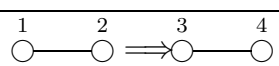
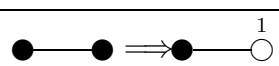
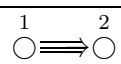
Type θ Cartan not.	θ-diagram	Type $\Phi(w_0)$	Type $\mathcal{L}(\theta)$
$D\mathrm{I}_b$ $(l \geq 4)$		D_l (l even) D_{l-1} (l odd)	$D(l)$ (l even) $D^*(l-1)$ (l odd)
$D\mathrm{III}_a$ $(l \geq 2)$		$l \cdot A_1$	$A(l)$
$D\mathrm{III}_b$ $(l \geq 2)$		$l \cdot A_1$	$A(l)$
$E\mathrm{I}$		D_4	$A(4)$
$E\mathrm{II}$		D_4	$A(4)$
$E\mathrm{III}$		$2 \cdot A_1$	$A(2)$

Table IV

Type θ Cartan not.	θ-diagram	Type $\Phi(w_0)$	Type $\mathcal{L}(\theta)$
EIV		$\emptyset$	$A(0)$
EV		E_7	$E(7)$
EVI		D_4	$A(4)$
EVII		$3 \cdot A_1$	$A(3)$
EVIII		E_8	$E(8)$
EIX		$D4$	$A(4)$
FI		F_4	$F(4)$
FII		A_1	$A(1)$
G		G_2	$B(2)$

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