

SYMMETRIC k -VARIETIES

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ABSTRACT. Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k , H a k -open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k -rational points of G (resp. H). The variety G_k/H_k is called a symmetric k -variety. For $k = \mathbb{R}$ and \mathbb{C} the representation theory of these varieties has been studied extensively. To study the representation theory over other fields, like local fields and finite fields, more needs to be known about their structure and geometry.

In this paper we discuss a number of recent results about symmetric k -varieties over fields, other than \mathbb{R} or \mathbb{C} . This includes a description of the orbits of minimal parabolic k -subgroups on G_k/H_k and also a partial classification of the k -involutions, together with all the fine structure of restricted root systems with Weyl groups and multiplicities.

1. INTRODUCTION

Symmetric varieties are the spherical homogeneous spaces G/H with G a reductive algebraic group and H the fixed point group of an involution of G . Symmetric k -varieties are the k -points of a symmetric variety, defined over a field k , which is not necessarily algebraically closed. These varieties occur in many areas. Examples include geometry (see [26] and [1]), singularity theory (see [31]) and the cohomology of arithmetic subgroups (this involves a study of \mathbb{Q} -involutions, see [35]). However, they are probably best known for their role in representation theory. The representation theory and Plancherel formulas of symmetric k -varieties over the real numbers (also called semisimple symmetric spaces) has been studied extensively in the last few decades. The first breakthrough was made when in the early fifties Harish-Chandra commenced his study of general semisimple Lie groups, which finally led to the Plancherel formula (published 1976). Before that in 1957, Harish-Chandra already found a Plancherel formula for any semisimple symmetric space G_k/H_k with H_k compact. (In this case no discrete series occur). The next step was to study the representation theory of the general semisimple symmetric spaces. For these most of the work was done in the late 70's and 80's by a number of

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mathematicians, including Faraut, Flensted-Jensen, Oshima, Sekiguchi, Matsuki, Schlichtkrul and van der Ban. Although a Plancherel formula has not been found yet, most of the work has been done.

The natural next step would be to study the representation theory of symmetric k -varieties over local fields. Until recently not much was known about their representation theory, but recently a number of promising results were obtained.

The representation theory of symmetric k -varieties is also of interest over base fields other than local fields. For example symmetric varieties over the complex numbers play an essential role in the study of Harish Chandra modules (see [36]). Symmetric k -varieties defined over a finite field were studied by Lusztig [22].

The representation theory of semisimple symmetric spaces depends heavily on the structure and geometry of these symmetric spaces. To study the representation theory of the symmetric k -varieties over local fields a lot of structure of these symmetric k -varieties will be needed. In this contribution we will give an exposition of what is known of the structure of symmetric k -varieties over an arbitrary base field of characteristic not 2. First we discuss a number of properties of the orbits of minimal parabolic k -subgroups on a symmetric k -variety. These orbits are of fundamental importance in the representation theory of these symmetric k -varieties (see [36] and [8, 25]).

The biggest part of this paper is devoted to the classification of the symmetric k -varieties together with their fine structure of restricted root systems with Weyl groups and multiplicities. The basic ideas behind this classification are as follows. Isomorphism classes of involutions for groups defined over algebraically closed fields were classified by reducing the problem to $W(T)$ -conjugacy classes of involutions normally related to a maximal torus T (see [12]). This means that there exists a map ρ from \mathcal{C} , the set of isomorphism classes of involutions of G , to $\mathcal{C}(T)$, the set of $W(T)$ -isomorphism classes of involutions of $(X^*(T), \Phi(T))$. Here T is a maximal torus of G , $X^*(T)$ the group of characters of T , $\Phi(T)$ the set of roots of T with respect to G and $W(T)$ the Weyl group of T with respect to G . In this case the map ρ is injective and the classification of isomorphism classes of involutions of G reduces to a classification of isomorphism classes of involutions θ of $(X^*(T), \Phi(T))$, which can be lifted to an involution $\tilde{\theta}$ of (G, T) , normally related to T (i.e. T contains a maximal $\tilde{\theta}$ -split torus of G). This characterization also gives all the fine structure of the symmetric varieties related to these involutions.

To characterize \mathcal{C}_k , the set of isomorphism classes of k -involutions of G , we can do something similar. Only this time, since we have k -involutions, we do not only characterize the involutions on a maximal torus, but also on a

maximal k -split torus A of G . Assume $T \supset A$ is a maximal k -torus of G and let $\mathcal{C}(T, A)$ denote the set of $W(T, A)$ -isomorphism classes of involutions of $(X^*(T), \Phi(T), \Phi(A))$. Then, as with the isomorphism classes of involutions of G , we get a map $\rho_k : \mathcal{C}_k \rightarrow \mathcal{C}(T, A)$, using the condition that the k -involutions θ of G have to be normally related to A and T (normally related to A meaning that A contains a maximal (θ, k) -split torus of G). The image of ρ_k consists of the $W(T, A)$ -isomorphism classes of involutions of $(X^*(T), \Phi(T), \Phi(A))$, which can be lifted to k -involutions of G , normally related to A and T . We call these admissible k -involutions of $(X^*(T), \Phi(T))$. Unfortunately the map ρ_k is in general not one-to-one. The fibers are characterized by quadratic elements and involutions of the anisotropic kernel. This means that the classification of k -involutions of G essentially reduces to the following 3 problems.

- (1) a classification of admissible k -involutions of $(X^*(T), \Phi(T))$.
- (2) a classification of k -involutions of k -anisotropic semisimple groups.
- (3) a classification of “quadratic elements” $a \in A/A^2$ (A a maximal (θ, k) -split torus of G) representing different isomorphism classes $\theta \text{Int}(a)$.

For G defined over the real numbers, a classification of the k -involutions of G , using this characterization, was given in [12]. The fine structure of symmetric k -varieties follows from the related admissible k -involutions of $(X^*(T), \Phi(T))$. A classification of the admissible k -involutions for G defined over the real numbers, p -adic numbers, finite fields and number fields is discussed in section 6.

Most of the results in this exposition about symmetric k -varieties come from [12, 13, 14, 15, 16, 17, 18, 20]. The results about the classification of semisimple k -groups are, with some modifications, from [34, 29].

A brief outline of this paper follows. In section 2 we set the notation and review some basic facts about symmetric k -varieties. This includes a discussion of the natural root system of a symmetric k -variety. Section 3 is devoted to a characterization of the orbits of minimal parabolic k -subgroups on the symmetric k -varieties. Part of this characterization consists of H -conjugacy classes of θ -stable maximal k -split tori of G . We discuss a characterization of these in terms of involutions of a root system.

The next 3 sections deal with the classification of k -involutions of G . In section 4 we discuss the actions of an involution θ and the Galois group Γ on $(X^*(T), \Phi(T))$ and combine these actions to characterize k -involutions of $(X^*(T), \Phi(T))$. All these actions can be represented by an index, which will be essential in the classification of the isomorphism classes of k -involutions of G . Section 5 is devoted to a characterization of the isomorphism classes of k -involutions of G . Also the classifications of semisimple k -groups and involutions over algebraically closed fields are reviewed briefly. In section 6 a

classification of the admissible k -involutions of (X, Φ) is discussed. It is shown that these isomorphism classes can be represented by a (Γ, θ) -index. The admissible (Γ, θ) -indices are classified for k the real numbers, p -adic fields, finite fields and number fields.

We conclude with some general remarks about the representation theory of symmetric k -varieties over local fields.

2. STRUCTURE OF SYMMETRIC k -VARIETIES

2.1. Preliminaries. In this subsection we set the notations and recall a few results from [20], [13] and [15]. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups. Our basic reference for reductive groups will be [5].

Given an algebraic group G , the identity component is denoted by G^0 . We use $L(G)$ (resp. \mathfrak{g} , the corresponding lower case German letter) for the Lie algebra of G . If H is a subset of G , $N_G(H)$ (resp. $Z_G(H)$) is the normalizer (resp. centralizer) of H in G . We write $Z(G)$ for the center of G . The commutator subgroup of G is denoted by $D(G)$ or $[G, G]$.

An algebraic group defined over k shall also be called an algebraic k -group. For an extension K of k , the set of K -rational points of G is denoted by G_K or $G(K)$. If G is an algebraic k -group and Y a subset of G , then we will denote the Zariski closure of Y in G by $cl(Y)$.

If G is a reductive k -group and A a torus of G then we denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of A (resp. one-parameter subgroups of A) and by $\Phi(G, A)$ (resp. $\Phi^\vee(G, A)$) the set of the roots (resp. coroots) of A in G . Let $W(G, A) = N_G(A)/Z_G(A)$ denote the Weyl group of G relative to A . If $\alpha \in \Phi(G, A)$, then let U_α denote the unipotent subgroup of G corresponding to α . If A is a maximal torus, then U_α is one-dimensional. Given a quasi-closed subset ψ of $\Phi(G, A)$, the group G_ψ (resp. G_ψ^*) is defined in [5, 3.8]. If G_ψ^* is unipotent, ψ is said to be unipotent and often one writes U_ψ for G_ψ^* .

2.2. k -involutions. Let G be a connected k -group and θ a k -automorphism of G of order two. Its fixed point group $G_\theta = \{g \in G \mid \theta(g) = g\}$ is a k -group, which is reductive if G is reductive. If G is semisimple and simply connected, then G_θ is connected, but in general G_θ is not necessarily connected. The k -automorphism θ will also be called a k -involution of G . To determine, whether an involution of G is a k -involution, it suffices to look at the fixed point group (see [20, 1.6]).

Proposition 2.2.1. *Let G be a connected semi-simple algebraic k -group and θ an involution of G . Then θ is defined over k if and only if G_θ^0 is defined over k .*

2.2.2. If G is reductive and H a k -open subgroup of G_θ , then we call the variety G/H a *symmetric variety* and the variety G_k/H_k a *symmetric k -variety*. The symmetric k -varieties over the real numbers are also called affine symmetric spaces. Symmetric varieties are spherical.

Given $g, x \in G$, the *twisted action* associated to θ is given by $(g, x) \mapsto g * x = gx\theta(g)^{-1}$. Let $Q = \{g^{-1}\theta(g) \mid g \in G\}$ and $Q' = \{g \in G \mid \theta(g) = g^{-1}\}$. The set Q is contained in Q' . Both Q and Q' are invariant under the twisted action associated to θ . There are only finite number of twisted G -orbits in Q' and each such orbit is closed (see [27]). In particular, Q is a connected closed k -subvariety of G . Define a morphism $\tau : G \rightarrow G$ by

$$(1) \quad \tau(x) = x\theta(x^{-1}), \quad (x \in G).$$

Then $Q = \tau(G)$ is a closed k -subvariety of G and $\tau(x) = \tau(y)$ if and only if $k = x^{-1}y \in G_\theta$. The morphism τ induces an k -isomorphism of the coset space G/G_θ onto Q . We note that $\theta(\tau(x)) = \tau(x)^{-1}$ for $x \in G$. This map is essential in the study of involutions of reductive algebraic groups and their symmetric k -varieties.

2.3. θ -stable k -tori. Let G be a connected reductive algebraic k -group and θ an k -involution of G . In this subsection we discuss θ -stable k -tori and the various restricted root systems associated with these symmetric varieties. First some notation.

2.3.1. Let T be a θ -stable torus of G . Write $T_\theta^+ = (T \cap G_\theta)^0$ and $T_\theta^- = \{x \in T \mid \theta(x) = x^{-1}\}^0$. Then it is easy to verify that the product map

$$\mu : T_\theta^+ \times T_\theta^- \rightarrow T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular $T = T_\theta^+ T_\theta^-$ and $T = T_\theta^+ \cap T_\theta^-$ is a finite group. The automorphisms of $\Phi(G, T)$ and $W(G, T)$ induced by θ will also be denoted by θ . The natural tori associated with the symmetric variety G/H are:

Definition 2.3.2. A torus A of G is called *θ -split* if $\theta(a) = a^{-1}$ for every $a \in A$.

The set of roots of a maximal θ -split torus gives a root system, which is the natural root system of a symmetric variety (see [27]). If $k = \mathbb{R}$, then G/H is the complexification of a Riemannian symmetric space. Therefore much of the fine structure of the symmetric varieties is similar to that of the Riemannian symmetric spaces. Similarly much of the fine structure of

the general symmetric k -varieties is similar to that of the affine symmetric spaces. There are again 3 restricted root systems. On the one hand one has the restricted root system of the symmetric variety G/H , which is the root system of a maximal θ -split torus S of G . On the other hand one has the restricted root system of G_k , which is the root system of a maximal k -split torus A_0 of G . The root system associated with the symmetric k -variety G_k/H_k is the root system of the following tori.

Definition 2.3.3. A k -torus A of G is called (θ, k) -split if it is both θ -split and k -split.

The set of roots of a maximal (θ, k) -split torus A_0 give a root system, which is the natural root system of a symmetric k -variety (see [20]). Since we can embed A_0 in a maximal k -split torus A and also in a maximal θ -split torus S , the set of root of A_0 are the restrictions of the roots of $\Phi(A)$ to A_0 and the restrictions of roots of $\Phi(S)$ to A_0 . From the following result it follows that all three these tori can be chosen inside a maximal torus of G (see [20]).

Proposition 2.3.4. *Let A_0 be a maximal (θ, k) -split torus of G , $A \subset Z_G(A_0)$ a maximal k -split torus, $S \supset A_0$ a maximal θ -split torus and let C , M_1 , M_2 denote the central, anisotropic and isotropic factors of $Z_G(A_0)$ over k respectively. Then A is θ -stable, $CM_1 \subset Z_G(A)$ and $M_2 \subset H$. Moreover there exists a maximal k -torus T of G , such that $T \supset A$ and $T \supset S$.*

This result is essential in the classification of the symmetric k -varieties with their restricted root systems and Weyl groups (see section 5).

Remark 2.3.5. The natural parabolic k -subgroups associated with the (θ, k) -split tori are the θ -split parabolic k -subgroups. These θ -split parabolic k -subgroups are of importance in the representation theory of the symmetric k -varieties. For example, for $k = \mathbb{R}$, the representations induced from a parabolic k -subgroup P contribute to the "Plancherel decomposition of $L^2(G_k/H_k)$ if P is a θ -split parabolic k -subgroup. The contributions to the most "continuous part" of the "Plancherel decomposition" come from minimal θ -split parabolic k -subgroups (see [11]).

The minimal θ -split parabolic k -subgroups of G are conjugate under H , but not necessarily conjugate under H_k . Similarly, the maximal (θ, k) -split tori are not conjugate under H_k . The best we can do is the following result, which depends on an analyses of the orbits of minimal parabolic k -subgroups on the symmetric k -varieties (see section 3).

Proposition 2.3.6. *Let A_1 and A_2 be maximal (θ, k) -split tori of G and $A \supset A_1$ a maximal k -split torus of G . Then there exists $g \in (H^0 Z_G(A))_k$ such that $gA_1g^{-1} = A_2$. In particular all maximal (θ, k) -split tori of G are conjugate under H .*

Remark 2.3.7. If $k = \mathbb{R}$, the maximal (θ, k) -split tori of G are conjugate under H_k , due to the conjugacy of the compact real forms of G . For other base fields, this is in general not true. For more details see [20, §4].

3. ORBITS OF PARABOLIC SUBGROUPS ON SYMMETRIC k -VARIETIES

In the study of symmetric k -varieties the orbits under the action of minimal parabolic k -subgroups play an essential role. In this section we will review a few facts about these orbits. Most of the results in this section are from [20] and [14].

3.1. Orbits of parabolic k -subgroups on symmetric k -varieties. Let G be a reductive k -group, θ an involution of G defined over k and H a k -open subgroup of G_θ . Define the morphism $\tau : G \rightarrow G$ as in (1). Let P be a minimal parabolic k -subgroup of G , A a θ -stable maximal k -split torus of P , $N = N_G(A)$, $Z = Z_G(A)$ and $W = W(A) = N_G(A)/Z_G(A)$ the corresponding Weyl group. The key lemma in the study of the orbits of parabolic k -subgroups on symmetric k -varieties is the following result (see [20, 2.6]).

Lemma 3.1.1. *Every minimal parabolic k -subgroup P of G contains a θ -stable maximal k -split torus, unique up to conjugation by an element from $(H \cap R_u(P))_k$.*

Using this result we can reduce the orbit decomposition to a question about θ -stable maximal k -split tori and their Weyl groups.

Proposition 3.1.2. *If $g \in G_k$ satisfies $\theta(g) = g^{-1}$, then there exists $x \in U_k$ such that $xg\theta(x)^{-1} \in N_G(A)$.*

3.1.3. The P_k orbits on G_k/H_k can be described now as follows. Set $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_k\}$. The group $Z_k \times H_k$ acts on \mathcal{V}_k by $(x, z)y = xyz^{-1}$, $(x, z) \in Z_k \times H_k$, $y \in \mathcal{V}_k$. Let V_k be the set of $(Z_k \times H_k)$ -orbits on \mathcal{V}_k . If $v \in V_k$, we let $x(v) \in \mathcal{V}_k$ be a representative of the orbit v in \mathcal{V}_k . The set V_k characterizes the double cosets (see [20]).

Theorem 3.1.4. *The inclusion map $\mathcal{V}_k \rightarrow G_k$ induces a bijection of the set V_k of $(Z_k \times H_k)$ -orbits on \mathcal{V}_k onto the set of $(P_k \times H_k)$ -orbits on G_k . In particular G_k is the disjoint union of the double cosets $H_k v P_k$, $v \in V_k$.*

For algebraically closed fields this result is due to Springer [32]. For $k = \mathbb{R}$ these double cosets were characterized in a slightly different way (see [23] and [28]). It is not hard to derive this characterization from the above result.

Corollary 3.1.5. *Let $\{A_i \mid i \in I\}$ be representatives of the H_k -conjugacy classes of θ -stable maximal k -split tori in G . Then*

$$H_k \backslash G_k / P_k \cong \bigcup_{i \in I} W_{H_k}(A_i) \backslash W_{G_k}(A_i)$$

Remark 3.1.6. In a number of cases one can show that there are only finitely many orbits. If k is algebraically closed, the finiteness of V_k was proved by Springer [32]. The finiteness of the orbit decomposition for $k = \mathbb{R}$ was discussed by Wolf [40], Rossmann [28] and Matsuki [23]. For general local fields this result can be found in Helminck-Wang [20]. However in most cases the set V_k is not finite, as can be seen from the following example.

Example 3.1.7. Let $G = SL(2)/\mathbb{Q}$, $\theta(x) = {}^t x^{-1}$, B = the Borel subgroup of upper triangular matrices and A the group of diagonal matrices. Then $\tau(V_{\mathbb{Q}})$ coincides with the set consisting of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a = x^2 + y^2$, $(x, y) \in \mathbb{Q}^2 - \{(0, 0)\}$. One readily checks that $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ are in the same twisted $B_{\mathbb{Q}}$ orbit if and only if $a^{-1}b \in (\mathbb{Q}^*)^2$. So it follows that $V_k \cong \bigoplus_{p \equiv 1(4)} \mathbb{Z} / 2\mathbb{Z}$.

3.1.8. *$P \times H$ orbits on G .* There is a natural inclusion map of $P_k \times H_k$ orbits on G_k onto $P \times H$ orbits on G . This map is of importance in the study of the $P_k \times H_k$ orbits on G_k . It basically maps $Z_k \times H_k$ orbits on \mathcal{V}_k to $Z \times H$ orbits on the set $\mathcal{V} = \{x \in G \mid \tau(x) \in N\}$. Similar as in 3.1.3 the group $Z \times H$ acts on \mathcal{V} by $(x, z)y = xyz^{-1}$, $(x, z) \in Z \times H$, $y \in \mathcal{V}$. Let V be the set of $(Z \times H)$ -orbits on \mathcal{V} . If $v \in V$, we let $x(v) \in \mathcal{V}$ be a representative of the orbit v in \mathcal{V} . The set V is related to the set of $H \times P$ orbits on G (see [15]). The inclusion map $\mathcal{V}_k \rightarrow \mathcal{V}$ induces a map $\eta : V_k \rightarrow V$, where η maps the orbit $Z_k g H_k$ onto $Z g H$.

3.2. **Action of W on V_k .** In this subsection we discuss the action of W on V_k and V . We use the same notion as in the previous subsection. In particular let P, A be as in 3.1. There is a natural (left) action of the Weyl group $W = W(A)$ on V_k and V , which is defined as follows.

3.2.1. Let $v \in V$ and let $x = x(v)$. If $n \in N$, then $nx \in \mathcal{V}$ and its image in V depends only on the image of n in W . We thus obtain a (left) action of W on V , denoted by $(w, v) \rightarrow w \cdot v$ ($w \in W$, $v \in V$). The action of W on V_k is defined similarly. The W orbits on V_k and V correspond with conjugacy classes of maximal (quasi) k -split tori, what can be seen as follows.

Let \mathcal{A}_k denote the set of maximal k -split tori of G and let \mathcal{A}_k^θ be the fixed point set of θ i.e., the set of θ -stable maximal k -split tori. If $x \in \mathcal{V}_k$, then xAx^{-1} is a θ -stable maximal k -split torus and conversely any θ -stable maximal

k -split tori in \mathcal{A}_k^θ can be written as xAx^{-1} for some $x \in \mathcal{V}_k$. The group H_k acts on \mathcal{A}_k^θ by conjugation. If $v \in V_k$, then $x(v)^{-1}Ax(v) \in \mathcal{A}_k^\theta$. This determines a map of V_k to the orbit set \mathcal{A}_k^θ/H_k . Since this map is constant on W -orbits, we also get a map of orbit sets

$$(2) \quad \gamma_k : V_k/W \rightarrow \mathcal{A}_k^\theta/H_k.$$

3.2.2. For V we can give a similar characterization as for V_k using quasi k -split tori instead of k -split tori. Recall that a torus A_1 of G is called a *quasi k -split torus* if A_1 is conjugate to a k -split torus of G . All maximal quasi k -split tori of G are conjugate, because all maximal k -split tori of G are conjugate. Let \mathcal{A} be the variety of maximal quasi k -split tori of G . This is an affine variety, isomorphic to $G/N_G(A)$, on which θ acts. Let \mathcal{A}^θ be the fixed point set of θ , i.e. the set of θ -stable maximal quasi k -split tori. It is an affine variety on which H acts by conjugation. If $v \in V$, then $x(v)^{-1}Ax(v) \in \mathcal{A}^\theta$. This determines a map of V to the orbit set \mathcal{A}^θ/H , which is constant on W -orbits. So we get a map of orbit sets

$$(3) \quad \gamma : V/W \rightarrow \mathcal{A}^\theta/H.$$

We have now the following characterization of the W orbits in V_k and V (see [20]).

Proposition 3.2.3. *Let G , \mathcal{A}^θ , \mathcal{A}_k^θ , γ and γ_k be as above. Then we have the following.*

- (i) $\gamma : V/W \rightarrow \mathcal{A}^\theta/H$ is bijective.
- (ii) $\gamma_k : V_k/W \rightarrow \mathcal{A}_k^\theta/H_k$ is bijective.

3.2.4. Let \mathcal{A}_0 denote the set of θ -stable quasi k -split tori of G , which are H -conjugate to a θ -stable maximal k -split torus. Then $\mathcal{A}_0/H \subset \mathcal{A}^\theta/H$ is the set of H -conjugacy classes of θ -stable maximal k -split tori of G . There is a natural map

$$(4) \quad \zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H,$$

sending the H_k conjugacy class of a θ -stable maximal k -split torus onto its H conjugacy class. \mathcal{A}_0/H is the image of ζ . On the other hand the inclusion map $\mathcal{V}_k \rightarrow \mathcal{V}$ induces a map $\eta : V_k \rightarrow V$, where η maps the orbit $Z_k g H_k$ onto ZgH . This map is W -equivariant. Denote the corresponding orbit map by $\delta : V_k/W \rightarrow V/W$ and write V_0 for the image of η in V . Denote the restriction of γ to V_0/W by γ_0 . Then γ_0 maps V_0/W onto \mathcal{A}_0/H . This all leads to the

following diagram:

$$\begin{array}{ccc} V_k/W & \xrightarrow{\gamma_k} & \mathcal{A}_k^\theta/H_k \\ \downarrow \delta & & \downarrow \zeta \\ V_0/W & \xrightarrow{\gamma_0} & \mathcal{A}_0/H \end{array}$$

Since γ and γ_k are bijections, it follows that there is a bijection between the fibers of δ and the fibers of ζ . For a θ -stable maximal k -split torus A , the fiber $\zeta^{-1}(A)$ consists of all θ -stable maximal k -split tori, which are H conjugate to A , but not H_k conjugate. That these fibers can be infinite can be seen from example 3.1.7.

Remarks 3.2.5. (1) For $k = \mathbb{R}$, a slightly different characterization of the orbits in \mathcal{A}_k^θ/H_k was given by Matsuki [23]. The results in [13], [14] and [15] give both a characterization and a classification of these orbits.

(2) To classify \mathcal{A}_k^θ/H_k one needs to classify the image and fibers of ζ . A characterization of the image \mathcal{A}_0/H will be discussed in the next subsection. In order to classify \mathcal{A}_0/H one needs also a classification of \mathcal{A}^θ/H . A classification of the fibers of ζ , for $k = \mathbb{R}$ can be found in [15]. A classification of the fibers of ζ , for G defined over \mathbb{Q}_p will be dealt with in a future paper.

3.3. Characterization of \mathcal{A}_0/H . The H_k -conjugacy classes of θ -stable maximal k -split tori are determined by the image and fibers of the map $\zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H$. In this section we give a characterization of the image of ζ , which consists of the H -conjugation classes of θ -stable maximal k -split tori. In order to find the H_k -conjugation classes all we have to do then is to characterize the fibers of ζ .

3.3.1. We use the notation of section 3.1. Given a θ -stable quasi k -split torus A , we reserve the notation A^+ and A^- for A_θ^+ and A_θ^- respectively. For other involutions of A , we shall keep the subscript.

Similar as for the conjugacy classes of θ -stable maximal tori, there is only one conjugacy class with A^+ (resp. A^-) maximal. From 2.3.6 we get the following.

Lemma 3.3.2. *All θ -stable maximal k -split tori containing a maximal (θ, k) -split torus (resp. a maximal k -split torus of H) are conjugate under H^0 .*

Definition 3.3.3. For $A_1, A_2 \in \mathcal{A}_0$, the pair (A_1, A_2) is called *standard* if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. In this case, we also say that A_1 is *standard* with respect to A_2 .

Similar as in the case of $k = \bar{k}$ (see [13]), the θ -stable maximal k -split tori of G can be put in a standard position.

Lemma 3.3.4. *Let $A_1, A_2 \in \mathcal{A}_0$ such that $A_1^+ \supset A_2^+$ (resp. $A_1^- \subset A_2^-$). Then there exists $x \in Z_H(A_2^+)$ (resp. $Z_H(A_1^-)$) such that (A_1, xA_2x^{-1}) is standard. In particular if A_1^+ and A_2^+ (resp. A_1^- and A_2^-) are H -conjugate, so are A_1 and A_2 .*

A standard pair (A_1, A_2) of θ -stable maximal k -split tori of G gives rise to an involution in $W(A_1)$ (resp. $W(A_2)$).

Lemma 3.3.5. *Let (A_1, A_2) be a standard pair of θ -stable maximal k -split tori of G . Then we have the following conditions:*

- (i) *There exists $g \in Z_G(A_1^- A_2^+)$ such that $gA_1g^{-1} = A_2$.*
- (ii) *If $n_1 = \theta(g)^{-1}g$ and $n_2 = \theta(g)g^{-1}$, then $n_1 \in N_G(A_1)$ and $n_2 \in N_G(A_2)$.*
- (iii) *If $w_1 \in W(A_1)$ and $w_2 \in W(A_2)$ are the images of n_1 and n_2 , then $w_1^2 = w_2^2 = e$, and $(A_1)_{w_1}^+ = (A_2)_{w_2}^+ = A_1^- A_2^+$, which characterizes w_1 and w_2 .*

By (iii) of Lemma 3.3.5, w_1 and w_2 are independent of the choice of $g \in Z_G(A_1^- A_1^+)$ with $gA_1g^{-1} = A_2$. We define standard involutions now as follows:

Definition 3.3.6. The involutions $w_1 \in W(A_1)$ (resp. $w_2 \in W(A_2)$) as in 3.3.5 are called the A_2 -standard involution (resp. A_1 -standard involution) of $W(A_1)$ (resp. $W(A_2)$).

3.3.7. For a θ -stable k -torus A of G , write $W(A, H)$ for $N_H(A)/Z_H(A)$. Let $A_0 \in \mathcal{A}_0$ (resp. $S \in \mathcal{A}_0$) be θ -stable maximal k -split tori of G , such that A_0^- (resp. S^+) is a maximal (θ, k) -split torus of G (resp. a maximal k -split torus of H). We can choose S to be standard with respect to A_0 . In the following we fix such a standard pair (S, A_0) . Then we have the following:

Lemma 3.3.8. *Let $A_1 \in \mathcal{A}_0$. Then A_1 is H -conjugate with a θ -stable maximal k -split torus A of G , which is standard with respect to A_0 and S .*

The conjugacy of θ -stable maximal k -split tori, standard with respect to A_0 (resp. S) can be reduced now to the conjugacy of the corresponding standard involutions under the Weyl group.

Proposition 3.3.9. *Assume that $A_1, A_2 \in \mathcal{A}_0$ such that they are standard with respect to A_0 (resp. S). Let w_1 and w_2 be the A_1 -standard and A_2 -standard involutions in $W(A_0)$ (resp. $W(S)$) respectively. Then A_1 and A_2 are H -conjugate if and only if w_1 and w_2 are conjugate under $W(A_0, H)$ (resp. $W(S, H)$).*

Remarks 3.3.10. (1) The orbit set \mathcal{A}^θ/H (i.e. H -conjugacy classes of θ -stable maximal quasi k -split tori) can also be characterized in terms of conjugacy classes of involutions in $W(A_0)$. Although the characterization of \mathcal{A}^θ/H is

similar to that of \mathcal{A}_0/H , the proof is much more complicated. A complete characterization of \mathcal{A}^θ/H and \mathcal{A}_0/H can be found in [14].

(2) The above proposition provides a sound criterion when elements in \mathcal{A}_0 are H -conjugate. To complete the characterization of H -conjugacy classes of \mathcal{A}_0 , it reduces to single out those $w \in W(A_0)$ (resp. $W(S)$) which corresponds to elements of \mathcal{A}_0/H and which corresponds to elements of \mathcal{A}^θ/H . A classification of these involutions, for a number of base fields k , can be found in [14].

4. (Γ, θ) -INDICES

In this section we study the fine structure associated with a symmetric k -variety. Since this is closely related to the fine structure of both the semisimple k -group and the corresponding symmetric variety, we also briefly review a few properties of the fine structure of these. Most of the results in this section come from [18], [29], [34] and [12]. Throughout this section let G be a reductive k -group and $\theta \in \text{Aut}(G)$ a k -involution. Let T be a θ -stable maximal k -torus of G , $X = X^*(T)$ and $\Phi = \Phi(T)$. There is a finite Galois extension K/k such that T splits over K . Denote the Galois group $\text{Gal}(K/k)$ of K/k by Γ and let $\Theta = \{1, -\theta\} \subset \text{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|_T$. Both Γ and Θ act on (X, Φ) . The action of Γ gives the restricted root system of a maximal k -split torus and the action of Θ gives the restricted root system of a maximal θ -split torus. One can combine these actions to get the restricted root system of a maximal (θ, k) -split torus. All three these actions on the root datum can be described in a similar manner. Therefore we will consider the action of a finite group \mathcal{E} on (X, Φ) . Both the groups $\mathcal{E} = \Gamma$ and $\mathcal{E} = \Theta$ give such an action.

4.1. Root data. To deal with the notion of root system in reductive groups it is quite useful to work with the notion of root datum (see [33, §1]). If T is a torus in a reductive group G , such that $\Phi(T)$ is a root system with Weyl group $W(T)$, then *the root datum associated to the pair (G, T)* is: $(X^*(T), \Phi(T), X_*(T), \Phi^\vee(T))$, where $X^*(T)$, $\Phi(T)$, $X_*(T)$ and $\Phi^\vee(T)$ are as defined in (2.1). So in each of the cases that T is either a maximal torus of G , a maximal k -split torus of G , a maximal θ -split torus of G or a maximal (θ, k) -split torus of G , then the above root datum exists.

If $\phi \in \text{Aut}(X, \Phi)$, then its transpose ${}^t\phi$ induces an automorphism of Φ^\vee , so Φ induces a unique automorphism in $\text{Aut}(\Psi)$, the set of automorphisms of the root datum Ψ . We shall frequently identify $\text{Aut}(X, \Phi)$ and $\text{Aut}(\Psi)$.

For any closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite group generated by the s_α for $\alpha \in \Phi_1$.

4.2. Group actions on root data. Let Ψ be a root datum with $\Phi \neq \emptyset$, as in (4.1) and let \mathcal{E} be a finite group acting on Ψ . For $\sigma \in \mathcal{E}$ and $\chi \in X$ we will also write χ^σ or $\sigma(\chi)$ for the element $\sigma.\chi \in X$. Write $W = W(\Phi)$ for the Weyl group of Φ . Now define the following:

$$(5) \quad X_0 = X_0(\mathcal{E}) = \left\{ \chi \in X \mid \sum_{\sigma \in \mathcal{E}} \chi^\sigma = 0 \right\}$$

Then X_0 is a co-torsion free submodule of X , invariant under the action of \mathcal{E} . Let $\Phi_0 = \Phi \cap X_0$. This is a closed subsystem of Φ invariant under the action of \mathcal{E} . Denote the Weyl group of Φ_0 by W_0 and identify it with the subgroup of $W(\Phi)$ generated by the reflections $s_\alpha, \alpha \in \Phi_0$. Put $W^\mathcal{E} = \{w \in W \mid w(X_0) = X_0\}$, $\bar{X}_\mathcal{E} = X/X_0(\mathcal{E})$ and let π be the natural projection from X to $\bar{X}_\mathcal{E}$. If we take $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ and $Y = X^*(A)$, then Y may be identified with $\bar{X}_\mathcal{E} = X/X_0$. Let $\bar{\Phi}_\mathcal{E} = \pi(\Phi - \Phi_0(\mathcal{E}))$ denote the set of *restricted roots of Φ relative to \mathcal{E}* .

Remark 4.2.1. In the case that $\mathcal{E} = \Gamma$, then X_0 is the annihilator of a maximal k -split torus A of T . Similarly in the case that $\mathcal{E} = \Theta$, then X_0 is the annihilator of a maximal θ -split torus A of T . In both these cases $\bar{\Phi}_\mathcal{E}$ is the root system of $\Phi(A)$ with Weyl group $\bar{W}_\mathcal{E}$.

We define now an order on (X, Φ) related to the action of \mathcal{E} as follows.

Definition 4.2.2. A linear order on X which satisfies

$$(6) \quad \text{if } \chi > 0 \text{ and } \chi \notin X_0, \text{ then } \chi^\sigma > 0 \text{ for all } \sigma \in \mathcal{E}$$

is called a \mathcal{E} -linear order. A fundamental system of Φ with respect to a \mathcal{E} -linear order is called a \mathcal{E} -fundamental system of Φ .

A \mathcal{E} -linear order on X induces linear orders on $Y = X/X_0$ and X_0 , and conversely, given linear orders on X_0 and on Y , these uniquely determine a \mathcal{E} -linear order on X , which induces the given linear orders (i.e., if $\chi \notin X_0$, then define $\chi > 0$ if and only if $\pi(\chi) > 0$). Instead of the above \mathcal{E} -linear order one can give a more general definition of a linear order on X , using only the fact that X_0 is a co-torsion free submodule of X (see [29, §2.1]).

In the following we give a number of properties of an \mathcal{E} -linear order on X .

4.2.3. Fix a \mathcal{E} -linear order $>$ on X , let Δ be a \mathcal{E} -fundamental system of Φ and let Δ_0 be a fundamental system of Φ_0 with respect to the induced order on X_0 . Define $\bar{\Delta}_\mathcal{E} = \pi(\Delta - \Delta_0)$. This is called a *restricted fundamental system* of Φ relative to A . The following proposition lists some properties of $X, X_0, \Phi, \Phi_0, \Delta, \Delta_0$, etc.

Proposition 4.2.4. *Let $X, X_0, \Phi, \Phi_0, \bar{\Phi}_\mathcal{E}, \Delta, \Delta_0$ and $\bar{\Delta}_\mathcal{E}$ be defined as above. Then we have the following*

- (1) $\Delta_0 = \Delta \cap \Phi_0$.
- (2) *If Δ' is another \mathcal{E} -fundamental system of Φ , $\Delta'_0 = \Delta' \cap \Phi_0$, and $\bar{\Delta}'_\mathcal{E} = \pi(\Delta' - \Delta'_0)$, then $\Delta = \Delta'$ if and only if $\Delta_0 = \Delta'_0$ and $\bar{\Delta}_\mathcal{E} = \bar{\Delta}'_\mathcal{E}$.*
- (3) *If Δ' is another \mathcal{E} -fundamental system of Φ such that $\bar{\Delta}_\mathcal{E} = \bar{\Delta}'_\mathcal{E}$, then there exists a unique $w_0 \in W_0$ such that $\Delta' = w_0\Delta$.*

4.2.5. There is a natural (Weyl) group associated with the set of restricted roots, which is related to $W^\mathcal{E}/W_0$. Since W_0 is a normal subgroup of $W^\mathcal{E}$, every $w \in W^\mathcal{E}$ induces an automorphism of $\bar{X}_\mathcal{E} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ ($\chi \in X$). Define $\bar{W}_\mathcal{E} = \{\pi(w) \mid w \in W^\mathcal{E}\}$. We call this the *restricted Weyl group*, with respect to the action of \mathcal{E} on X . It is not necessarily a Weyl group in the sense of Bourbaki [6, Ch.VI,no.1]. However we can show the following.

Proposition 4.2.6. *Let $X, X_0, \Phi, \Delta, W_0, W^\mathcal{E}, \bar{W}_\mathcal{E}$ be defined as above and let $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ be the annihilator of X_0 . Then $\bar{W}_\mathcal{E} \cong W^\mathcal{E}/W_0 \cong N_G(A)/Z_G(A)$.*

Remark 4.2.7. In the case that A is a maximal k -split, θ -split or (θ, k) -split torus, then $\bar{\Phi}_\mathcal{E}$ is actually a root system with Weyl group $\bar{W}_\mathcal{E}$. The general question when $\bar{\Phi}_\mathcal{E}$ is a root system in $Y = X/X_0$ was studied in [30].

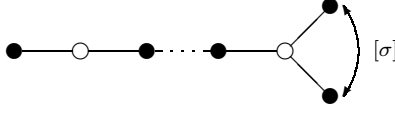
4.2.8. From 4.2.6 it follows that $W^\mathcal{E}$ acts on the set of \mathcal{E} -fundamental systems of Φ . There is also a natural action of \mathcal{E} on this set. If Δ is a \mathcal{E} -fundamental system of Φ , and $\sigma \in \mathcal{E}$, then the \mathcal{E} -fundamental system $\Delta^\sigma = \{\alpha^\sigma \mid \alpha \in \Delta\}$ gives the same restricted basis as Δ , i.e. $\bar{\Delta}^\sigma = \bar{\Delta}$. This follows from the fact that $\alpha_i \equiv \alpha_i^\sigma \pmod{X_0}$ for all $\alpha_i \in \Delta$, $\sigma \in \mathcal{E}$. From 4.2.4 it follows that there is a unique element $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma\Delta$. This means we can define a new operation of \mathcal{E} on X as follows:

$$(7) \quad \chi^{[\sigma]} = w_\sigma^{-1}\chi^\sigma, \quad \chi \in X, \quad \sigma \in \mathcal{E}.$$

It is easily verified that $\chi \rightarrow \chi^{[\sigma]}$ is an automorphism of the triple (X, Φ, Δ) and that $\chi^{[\sigma][\tau]} = \chi^{[\sigma\tau]}$ for all $\sigma, \tau \in \mathcal{E}$, $\chi \in X$. This action of \mathcal{E} on Ψ is essentially determined by Δ, Δ_0 and $[\sigma]$. Following Tits [34] we will call the quadruple $(X, \Delta, \Delta_0, [\sigma])$ an *index of \mathcal{E}* or an \mathcal{E} -index. We will also use the name \mathcal{E} -diagram, following the notation in Satake [29, 2.4].

As in [34] we make a diagrammatic representation of the index of \mathcal{E} by coloring black those vertices of the ordinary Dynkin diagram of Φ , which represent roots in $\Delta_0(\mathcal{E})$ and indicating the action of $[\sigma]$ on Δ by arrows. An

example in type D_l is:



To use these \mathcal{E} -indices in the characterization of isomorphism classes of reductive k -groups or involutions, we need a notion of isomorphism between these indices.

Definition 4.2.9. Let Ψ and Ψ' be semisimple root data and \mathcal{E} a group acting on them. A *congruence* φ of the \mathcal{E} -index $(X, \Delta, \Delta_0, [\sigma])$ of Ψ onto the \mathcal{E} -index $(X', \Delta', \Delta'_0, [\sigma'])$ of Ψ' is an isomorphism which maps $(X, \Delta, \Delta_0) \rightarrow (X', \Delta', \Delta'_0)$, and satisfies $[\sigma]' = \varphi[\sigma]\varphi^{-1}$.

For k -involutions it suffices to consider two actions of \mathcal{E} on the same root datum. In that case we will also use the term *isomorphic* \mathcal{E} -indices instead of congruent \mathcal{E} -indices. In this case one can differentiate between inner and outer automorphisms.

Definition 4.2.10. Let Ψ be a root datum and $\mathcal{E}_1, \mathcal{E}_2 \subset \text{Aut}(\Psi)$ the subgroups of $\text{Aut}(\Psi)$ corresponding to actions of \mathcal{E} on Ψ . Two indices $(X, \Delta, \Delta_0(\mathcal{E}_1), [\sigma]_1)$ and $(X, \Delta', \Delta'_0(\mathcal{E}_2), [\sigma]_2)$ are said to be $W(\Phi)$ - (resp. $\text{Aut}(\Phi)$)-*isomorphic* if there is a $w \in W(\Phi)$ (resp. $w \in \text{Aut}(\Phi)$), which maps $(\Delta, \Delta_0(\mathcal{E}_1))$ onto $(\Delta', \Delta'_0(\mathcal{E}_2))$ and satisfies $w[\sigma]_1 w^{-1} = [\sigma]_2$. Instead of $W(\Phi)$ -isomorphic we will also use the term *isomorphic*.

Remark 4.2.11. An index of \mathcal{E} may depend on the choice of the \mathcal{E} -basis of Φ , i.e. for two \mathcal{E} -bases Δ, Δ' , the corresponding indices $(X, \Delta, \Delta_0(\mathcal{E}), [\sigma])$ and $(X, \Delta', \Delta'_0(\mathcal{E}), [\sigma]')$ need not be isomorphic. However this cannot happen if $\bar{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\bar{W}_{\mathcal{E}}$:

Lemma 4.2.12. *Let Ψ be a semisimple root datum and $\mathcal{E} \subset \text{Aut}(\Psi)$ a group acting on Ψ such that $\bar{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\bar{W}_{\mathcal{E}}$. If Δ, Δ' are \mathcal{E} -bases of Φ , then $(X, \Delta, \Delta_0(\mathcal{E}), [\sigma])$ and $(X, \Delta', \Delta'_0(\mathcal{E}), [\sigma]')$ are isomorphic.*

Remark 4.2.13. In the case that $\bar{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\bar{W}_{\mathcal{E}}$, then the restricted root system together with the multiplicities of the roots can be easily determined from the \mathcal{E} -index. See for example [12] and [18].

In the cases of $\mathcal{E} = \Theta$ and $\mathcal{E} = \Gamma$ we get the well known θ -index and Γ -index, which are essential in the respective classifications. Since the classification of k -involutions depends on a classification of these, we will briefly review these in the next subsection. First we need still a notion of irreducibility for \mathcal{E} -indices.

Definition 4.2.14. Let $\mathcal{E} \subset \text{Aut}(X, \Phi)$ be a subgroup and Δ a \mathcal{E} -basis of Φ . An index $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ is \mathcal{E} -irreducible if Δ is not the union of two mutually orthogonal $[\sigma]$ -invariant (non-empty) subsystems Δ', Δ'' . The system \mathcal{D} is *absolutely irreducible* if Δ is connected. In the case $\mathcal{E} = \Gamma$ (resp. Θ) we will also call an \mathcal{E} -irreducible index an k -irreducible index (resp. θ -irreducible index).

4.3. θ -index. In this subsection we discuss the index associated with an involutorial automorphism of a reductive algebraic group. Let G be a reductive algebraic group, $\theta \in \text{Aut}(G)$ an involution and T a θ -stable maximal torus of G . Write $X = X^*(T)$, $\Phi = \Phi(T)$ and let $\Theta = \{1, -\theta\} \subset \text{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta \upharpoonright T$. In this case we will also write $X_0(\theta)$, \bar{X}_θ , $\Phi_0(\theta)$, $\bar{\Phi}_\theta$, $W_1(\theta)$, \bar{W}_θ , $\Delta_0(\theta)$, $\bar{\Delta}_\theta$ instead of, respectively, $X_0(\Theta)$, \bar{X}_Θ , $\Phi_0(\Theta)$, $\bar{\Phi}_\Theta$, $W_0(\Theta)$, $W_1(\Theta)$, \bar{W}_Θ , $\Delta_0(\Theta)$, $\bar{\Delta}_\Theta$. A Θ -order on X will also be called a θ -order on X , a Θ -basis of Φ a θ -basis of Φ and a Θ -index a θ -index.

Let Δ be a θ -basis of Φ . To find the θ -index we need to find the action of $[-\theta]$ on (X, Φ, Δ) . Since $\theta(-\Delta)$ is also a θ -basis of Φ with the same restricted basis, it follows from (4.2.4) that there is $w_0(\theta) \in W_0(\theta)$ such that $w_0(\theta)\theta(\Delta) = -\Delta$. Put $\theta^* = \theta^*(\Delta) = -w_0(\theta)\theta$. Then $\theta^* = [-\theta]$. Note that $\theta^*(\Delta) \in \text{Aut}(X, \Phi, \Delta) = \{\Phi \in \text{Aut}(X, \Phi) \mid \Phi(\Delta) = \Delta\}$, $\theta^*(\Delta)^2 = \text{id}$ and $\theta^*(\Delta_0(\theta)) = \Delta_0(\theta)$.

Remarks 4.3.1. (1) $\theta^* = [-\theta]$ can be described by its action on the Dynkin diagram of Δ . Notice that

- (a) if Φ is irreducible, then θ^* is either the identity or a diagram automorphism of order 2. The latter happens only if Φ is either of type $A_l(l \geq 2)$, $D_{2l+1}(l \geq 2)$ or E_6 .
- (b) if $\Phi = \Phi_1 \cup \Phi_2$ with Φ_1, Φ_2 irreducible and $\theta(\Phi_1) = \Phi_2$, then θ^* exchanges the Dynkin diagrams of Φ_1 and Φ_2 . In particular $\Phi_0(\theta) = \emptyset$, so $w_0(\theta) = \text{id}$ and $\theta = -\theta^*$

(2) If $\theta = \text{id}$ and Δ is a basis of Φ , then $\theta^*(\Delta) = -w_0(\text{id})$ is called the *opposition involution* of Δ . In this case we shall also write $\text{id}^*(\Delta)$ for $\theta^*(\Delta)$. For Φ irreducible the opposition involution is non-trivial if and only if Φ is either of type $A_l(l \geq 2)$, $D_{2l+1}(l \geq 2)$ or E_6 . Since $\theta^* \upharpoonright \Delta_0(\theta) = -w_0(\theta)$ is the opposition involution of $\Delta_0(\theta)$, the action of θ^* on $\Delta_0(\theta)$ is uniquely determined on each irreducible component of $\Phi_0(\theta)$. So for the θ -index we can omit the action of θ^* on $\Delta_0(\theta)$.

Since all maximal θ -split tori are H -conjugate it follows that the θ -index is uniquely determined by the G -isomorphism class of θ .

Proposition 4.3.2. *Let A be a maximal θ -split torus of G , $T \supset A$ a maximal torus and Δ a θ -basis of $\Phi(T)$. The θ -index $(X, \Delta, \Delta_0, \theta^*)$ is uniquely determined (up to congruence) by the isomorphism class of θ .*

Remark 4.3.3. We will see in 5.1 that for involutions the converse of 4.3.2 also holds. For isomorphism classes of reductive k -groups or k -involutions, the respective indices do not characterize the isomorphism classes.

4.4. Γ -index. In this subsection we introduce the index related to the isomorphism classes of semisimple k -groups. Throughout this subsection let G be a reductive k -group, A a k -split torus of G , $T \supset A$ a maximal k -torus, $\Gamma = \text{Gal}(K/k)$ the Galois group of K/k , $X = X^*(T)$, $\Phi = \Phi(T)$, $X_0 = X_0(\Gamma)$, $\Phi_0 = \Phi_0(\Gamma)$, etc.

4.4.1. Let Δ be a Γ -basis of Φ , and let $\Delta_0 = \Delta \cap X_0$. As in (7) we have an action of Γ on Δ , which we denote by $[\sigma]$. The 4-tuple $(X, \Delta, \Delta_0, [\sigma])$ is called the Γ -index of (G, T, A) . If A is a maximal k -split torus of G , then we will also call this the Γ -index of G . From (4.2.12) it follows again that for A maximal k -split the Γ -index does not depend on the choice of Δ . Using 4.2.12 it easily follows now that the Γ -index of G is uniquely determined by the k -isomorphism class of G .

Proposition 4.4.2. *The k -isomorphism class of G uniquely determines (up to isomorphism) $(X, \Phi, \Delta_0, [\sigma])$, the Γ -index of G .*

4.5. (Γ, θ) -index. In this subsection we discuss an index related to the isomorphism classes of k -involutions.

4.5.1. Γ_θ -index. Let G be a connected semisimple k -group, $\theta \in \text{Aut}(G)$ an k -involution, A a (θ, k) -split torus of G , $T \supset A$ a θ -stable maximal k -torus of G and $X = X^*(T)$, $\Phi = \Phi(T)$. Let Γ be the Galois group of K/k as in subsection 4.4 and $\Theta = \{1, -\theta\} \subset \text{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta \mid T$ as in 4.3. Let $\mathcal{E}_\Gamma \subset \text{Aut}(X, \Phi)$ be the subgroup corresponding to the action of Γ on (X, Φ) and let $\Gamma_\theta = \mathcal{E}_\Gamma \cdot \Theta$ the subgroup of $\text{Aut}(X, \Phi)$ generated by \mathcal{E}_Γ and Θ . As in 4.4 let $X_0 = X_0(\Gamma_\theta)$, $\Phi_0 = \Phi_0(\Gamma_\theta)$, etc. We will also use the notation $\Phi_0(\Gamma, \theta)$ (resp. $\Delta_0(\Gamma, \theta)$) for $\Phi_0(\Gamma_\theta)$ (resp. $\Delta_0(\Gamma_\theta)$). In addition, let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$. Moreover $\bar{\Phi}_{\Gamma_\theta} = \Phi(A)$ is the set of restricted roots of A , which, by [20, 5.9] is a root system if A is a maximal (θ, k) -split torus of G . Let Δ be a Γ_θ -bases of Φ , and let $\Delta_0 = \Delta \cap X_0$. Similar as in (7) we have an action of Γ_θ on Δ , which we denote by $[\sigma]$. The 4-tuple $(X, \Delta, \Delta_0, [\sigma])$ is called the Γ_θ -index of (G, T, A) .

If A is a maximal (θ, k) -split torus of G , then we will also call this the Γ_θ -index of G . From the Γ_θ -index $(X, \Delta, \Delta_0, [\sigma])$ of G one easily determines the restricted root system of a maximal (θ, k) -split torus of G .

4.5.2. (Γ, θ) -order. The Γ_θ -index, as defined above, corresponds to a Γ_θ -order on (X, Φ) . However there is a lot of additional structure present, which is not represented in the Γ_θ -index. We also have a θ -index and a Γ -index. This can be seen as follows. Assume A is a maximal (θ, k) -split torus of G . Then there exists a maximal k -split torus $A_1 \supset A$, which defines a Γ -order on (X, Φ) . On the other hand A is contained in a maximal θ -split torus S , which defines a θ -order on (X, Φ) . By 2.3.4 there exists a maximal k -torus $T \supset A$ such that $T \supset S$ and $T \supset A_1$. The Γ_θ -order, Γ -order and θ -order can now all be defined simultaneously on (X, Φ) as follows.

Definition 4.5.3. Let Ψ be a semisimple root datum and let Γ, θ act on (X, Φ) as in 4.5.1. A linear order on X which is simultaneously a Γ -, θ - and Γ_θ -order is called a (Γ, θ) -order. A fundamental system of Φ with respect to a (Γ, θ) -order is called a (Γ, θ) -fundamental system of Φ .

From the above remarks it follows that if A, A_1, S, T are as above, then a (Γ, θ) -order on (X, Φ) exists. However not every Γ_θ -order is a (Γ, θ) -order. Another characterization of a (Γ, θ) -order is given in the following result.

Proposition 4.5.4. *Let $>$ be a Γ_θ -order on (X, Φ) . The following are equivalent:*

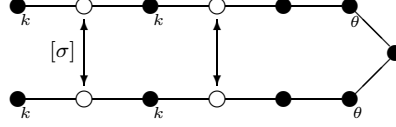
- (1) (X, Φ) has a (Γ, θ) -order.
- (2) $\Phi_0(\Gamma, \theta) = \Phi_0(\sigma) \cup \Phi_0(\theta)$.
- (3) If $\Phi_1 \subset \Phi_0(\Gamma, \theta)$ irreducible component then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.

Remarks 4.5.5. (1) A (Γ, θ) -order, as above, is completely determined by the sextuple $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$. We will call this sextuple an *index of (Γ, θ)* or an (Γ, θ) -index. This terminology follows again Tits [34]. We will also use the name (Γ, θ) -diagram, following the notation in Satake [29, 2.4].

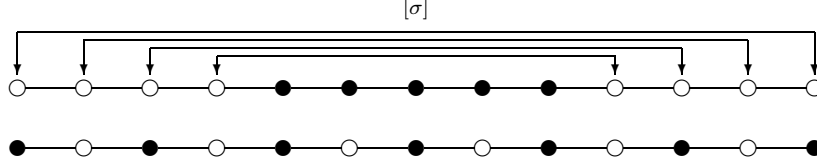
(2) The above index of (Γ, θ) determines the indices of both Γ and θ and vice versa.

(3) We can make a diagrammatic representation of the (Γ, θ) -index by coloring black those vertices of the ordinary Dynkin diagram of Φ , which represent roots in $\Delta_0(\Gamma, \theta)$ and giving the vertices of $\Delta_0(\Gamma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\Gamma) \cap \Delta_0(\theta)$ a label k or θ if $\alpha \in \Delta_0(\Gamma) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ or $\alpha \in \Delta_0(\theta) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ respectively. The actions of $[\sigma]$ and θ^* are indicated by arrows.

Here is an example with Φ of type A_{13} :



This (Γ, θ) -index is obtained by gluing together the indices



of Γ resp. θ with the above recipe.

(4) An index of (Γ, θ) may depend again on the choice of the (Γ, θ) -basis of Φ . However if Φ_{Γ_θ} is a root system, then it follows similar as in 4.2.12 that the (Γ, θ) -index is independent of the choice of the (Γ, θ) -basis.

The isomorphism between these indices is defined similar as in 4.2.9.

Definition 4.5.6. Let Γ act on (X, Φ) as in 4.5.1 and let $\theta_1, \theta_2 \in \text{Aut}(X, \Phi)$ be involutions. The (Γ, θ) -index $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta_1), [\sigma], \theta_1^*(\Delta))$ of (Γ, θ_1) and the (Γ, θ) -index $(X', \Delta', \Delta'_0, [\sigma]')$ of (Γ, θ_2) are said to be *isomorphic* if there is a $w \in W(\Phi)$, which maps $(X, \Delta, \Delta_0(\sigma_1), \Delta_0(\theta_1))$ onto $(X, \Delta, \Delta'_0(\sigma_2), \Delta'_0(\theta_2))$ and which satisfies: $w\theta_1^*(\Delta)w^{-1} = \theta_2^*(\Delta')$ and $w[\sigma]w^{-1} = [\sigma]'$.

5. CHARACTERIZATION OF THE ISOMORPHY CLASSES

In each of the cases of symmetric varieties, symmetric k -varieties and semisimple k -groups there is a natural fine structure associated with these spaces. For a study of these spaces and their representation theory it is important to have a classification of these spaces together with this fine structure of restricted root systems with multiplicities and Weyl groups. This fine structure easily follows from the index as defined in section 4. On the other hand this index can also be used as an invariant to characterize the isomorphism classes. In the case of isomorphism classes of involutions these indices completely characterize the isomorphism classes. In the case of isomorphism classes of semisimple k -groups one needs a second invariant to characterize the isomorphism classes and in the case of isomorphism classes of k -involutions three invariants are needed.

Since the classification of k -involutions depends on the classifications of semisimple k -groups (see [34]) and the classification of involutions over algebraically closed fields (see [12]), we will first briefly review some facts about both these classifications, which will be needed later in the classification of the k -involutions.

5.1. Characterization of the isomorphism classes of involutions. The classification of isomorphism classes of involutions can be reduced to a classification of $W(T)$ -conjugacy classes of involutions normally related to a maximal torus T (see [12]). In this subsection we briefly review these results. We use the same notation as in 4.3. In particular let G be a reductive algebraic group, $\theta \in \text{Aut}(G)$ an involution and T a maximal torus of G . Write $X = X^*(T)$ and $\Phi = \Phi(T)$. To relate the isomorphism classes of involutions to the indices as in 4.3, we define the following:

Definition 5.1.1. Let T be a maximal torus of G . An automorphism θ of G of order ≤ 2 is said to be *normally related to T* if $\theta(T) = T$ and T_θ^- is a maximal θ -split torus of G .

Note that, since all maximal tori of G are conjugate under $\text{Int}(G)$, every involutorial automorphism of G is conjugate to one which is normally related to T . The involutions normally related to T can be characterized now as follows (see [12, 3.7]).

Theorem 5.1.2. Let $\theta_1, \theta_2 \in \text{Aut}(G)$ be such that $\theta_1^2 = \theta_2^2 = \text{id}$ and assume θ_1, θ_2 are normally related to T . Then θ_1 and θ_2 are conjugate under $\text{Int}(G)$ if and only if $\theta_1 \mid T$ and $\theta_2 \mid T$ are conjugate under $W(T)$.

We showed in (4.3.2) that the G -isomorphism class determines the θ -index up to congruence. From 5.1.2 it follows now that these indices actually completely characterize the isomorphism classes. To formulate this result we need to define first a notion of admissibility.

Definition 5.1.3. Let $\theta \in \text{Aut}(X, \Phi)$ be an involution. Then θ is called *admissible* if there exists an involution $\tilde{\theta} \in \text{Aut}(G, T)$ such that $\tilde{\theta}|T = \theta$ and $T_{\tilde{\theta}}^-$ is a maximal $\tilde{\theta}$ -split torus of G . If X is semisimple, then the indices of admissible involutions of (X, Φ) are called *admissible θ -indices*.

We have now the following characterization of the isomorphism classes of involutions in terms of θ -indices. Note that these θ -indices yield most of the fine structure of the corresponding symmetric variety G/H .

Theorem 5.1.4. Let G, T be as above and assume G is semisimple. Then there is a bijection of the set of $\text{Int}(G)$ conjugacy classes of involutorial automorphisms of G and the isomorphism classes of indices of admissible involutions of $(X^*(T), \Phi(T))$.

5.2. Characterization of the isomorphism classes of semisimple k -groups. In this subsection we give a characterization of the isomorphism classes of semisimple k -groups. Most of these results can be found in [34] and [29].

5.2.1. We use the same notation as in 4.4. In 4.4.2 we demonstrated that the Γ -index is an invariant for the isomorphism classes of semisimple k -groups. Another invariant is the following. Let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$ and is k -anisotropic if A is maximal k -split. Let $T_0 = T \cap G_0$. This is a maximal k -torus of G_0 . Since all maximal k -split tori of G are conjugate under G_k , it follows that G_0 is uniquely determined (up to k -isomorphism) by the k -isomorphism class of G . We will call G_0 the *k -anisotropic kernel* of G .

We have shown now that the k -isomorphism class of G uniquely determines the Γ -index $(X, \Phi, \Delta_0, [\sigma])$ of G and the k -anisotropic kernel G_0 of G . The following result shows that these two actually suffice to characterize the isomorphism classes (see [34] or [29]).

Theorem 5.2.2. *Let G, G' be connected semi-simple algebraic groups defined over k . Let T, A, X, G_0, T_0 , etc., T', A', X', G'_0, T'_0 etc. be as defined above, and corresponding to G and G' , respectively. There exists a k -isomorphism $\varphi : (G, T, A) \rightarrow (G', T', A')$ if and only if the following conditions are satisfied:*

- (i) *There exists a congruence $\phi : (X, \Delta, \Delta_0, [\sigma]) \rightarrow (X', \Delta', \Delta'_0, [\sigma]')$ of the Γ -index of G onto the Γ -index of G'*
- (ii) *there exists a k -isomorphism $\varphi_0 : (G_0, T_0) \rightarrow (G'_0, T'^0_0)$ such that the restriction ϕ_0 of ϕ to $(X^0, \Delta_0, [\sigma] \mid X^0)$ is associated to φ_0 (i.e., $\varphi_0^{[*]} = \phi_0$).*

The Γ -indices, which belong to connected semi simple groups will be called admissible. They are defined as follows:

Definition 5.2.3. If X is a free module of rank X , Δ a fundamental system of a root system Φ in X , Δ_0 a subset of Δ , and $[\]$ a homomorphism of the Galois group Γ into $\text{Aut}(X, \Delta, \Delta_0)$, we will say that the system $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ is *admissible* if there exists a connected semi-simple group G defined over k having \mathcal{D} as Γ -index.

Remark 5.2.4. This result reduces the problem of classifying connected semi-simple algebraic groups defined over k to the following two problems:

- (1) classification of all admissible Γ -indices.
- (2) classification of all k -anisotropic semisimple algebraic groups.

For arbitrary base fields not much is known about the k -anisotropic semisimple algebraic groups. The first problem is discussed in Tits [34]. See also 6.2.

5.3. Characterization of the isomorphism classes of k -involutions. In this subsection we give a characterization of the G_k -isomorphism classes of k -involutions of G . Most of the results in this subsection come from [18].

5.3.1. Let G be a reductive k -group and θ a k -involution of G . We consider isomorphism classes of k -involutions under the action of G_k . We want to characterize the isomorphism classes in a such a way that we also get a classification of the natural root systems of the symmetric k -varieties. This means we need to characterize the isomorphism classes of the k -involutions on a fixed maximal k -split torus.

Definition 5.3.2. Let A be a maximal k -split torus of G . A k -involution θ of G is *normally related* to A if $\theta(A) = A$ and A_θ^- is a maximal (θ, k) -split, torus of G .

The Weyl group of A acts on the isomorphism classes, what essentially reduces the problem to involutions of $Z_G(A)$. If we take involutions normally related to A we can also restrict to the centralizer of a maximal (θ, k) -split torus.

Lemma 5.3.3. *Let A be a maximal k -split torus of G , $\theta_1, \theta_2 \in \text{Aut}_k(G)$ k -involutions, normally related to A and $\tilde{A} = A_{\theta_1}^-$. Then $\theta_1 \mid Z_G(A) = \theta_2 \mid Z_G(A)$ if and only if $\theta_1 \mid Z_G(\tilde{A}) = \theta_2 \mid Z_G(\tilde{A})$*

Involutions which coincide on $Z_G(A)$ can now be characterized as follows.

Proposition 5.3.4. *Let A be a maximal k -split torus of G and $\theta_1, \theta_2 \in \text{Aut}_k(G)$ k -involutions, normally related to A . If $\theta_1 \mid Z_G(A) = \theta_2 \mid Z_G(A)$, then there exists $a \in A_{\theta_1}^-$ such that $\theta_2 = \theta_1 \text{Int}(a)$.*

To characterize the isomorphism classes we need some more notation.

Notation 5.3.5. Let A be a maximal k -split torus of G , θ a k -involution of G , normally related to A and $a \in A$ such that $\theta \text{Int}(a)$ is a k -involution. Write $H_a = G_{\theta \text{Int}(a)}$ and let

$$Z_{a,\theta} = \{z \in Z_G(A) \mid \exists h \in G_{\theta \text{Int}(a)} \text{ such that } hz \in (G_{\theta \text{Int}(a)}^0 Z_G(A))_k\}$$

$$N_{a,\theta} = N_{G_k}(A).Z_{a,\theta}$$

A more detailed description of the set $Z_{a,\theta}$ is obtained as follows. The torus A_θ^- is a maximal $(\theta \text{Int}(a), k)$ -split torus of G . The set $(G_{\theta \text{Int}(a)}^0 Z_G(A))_k$ is related to the $P_k \times H_k$ orbits in the open orbit $P.H \subset G$, where $P \supset A$ is a minimal parabolic k -subgroup of G contained in a minimal θ -split parabolic k -subgroup of G . For more details, see [20, §9]. Now let $V_{a,\theta} = \{v_i \mid i \in I_{a,\theta}\} \subset G_k$ be a set of representatives of $(G_{\theta \text{Int}(a)})_k \times Z_{G_k}(A)$ orbits in $(G_{\theta \text{Int}(a)}^0 Z_G(A))_k$. For each $v_i \in V_{a,\theta}$, ($i \in I_{a,\theta}$) choose $h_i \in G_{\theta \text{Int}(a)}^0$, $z_i \in Z_G(A)$ such that $v_i = h_i z_i$. Let $\mathcal{Z}_{a,\theta} = \{z_i \mid i \in I_{a,\theta}\}$. Then we have the following characterization of $Z_{a,\theta}$.

$$Z_{a,\theta} = \{z \in Z_G(A) \mid z = h z_1 z_2 \text{ with } h \in G_{\theta \text{Int}(a)} \cap Z_G(A), z_1 \in \mathcal{Z}_{a,\theta} z_2 \in Z_{G_k}(A)\}$$

The isomorphism classes of k -involutions can be characterized now as follows:

Theorem 5.3.6. *Let A be a maximal k -split torus of G and θ_1, θ_2 k -involutions of G , normally related to A . The following are equivalent.*

- (i) $\theta_2 \mid Z_G(A)$ and $\theta_1 \mid Z_G(A)$ are isomorphic under $N_{a,\theta}$
- (ii) θ_2 is isomorphic under G_k with $\theta_1 \text{Int}(a)$ for some $a \in A_{\theta_1}^-$
- (iii) θ_2 is isomorphic under N_{a,θ_2} with $\theta_1 \text{Int}(a)$ for some $a \in A_{\theta_1}^-$

As for the isomorphism between the involutions $\theta \text{Int}(a)$, $a \in A_{\theta}^-$ we note:

Proposition 5.3.7. *Let A be a maximal k -split torus of G , θ a k -involution of G , normally related to A and $a \in A_{\theta}^-$. Then the following are equivalent:*

1. θ and $\theta \text{Int}(a)$ are isomorphic under G_k
2. θ and $\theta \text{Int}(a)$ are isomorphic under $(H^0 Z_G(A))_k$.
3. θ and $\theta \text{Int}(a)$ are isomorphic under $(AG_{\theta})_k$.

We need again a notion of admissibility.

Definition 5.3.8. Let G be a reductive k -group, A a maximal k -split torus of G and $T \supset A$ a maximal k -torus of G . An involution $\theta \in \text{Aut}(X^*(T), \Phi(T))$ is said to be an *admissible k -involution* (with respect to (G, T, A)) if there exists a k -involution $\tilde{\theta}$ of G , normally related to A and T and such that $\tilde{\theta} \mid T = \theta$.

5.3.9. Using the above results we can characterize the isomorphism classes of k -involutions as follows. Fix a maximal k -split torus A of G and write $Z = Z_G(A)$, $N = N_G(A)$. Denote the family of all k -involutions of G by \mathcal{F}_k and the family of all k -involutions of G , which are normally related to A by $\mathcal{F}_k(A)$. Denote the set of G_k -isomorphism classes in \mathcal{F}_k by \mathcal{C}_k . From 2.3.4 and the conjugacy of the maximal k -split tori of G it follows that every k -involution of G is G_k -isomorphic to one normally related to A , so every class in \mathcal{C}_k has a representative in $\mathcal{F}_k(A)$. By (5.3.6) we can restrict then to $N_{a,\theta}$ -isomorphism classes of involutions in $\mathcal{F}_k(A)$. The N -isomorphism classes are related to conjugacy classes of admissible k -involution, as can be seen as follows. Let $T \supset A$ be a maximal k -torus of G , $W(A, T) = \{w \in W(T) \mid w(A) = A\}$ and \mathcal{T}_k the set of $W(A, T)$ -isomorphism classes of involutions of $(X^*(T), \Phi(T), \Phi(A))$. Denote the set of N -isomorphism classes of k -involutions in $\mathcal{F}_k(A)$ by $\mathcal{C}_k(A, G)$.

From the conjugacy of the maximal θ -split tori of G it follows then that every involution in $\mathcal{F}_k(A)$ is isomorphic under $Z_G(A)$ with one normally related to T . So we have a natural map

$$\rho : \mathcal{C}_k(A, G) \longrightarrow \mathcal{T}_k.$$

From [12, 3.7] it follows that ρ is one to one. Denote the image of ρ by $\mathcal{T}(A)$. These are the $W(A, T)$ -isomorphism classes of admissible k -involutions. The

$W(A, T)$ -isomorphism classes of admissible k -involutions can be described by an (Γ, θ) -index (see 6.3).

5.3.10. The next step is to characterize the isomorphism classes, which correspond with an $W(A, T)$ -isomorphism class of admissible k -involutions. We also have a natural map of \mathcal{C}_k into $\mathcal{C}_k(A, G)$ by taking restrictions of k -involutions in $\mathcal{F}_k(A)$ to $Z_G(A)$. This map can be split into two parts as follows. Let $\mathcal{F}_k(A, Z) = \{\theta \mid Z \in \text{Aut}(Z, G) \mid \theta \in \mathcal{F}_k(A)\}$ the restrictions of the k -involutions in $\mathcal{F}_k(A)$ to $Z_G(A)$. So we essentially identify all the involutions $\theta \text{Int}(a)$, ($a \in A/A^2$). Let $\mathcal{C}_k(Z, G)$ denote the isomorphism classes of the involutions in $\mathcal{F}_k(A, Z)$, which are isomorphic under G_k . The restriction map from $\mathcal{F}_k(A)$ to $\mathcal{F}_k(A, Z)$ induces a map $\mu : \mathcal{C}_k \rightarrow \mathcal{C}_k(Z, G)$, which is surjective by 5.3.6. The fibers can be characterized by $\{a_i \in A_{\theta}^-/A_{\theta}^{-2} \mid i \in I\}$. These will be called *the quadratic elements of A*.

On the other hand it follows from (5.3.6) that two involutions in $\mathcal{F}_k(A, Z)$, which are isomorphic under G_k , are also isomorphic under $N_G(A)$. So we also have a map $\nu : \mathcal{C}_k(Z, G) \rightarrow \mathcal{C}_k(A, G)$ sending each G_k -isomorphism class in its N -isomorphism class. This map is surjective and its fibers are essentially the G_k -isomorphism classes of k -involutions of $Z_G(A)$ (coming from involutions of G), which give the same N -isomorphism class. We have now the following sequence:

$$\mathcal{C}_k \xrightarrow{\mu} \mathcal{C}_k(Z, G) \xrightarrow{\nu} \mathcal{C}_k(A, G) \xrightarrow{\rho} \mathcal{T}_k$$

For a k -involution θ of G , normally related to A we denote its G_k -isomorphism class in \mathcal{C}_k (resp. $\mathcal{C}_k(Z, G)$) by $[\theta]$ (resp. $[\theta]_Z$) and its N -isomorphism class in $\mathcal{C}_k(A, G)$ by $[\theta]_N$. For an admissible k -involution θ we denote the k -involution in $\text{Aut}(G)$ representing the isomorphism class $\rho^{-1}(\theta) = [\theta]_N$ in $\mathcal{C}(A, G)$ also by θ . Denote the fiber of ν above $[\theta]_N = \rho^{-1}(\theta)$ by $\mathcal{C}(\theta) = \nu^{-1}\rho^{-1}(\theta)$. Finally for an isomorphism class $[\theta]_Z \in \mathcal{C}_k(Z, G)$ denote the fiber of μ by $\mathcal{C}_A(\theta)$.

We have now the following characterization of the isomorphism classes of k -involutions.

Theorem 5.3.11. *Let A be a maximal k -split torus of G and $T \supset A$ a maximal k -torus of G . Write $Z = Z_G(A)$, $N = N_G(A)$.*

- (i) *There is a bijection between the $W(A, T)$ -isomorphism classes of admissible k -involutions of $(X^*(T), \Phi(T), \Phi(A))$ and the N -isomorphism classes of k -involutions in $\mathcal{C}_k(A, G)$.*
- (ii) *The G_k -isomorphism classes in $\mathcal{C}(\theta)$ (θ an admissible k -involution) consists of $\{[\theta_i]_Z \mid i \in I\}$, where the θ_i are representatives of the G_k -isomorphism classes of involutions of Z , which are N -isomorphic to θ .*

(iii) The isomorphism classes in $\mathcal{C}_A(\theta_i)$ ($[\theta_i]_Z \in \mathcal{C}(\theta)$, with θ an admissible k -involution) are represented by a set of quadratic elements $\{a_{i,j} \in A_\theta^-/A_\theta^{-2} \mid j \in J_i\}$.

Remark 5.3.12. The above result reduces the classification of k -involutions of G to the following 3 problems.

- (1) a classification of admissible k -involutions.
- (2) a classification of k -involutions of k -anisotropic semisimple groups.
- (3) for each k -involution of $(G, Z_G(A))$ a classification of the quadratic elements characterizing the isomorphism classes in $\mathcal{C}_A(\theta)$.

Remarks 5.3.13. (1) The isomorphism classes of admissible k -involutions can be represented by a (Γ, θ) -index. A classification of these for a number of base fields, including finite fields, number fields, p -adic fields and the real numbers will be discussed in more detail in 6.3.

(2) A classification of the quadratic elements representing the isomorphism classes in $\mathcal{C}_A(\theta_i)$ (see 5.3.11(iii)) is a much more difficult problem. For most fields $k^*/(k^*)^2$ is infinite and hence also the set of quadratic elements of a k -split torus. If k is a finite field, p -adic field or the real numbers then there are only finitely many quadratic elements and a classification is visible. For $k = \mathbb{R}$ the quadratic elements were classified in [12]. The classification of the quadratic elements for $k = \mathbb{Q}_p$ will be dealt with in a future paper.

(3) The classification of the quadratic elements is somewhat simpler in a number of cases. This includes the case when G_θ is k -anisotropic, k -split, θ -split or (θ, k) -split (i.e. a maximal (θ, k) -split torus of G is also maximal θ -split). The main reason for this is that the description of the k -orbits contained in the open orbit of a minimal parabolic k -subgroup on G/H is simpler and the underlying geometry is more transparent. For more details, see [18] and [20].

(4) For $k = \mathbb{R}$ the classification of involutions of compact real groups is the same as the classification of involutions of complex semisimple groups. This means that the classification of the k -involutions reduces to the first and third problem in 5.3.12. A classification of the isomorphism classes of k -involutions, for $k = \mathbb{R}$, together with all the fine structure, can be found in [12].

6. CLASSIFICATION OF THE ADMISSIBLE INDICES

In this section we will show that the isomorphism classes of admissible k -involutions of a semisimple root datum Ψ can be represented by a (Γ, θ) -index and we will classify these for k the real numbers, p -adics fields, finite fields and number fields. For this classification we will need the classifications of θ -indices and Γ -indices. Therefore we first briefly review a few facts from these classifications.

6.1. Admissible θ -indices. In this subsection we discuss the classification of admissible θ -indices related to conjugacy classes of involutions of G . Our notations remain as in 4.3 and 5.1. In particular let G be a reductive algebraic group, T a maximal torus of G , $X = X^*(T)$ and $\Phi = \Phi(T)$.

6.1.1. The question whether an involution $\theta \in \text{Aut}(X, \Phi)$ is admissible can be split in two parts. First one needs to determine whether θ can be lifted, i.e. if there is an involutorial automorphism $\varphi \in \text{Aut}(G, T)$ inducing θ on (X, Φ) , and second the normality condition 5.1.1 must be satisfied. From the isomorphism theorem it follows that there exists always a possibly non involutorial $\varphi \in \text{Aut}(G, T)$, inducing θ on (X, Φ) . The question is when φ is an involution. Any automorphism of (G, T) can be characterized by an automorphism of the underlying root datum and a set of scalars. This can be seen as follows. Let $\{x_\alpha\}_{\alpha \in \Phi(T)}$ be a realization of $\Phi(T)$ in G as in [12] and let $\varphi \in \text{Aut}(G, T)$. Then $\varphi^* = {}^t(\varphi | T)^{-1}$ is an isomorphism of (X, Φ) satisfying $\varphi(U_\alpha) = U_{\varphi^*(\alpha)}$. Moreover there exists $c_{\alpha, \varphi} \in k^*$ such that for $\xi \in k$

$$(8) \quad \varphi(x_\alpha(\xi)) = x_{\varphi(\alpha)}(c_{\alpha, \varphi} \xi).$$

So φ is uniquely determined by φ^* and $\{c_{\alpha, \varphi} \mid \alpha \in \Phi\}$. Thus we will write $\varphi \leftrightarrow \{\varphi^*, c_{\alpha, \varphi}(\alpha \in \Phi)\}$. We will call a system $\{\varphi^*, c_{\alpha, \varphi}(\alpha \in \Phi)\}$ *admissible* if there is an automorphism $\varphi \in \text{Aut}(G, T)$ such that $\varphi \leftrightarrow \{\varphi^*, c_{\alpha, \varphi}(\alpha \in \Phi)\}$.

6.1.2. If $\varphi \in \text{Aut}(G, T)$ induces an involution θ on (X, Φ) , then φ is an involution if and only if

$$(9) \quad (\varphi | T)^2 = \text{id}_T \quad \text{and} \quad c_{\alpha, \varphi} c_{\theta(\alpha), \varphi} = 1 \quad \text{for all } \alpha \in \Phi(T).$$

Moreover θ is admissible if and only if it can be lifted to $\varphi \in \text{Aut}(G, T)$ satisfying

$$(10) \quad c_{\alpha, \varphi} = 1 \quad \text{for all } \alpha \in \Phi_0(\theta).$$

This last condition means that $\Phi(T)$ has no roots which are noncompact imaginary with respect to θ (see [12, 3.4]).

6.1.3. *Rank one restriction.* Recall that the *restricted rank* of an involution $\theta \in \text{Aut}(X, \Phi)$ is defined as the rank of the set of restricted roots $\bar{\Phi}_\theta$. The classification of admissible involutions can be reduced to admissible involutions of restricted rank one as follows. For each $\lambda \in \bar{\Phi}_\theta$ such that $\frac{1}{2}\lambda \notin \bar{\Phi}_\theta$, let $\Phi(\lambda)$ denote the set of all roots $\beta \in \Phi$ such that the restriction of β to \bar{X}_λ is an integral multiple of λ . Then $\Phi(\lambda)$ is a θ -stable closed subsystem of Φ (See [5, p.71]). Let $X(\lambda)$ denote the projection of X on the subspace of $E = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(\lambda)$.

Proposition 6.1.4. *Let $\theta \in \text{Aut}(X, \Phi)$ be an involution and Δ a θ -basis of Φ . Then θ is admissible if and only if $\theta \mid X(\lambda) \in \text{Aut}(X(\lambda), \Phi(\lambda))$ is admissible for all $\lambda \in \bar{\Delta}_\theta$.*

For a proof of this result see [12, 4.5]. This reduces the classification of admissible θ -indices to θ -indices of restricted rank one. From 4.2.14 it follows that it suffices to consider irreducible θ -indices. The irreducible, but not absolutely irreducible θ -indices are those for which $\theta^* = -\theta$ exchanges the connected components (see 4.3.1). There are 17 absolutely irreducible θ -indices of restricted rank one and the question which of these are admissible is a matter of manipulating the structure constants satisfying the conditions (9) and (10). For more details, see [12, §4].

Remark 6.1.5. The classification of isomorphism classes of involutions automorphisms of G is independent of the base field k . For G of adjoint type it is equivalent to the classification of real forms of a semisimple Lie algebra over \mathbb{C} , as is carried out by Araki [3]. See also Sugiura [29, appendix] for a simplification of this method. A further simplification of Araki's classification can be found in [12, §4].

6.2. Admissible Γ -indices. In this subsection we discuss the admissible Γ -indices related to the isomorphism classes of semisimple k -groups. Our notations remain as in 4.4. If $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ is an admissible Γ -index and (G, T) is a k -group corresponding to \mathcal{D} , then we will write $(G, T) \leftrightarrow \mathcal{D}$ to indicate the relation between \mathcal{D} and (G, T) .

6.2.1. The classification of admissible Γ -indices can also be reduced to the case of absolutely irreducible indices of k -rank = 1. That it suffices to consider absolutely irreducible indices can be seen as follows. Suppose $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ and X is simply connected. If \mathcal{D} is k -irreducible, but not absolutely irreducible, then $\Delta = \Delta_1 \cup \dots \cup \Delta_s$, where the Δ_i are mutually disjoint connected components of Δ and correspondingly one has $X = X_1 + \dots + X_s$. Define $\Gamma_1 = \{\sigma \in \Gamma \mid \Delta_1^{[\sigma]} = \Delta_1\}$. Then $\Gamma = \bigcup_{i=1}^s \Gamma_1 \sigma_i$, where $\Delta_i = \Delta_1^{[\sigma_i]}$. Let $\mathcal{D}_1 = (X_1, \Delta_1, \Delta_1 \cap \Delta_0, [\sigma])$, where $\sigma \in \Gamma_1$ and let k_1 be the fixed field of Γ_1 . Now we have the following:

Lemma 6.2.2. *\mathcal{D} is admissible as a Γ -index if and only if \mathcal{D}_1 is admissible as a Γ_1 -index.*

Note that if $(G_1, T_1)/k_1 \leftrightarrow \mathcal{D}_1$ then $(G, T) = R_{k_1/k}(G_1, T_1) \leftrightarrow \mathcal{D}$. Here $R_{k_1/k}$ is the functor of "descent" from the field k_1 to the field k (see [39]).

6.2.3. The above result reduces the classification to absolutely irreducible indices. The next thing we need is a classification of admissible Γ -indices for

k -anisotropic groups. Namely, if $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ is an admissible Γ -index, then one gets a subsystem $\mathcal{D}_0 = \{X^0, \Delta_0, [\sigma]\}$ where X^0 is the projection of X on $\{\Delta_0\}_{\mathbb{Q}}$ and \mathcal{D}_0 is the Γ -index of (G_0, T_0) , where G_0 is the k -anisotropic kernel. So a necessary condition for an Γ -index \mathcal{D} to be admissible is that the subindex \mathcal{D}_0 is a admissible Γ -index of a k -anisotropic group.

In the following let \mathcal{D} be a Γ -index, for which the subindex \mathcal{D}_0 is admissible and corresponds to a k -anisotropic group (G_0, T_0) where T_0 splits over K . We redefine the notion of admissibility for these indices now as follows.

Definition 6.2.4. The Γ -index \mathcal{D} is said to be *admissible over* $(G_0, T_0) \leftrightarrow \mathcal{D}_0$ if there is a connected semi-simple algebraic group G defined over k and a maximal torus T defined over k such that (G, T) is the k -anisotropic kernel of (G, T) , and \mathcal{D} is the Γ -index of G .

6.2.5. From Theorem 5.2.2 it follows now that if a Γ -index $\mathcal{D} = \{X, \Delta, \Delta_0, [\sigma]\}$ \mathcal{D} is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$, then the group (G, T) above, is unique up to k -isomorphism. Whether a Γ -index \mathcal{D} is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$ or not is again a matter of structure constants, like in 6.1.1.

Finally, similar as for involutions (see 6.1.3), the problem of classifying admissible Γ -indices reduces to the case of Γ -indices of groups having k -rank = 1. For more details on both these problems, see [34] or [29, §3.1].

Remark 6.2.6. From the above discussion it follows that the classification of admissible Γ -indices reduces to a classification of absolutely irreducible Γ -indices of k -rank= 1. For a number of base fields the semisimple algebraic k -groups have been classified. For $k = \mathbb{R}$ the classification was already known to Cartan (see [7]). In this case the Γ -indices were classified by Araki [3]. See also [29] and [12] for simplifications of this classification. The admissible Γ -indices have also been classified for p -adic fields, finite fields and number fields. For more details see [34] and [29]. The classification of the semisimple k -groups over a p -adic fields was studied by a number of people, including Tits [34], Satake [29] and Veisfeiler [38]. The classification in the case of number fields was completed only recently. For simply connected semisimple algebraic groups Kneser [21] and Harder [9, 10] determined $H^1(k, G)$. The problem of constructing all central simple Lie algebras of a given type over a number field was solved by a number of people, including N. Jacobson, J.C. Ferrar and B.N. Allison, see [2] for a detailed discussion of this.

6.3. Admissible (Γ, θ) -indices. In this subsection we will show that the $W(A, T)$ -isomorphism classes of admissible k -involutions of $(X^*(T), \Phi(T), \Phi(A))$ as in Theorem 5.3.9 correspond with congruence classes of admissible (Γ, θ) -indices. Our notations remain as in 4.5.1 and 5.3.9. Admissible (Γ, θ) -indices are defined as follows:

Definition 6.3.1. Let G be a reductive k -group, T a maximal k -torus of G , $X = X^*(T)$, $\Phi = \Phi(T)$, A the subtorus of T annihilated by $X_0(\Gamma)$ and $\theta \in \text{Aut}(X, \Phi)$ an involution. If $>$ is a (Γ, θ) -order on (X, Φ) and $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ the corresponding (Γ, θ) -index, then \mathcal{D} is said to be an *admissible (Γ, θ) -index* (with respect to (G, T)) if A is a maximal k -split torus of G and if there exists a k -involution $\tilde{\theta}$ of G , normally related to A and T and such that $\tilde{\theta} \upharpoonright T = \theta$.

The $W(A, T)$ -isomorphism classes of admissible k -involutions of $(X^*(T), \Phi(T), \Phi(A))$ correspond with the congruence classes of admissible (Γ, θ) -indices as follows.

Proposition 6.3.2. *Let A be a maximal k -split torus of G and $T \supset A$ a maximal k -torus of G . There is a bijection between the $W(A, T)$ -isomorphism classes of admissible k -involutions of $(X^*(T), \Phi(T), \Phi(A))$ and the congruence classes of admissible (Γ, θ) -indices of $(X^*(T), \Phi(T))$*

6.3.3. To classify the admissible (Γ, θ) -indices we can first determine all the possible (Γ, θ) -indices. From 4.2.12 it follows that a Γ_θ -index is a (Γ, θ) -index if and only if

(11)

If $\Phi_1 \subset \Phi_0(\Gamma_\theta)$ irreducible component, then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.

The problem which remains then is to determine which of these (Γ, θ) -indices are admissible. One easily shows that an admissible (Γ, θ) -index also satisfies the following conditions:

(12) θ^* and $[\sigma]$ commute ($\sigma \in \Gamma$)

(13) $\Delta_0(\Gamma)$ is θ^* -stable and $\Delta_0(\theta)$ is $[\sigma]$ -stable ($\sigma \in \Gamma$)

This leads to the following definition.

Definition 6.3.4. Let Ψ be a semisimple root datum and let Γ, θ act on (X, Φ) as in 4.5.1. An index $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ is called a *basic (Γ, θ) -index* if it satisfies (11), (12) and (13).

Notation 6.3.5. A basic (Γ, θ) -index $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ contains both a Γ -index and a θ -index. Denote the Γ -index by $\mathcal{D}_\Gamma = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ and the θ -index by $\mathcal{D}_\theta = (X, \Delta, \Delta_0(\theta), \theta^*)$.

6.3.6. Similarly as for admissible Γ -indices one can restrict an admissible (Γ, θ) -index $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ to the k -anisotropic kernel G_0 and obtain an admissible (Γ, θ) -index $\mathcal{D}_0 = (X_0(\Gamma), \Delta_0(\Gamma), \Delta_0(\theta), [\sigma] \upharpoonright \Delta_0(\Gamma), \theta^* \upharpoonright \Delta_0(\Gamma))$ of the pair $(X_0(\Gamma), \Phi_0(\Gamma))$. One would expect that, as in 6.2.3, the admissibility of a (Γ, θ) -index \mathcal{D} would depend on the admissibility

of the restriction index \mathcal{D}_0 and one would need to define a notion of admissibility over \mathcal{D}_0 , like in 6.2.4. This appears not to be necessary as follows from the following result.

Proposition 6.3.7. *Let G be a semisimple k -group, A a maximal k -split torus of G , $T \supset A$ a maximal k -torus of G and let $X = X^*(T)$, $\Phi = \Phi(T)$. Let $\theta \in \text{Aut}(G, T)$ be an involution and assume there exists a fundamental basis of (X, Φ) such that $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ is a basic (Γ, θ) -index as in 6.3.4. Then $\theta(A) = A$ and $\theta|_{Z_G(A)}$ is a k -involution.*

All we need to do now is determine when a basic (Γ, θ) -index is an (Γ, θ) -index and when these are admissible. These problems can be solved simultaneously.

Theorem 6.3.8. *Let (X, Φ) be as above and let Γ, θ act on (X, Φ) as in 4.5.1. Let Δ be a Γ_θ -fundamental basis of (X, Φ) and let $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$. Then the 6-tuple \mathcal{D} is an admissible (Γ, θ) -index if and only if the following conditions are satisfied*

- (1) \mathcal{D} is a basic (Γ, θ) -index
- (2) \mathcal{D}_k is an admissible Γ -index.
- (3) \mathcal{D}_θ is an admissible θ -index

The above results gives a simple condition, for when an involution of G is a k -involution.

Corollary 6.3.9. *Let $\theta \in \text{Aut}(G)$ be an involution and T a maximal torus of G containing a maximal k -split torus A of G and such that T^- (resp. A^-) is a maximal θ -split (resp. (θ, k) -split) torus of G . Then θ is a k -involution of G if and only if the 6-tuple $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ is a basic (Γ, θ) -index.*

6.4. Classification of the (Γ, θ) -indices. It follows from 6.3.8 that the classification of (Γ, θ) -indices depends only on a combinatorial condition, once the classifications of the Γ -indices and θ -indices are known. Similarly as in the case of Γ -indices (see 6.2.1) the classification of the (Γ, θ) -indices can be reduced to a classification of absolutely irreducible (Γ, θ) -indices. In Table I we list the absolutely irreducible (Γ, θ) -indices for k the real numbers, a p -adic field \mathbb{Q}_p , a number field \mathfrak{n} and a finite field \mathbb{F}_q . In this table we use the diagrammatic representation of the (Γ, θ) -indices as in 4.5.5.

In order to be able to refer to the absolutely irreducible (Γ, θ) -indices in this table, we will use the following notation, which combines the notation of Tits [34] for Γ -indices and the notation in [12] for θ -indices. In particular let $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ be a basic Γ_θ -index as in 6.3.4 and let \mathcal{D}_k and \mathcal{D}_θ be the corresponding Γ -index and the θ -index as in 6.3.5. For the Γ -indices we use the notation ${}^g X_{n,r}^t$. Here X denotes the type of Φ , i.e. one of

A, B, \dots, G, n the rank of Φ, r the rank of $\bar{\Delta}_\Gamma$ and g the order of the action of Γ on the Dynkin diagram. In the case that $g = 1$ (i.e. the Dynkin diagram has no nontrivial automorphism) we will omit it in the notation. Finally t denotes either the degree of the division algebra, which occurs in the definition of the considered form or the dimension of the anisotropic kernel. To differentiate between these two cases we put t between parentheses when it stands for the degree of the division algebra. In fact the degree of the division algebra is only used if X is of classical type.

As for the θ -indices they can be described by the type in the Cartan notation together with the rank of the restricted root system $\bar{\Delta}_\theta$, see [12, Table II]. We will use a superindex to indicate the rank of $\bar{\Delta}_\theta$. So combining these two we will denote a (Γ, θ) -index by ${}^g X_{n,r}^t(\text{type } \theta^p)$, where ${}^g X_{n,r}^t$ is as above, type θ is the Cartan notation of the involution and p denotes the rank of $\bar{\Delta}_\theta$. For example ${}^2 A_{2n+1,n+1}^{(1)}(III^p)$ means that Φ is of type A_{2n+1} , θ is of type $AIII_b$, the action of Γ on the Dynkin diagram is the diagram automorphism, the degree of the division algebra is 1 and $\text{rank } \bar{\Delta}_\Gamma = n + 1, \text{rank } \bar{\Delta}_\theta = p$.

In the last 4 columns of this table we indicate if a particular (Γ, θ) -index is admissible or not for the 4 different types of fields we consider. Here a “+” means that this (Γ, θ) -index is admissible for at least one field of that type (say for example number fields). Similarly a “-” means that this (Γ, θ) -index is not admissible for all fields of that type.

For the isomorphy of the k -involutions of G we will use a notation similar to that of the (Γ, θ) -indices. Since a (Γ, θ) -index \mathcal{D} determines only the isomorphy class of an k -involution under $N_G(A)$ (see 5.3.11) we have to add some notation to represent these k -involutions. If σ is a k -involution of G , normally related to (T, A) as in 5.3.2 with (Γ, θ) -index \mathcal{D} , then the other isomorphy classes in $\mathcal{C}_A(\sigma)$ differ at most a quadratic element. Therefor we will denote the k -involutions in a class $\mathcal{C}_A(\sigma)$ by: ${}^g X_{n,r}^t(\text{type } \theta^p)(\sigma, \epsilon_i)$, where ${}^g X_{n,r}^t(\text{type } \theta^p)$ represents the (Γ, θ) -index \mathcal{D} and $\{\epsilon_i \mid i \in I\}$ is a set of quadratic elements in A representing the different isomorphy classes in $\mathcal{C}_A(\sigma)$. All these involutions have the same (Γ, θ) -index \mathcal{D} .

From the (Γ, θ) -indices one easily determines the restricted root systems with Weyl groups and multiplicities. For more details see [18].

Table 1: Absolutely irreducible (Γ, θ) -indices

Type G	(Γ, θ) -index	admissible			
		\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$A_{n,n}^{(1)}(I)$	$\begin{array}{ccccccc} 1 & & 2 & & \dots & & n-1 & & n \\ \circ & \text{---} & \circ & & \dots & & \circ & \text{---} & \circ \end{array}$	+	+	+	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$A_{2n+1, 2n+1}^{(1)}(II)$		+	+	+	+
$A_{n, n}^{(1)}(III_a^p)$ $n - 2p \geq 1$		+	+	+	+
$A_{2n, 2n}^{(1)}(III_b), \sigma = \theta$ ${}^2A_{2n, n}^{(1)}(III_b), \sigma = \Gamma \& \theta$ ${}^2A_{2n, n}^{(1)}(I), \sigma = \Gamma$		+	+	+	+
$A_{2n-1, 2n-1}^{(1)}(III_b), \sigma = \theta$ ${}^2A_{2n-1, n}^{(1)}(III_b), \sigma = \Gamma \& \theta$ ${}^2A_{2n-1, n}^{(1)}(I), \sigma = \Gamma$		+	+	+	+
$A_{2n+1, n}^{(2)}(I)$		+	+	-	+
$A_{2n+1, n}^{(2)}(II)$		+	+	-	+
$A_{2n+1, n}^{(2)}(III_a^{2p})$ $1 \leq 2p < n$		+	+	-	+
$A_{4n-1, 2n-1}^{(2)}(III_b)$		+	+	-	+
$A_{4n+1, 2n}^{(2)}(III_b)$		+	+	-	+
$A_{n, p}^{(2d)}(II)$ $2d = \frac{n+1}{p+1} > 2$		-	+	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$A_{n,p}^{(d)}(I)$ $d = \frac{n+1}{p+1} > 2$		-	+	-	+
${}^2A_{4n-1,2n}^{(1)}(II)$		+	+	+	+
${}^2A_{4n+1,2n+1}^{(1)}(II)$		+	+	+	+
${}^2A_{2n,n}^{(1)}(III^p)$ $1 \leq p < n$		+	+	+	+
${}^2A_{2n+1,n+1}^{(1)}(III^p)$ $1 \leq p \leq n$		+	+	+	+
${}^2A_{n,p}^{(1)}(I)$ $n - 2p > 1$		+	-	-	+
${}^2A_{2n+1,n}^{(1)}(I)$		+	+	-	+
${}^2A_{4n+1,2n}^{(1)}(II)$		+	+	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^2A_{2n+1,n}^{(1)}(III^p)$ $1 \leq p \leq n$		+	+	-	+
${}^2A_{2n+1,n}^{(1)}(III_b)$		+	+	-	+
${}^2A_{2n+1,2p}^{(1)}(II)$ $1 \leq 2p < n$		+	-	-	+
${}^2A_{n,p}^{(1)}(III^q)$ $1 \leq p < q \leq \frac{1}{2}(n+1)$ $p < \frac{1}{2}(n-1)$		+	-	-	+
${}^2A_{n,q}^{(1)}(III^p)$ $1 \leq p \leq q \leq \frac{1}{2}(n-1)$		+	-	-	+
${}^2A_{n,p}^{(d)}(I), \sigma^* = \Gamma^*$ ${}^2A_{n,p}^{(d)}(III_b), \sigma^* = \Gamma^* \& \theta^*$ $d \geq 2$ $2pd \leq n+1$ $d n+1$		-	-	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^2A_{2n-1,p}^{(2d)}(II)$ $d \geq 1$ $2pd \leq n$ $d n$		-	-	-	+
${}^2A_{n,p}^{(d)}(III_a^q)$ $d \geq 2$ $2pd \leq n+1$ $d n+1$ $q \geq pd$		-	-	-	+
${}^2A_{n,p}^{(d)}(III_a^q)$ $d \geq 2$ $2pd \leq n+1$ $d n+1$ $q = rd, r < p$		-	-	-	+
$B_{n,n}(I^p)$		+	+	+	+
$B_{n,n-1}(I^p)$ $p \leq n-1$		+	+	-	+
$B_{n,q}(I^p)$ $1 \leq p \leq q \leq n-2$		+	-	-	+
$B_{n,p}(I^q)$ $1 \leq p \leq q < n$		+	-	-	+
$C_{n,n}^{(1)}(I)$		+	+	+	+
$C_{2n,2n}^{(1)}(II_b)$		+	+	+	+
$C_{n,n}^{(1)}(II^p)$ $1 \leq p \leq \frac{1}{2}(n-1)$		+	+	+	+
$C_{2n,n}^{(2)}(I)$		+	+	-	+
$C_{2n,n}^{(2)}(II_b)$		+	+	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$C_{2n,n}^{(2)}(II^p)$ $1 \leq p \leq (n-1)$		+	+	-	+
$C_{2n+1,n}^{(2)}(I)$		+	+	-	+
$C_{2n+1,n}^{(2)}(II^p)$ $1 \leq p \leq n$		+	+	-	+
$C_{n,p}^{(2)}(I)$ $p < \lfloor \frac{n-1}{2} \rfloor$		+	-	-	+
$C_{2n,p}^{(2)}(II_b)$ $p < n-1$		+	-	-	+
$C_{n,p}^{(2)}(II^q), \sigma = \theta$ $C_{n,q}^{(2)}(II^p), \sigma = \Gamma$ $1 \leq p < q < \frac{1}{2}n$		+	-	-	+
$D_{n,n}^{(1)}(I^p)$ $p \neq n-1$		+	+	+	+
$D_{2n,2n}^{(1)}(III_a)$		+	+	+	+
$D_{2n+1,2n+1}^{(1)}(III_b)$		+	+	+	+
$D_{n,p}^{(1)}(I_a^q)$ $1 \leq p \leq q \leq n-2$ $n-p=2m \geq 2$		+	$p=n-2$	-	+
$D_{n,q}^{(1)}(I_a^p)$ $1 \leq p < q \leq n-2$ $n-p=2m \geq 2$		+	$q=n-2$	-	+
$D_{n,p}^{(1)}(I_b)$ $n-q=2m \geq 2$		+	$p=n-2$	-	+
$D_{2n,2p}^{(1)}(III_a)$ $p < n$		+	$p=n-1$	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$D_{2n,n}^{(2)}(I_a^{2p})$ $p \leq n-1$		+	+	-	+
$D_{2n,n}^{(2)}(I_b)$		+	+	-	+
$D_{2n,n}^{(2)}(III_a)$		+	+	-	+
$D_{2n,n}^{(2)}(III_a)'$		+	+	-	+
$D_{2n+3,n}^{(2)}(I_a^{2p})$ $1 \leq p \leq n$		-	+	-	+
$D_{2n+3,n}^{(2)}(I_b)$		-	+	-	+
$D_{2n+3,n}^{(2)}(III_b)$		-	+	-	+
$D_{2n,p}^{(2)}(I_a^q)$ $1 \leq 2p \leq q < 2n-1$		-	-	-	+
$D_{2n,q}^{(2)}(I_a^{2p})$ $1 \leq 2p < 2q \leq 2n-1$		-	-	-	+
$D_{2n,p}^{(2)}(I_b)$ $n-p \geq 1$		-	-	-	+
$D_{2n,p}^{(2)}(III_a)$ $n-p \geq 1$		-	-	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^2D_{n+1,n}^{(1)}(I^p)$ $p \neq n$		+	+	+	+
${}^2D_{2n+1,2n}^{(1)}(III_b)$		+	+	+	+
${}^2D_{n,p}^{(1)}(I_a^q)$ $p \leq q \leq n-2$		$n-p=$ $2l+1$	-	-	+
${}^2D_{n,q}^{(1)}(I_a^p)$ $1 \leq p < q \leq n-2$ $n-p=2m \geq 2$		$n-q=$ $2l+1$	-	-	+
${}^2D_{n,p}^{(1)}(I_b)$ $n-p > 1$		$n-p=$ $2l+1$	-	-	+
${}^2D_{2n+1,2p}^{(1)}(III_b)$ $p < n$		+	-	-	+
${}^2D_{2n+2,n}^{(2)}(I_a^{2p})$ $p \leq n$		-	+	-	+
${}^2D_{2n+2,n}^{(2)}(I_b)$		-	+	-	+
${}^2D_{2n+1,n}^{(2)}(I_a^{2p})$ $1 \leq p < n$		+	+	-	+
${}^2D_{2n+1,n}^{(2)}(I_b)$		+	+	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^2D_{2n+1,n}^{(2)}(III_b)$		+	+	-	+
${}^3D_{4,2}^2(I_b)$		-	+	+	+
${}^6D_{4,2}^2(I_b)$		-	+	-	+
${}^3D_{4,1}^9(I_b)$		-	-	-	+
${}^6D_{4,1}^9(I_b)$		-	-	-	+
${}^3D_{4,0}^{28}(I_b)$		-	-	-	+
${}^6D_{4,0}^{28}(I_b)$		-	-	-	+
${}^1E_{6,6}^0(I)$		+	+	+	+
${}^1E_{6,6}^0(II), \sigma^* = \theta^*$ ${}^2E_{6,4}^{16}(I), \sigma^* = \Gamma^*$ ${}^2E_{6,4}^{16}(II), \sigma^* = \Gamma^* \& \theta^*$		+	+	+	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^1E_{6,6}^0(III)$		+	+	+	+
${}^1E_{6,6}^0(IV)$		+	+	+	+
${}^1E_{6,2}^{16}(II)$		-	+	-	+
${}^1E_{6,2}^{16}(I)$		-	+	-	+
${}^1E_{6,2}^{28}(I)$		+	-	-	+
${}^1E_{6,2}^{28}(II)$		+	-	-	+
${}^1E_{6,2}^{28}(III), \sigma^* = \theta^*$ ${}^2E_{6,2}^{16'}(IV), \sigma^* = \Gamma^*$		+	-	-	+
${}^1E_{6,2}^{28}(IV)$		+	-	-	+
${}^1E_{6,0}^{78}(I)$		-	-	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^1E_{6,0}^{78}(II)$		-	-	-	+
${}^1E_{6,0}^{78}(III)$		-	-	-	+
${}^1E_{6,0}^{78}(IV)$		-	-	-	+
${}^2E_{6,4}^{16}(IV)$		+	+	+	+
${}^2E_{6,4}^{16}(III)$		+	+	+	+
${}^2E_{6,2}^{16'}(I)$		+	-	-	+
${}^2E_{6,2}^{16'}(II)$		+	-	-	+
${}^2E_{6,2}^{16'}(III)$		+	-	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
${}^2E_{6,2}^{16''}(I)$		-	-	-	+
${}^2E_{6,2}^{16''}(II)$		-	-	-	+
${}^2E_{6,1}^{29}(I)$		-	-	-	+
${}^2E_{6,1}^{29}(III)$		-	-	-	+
${}^2E_{6,1}^{29}(II)$		-	-	-	+
${}^2E_{6,1}^{29}(IV)$		-	-	-	+
${}^2E_{6,1}^{35}(I)$		-	-	-	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathfrak{n}
${}^2E_{6,1}^{35}(II)$		-	-	-	+
${}^2E_{6,1}^{35}(III)$		-	-	-	+
${}^2E_{6,0}^{78}(I)$		+	-	-	+
${}^2E_{6,0}^{78}(II)$		+	-	-	+
${}^2E_{6,0}^{78}(III)$		+	-	-	+
${}^2E_{6,0}^{78}(IV)$		+	-	-	+
$E_{7,7}^0(V)$		+	+	+	+
$E_{7,7}^0(VI)$		+	+	+	+
$E_{7,7}^0(VII)$		+	+	+	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$E_{7,4}^9(V)$		+	+	-	+
$E_{7,4}^9(VI)$		+	+	-	+
$E_{7,4}^9(VII)$		+	+	-	+
$E_{7,3}^{28}(V)$		+	-	-	+
$E_{7,3}^{28}(VI)$		+	-	-	+
$E_{7,3}^{28}(VII)$		+	-	-	+
$E_{7,2}^{31}(V)$		-	-	-	+
$E_{7,0}^{133}(V)$		+	-	-	+
$E_{7,0}^{133}(VI)$		+	-	-	+
$E_{7,0}^{133}(VII)$		+	-	-	+
$E_{8,8}^0(VIII)$		+	+	+	+

Table continued

Table 1: (continued)

Type G	(Γ, θ) -index	\mathbb{R}	\mathbb{Q}_p	\mathbb{F}_q	\mathbf{n}
$E_{8,8}^0(IX)$		+	+	+	+
$E_{8,4}^{28}(VIII)$		+	-	-	+
$E_{8,4}^{28}(IX)$		+	-	-	+
$E_{8,0}^{248}(VIII)$		+	-	-	+
$E_{8,0}^{248}(IX)$		+	-	-	+
$F_{4,4}^0(I)$		+	+	+	+
$F_{4,4}^0(II)$		+	+	+	+
$F_{4,1}^{21}(I)$		+	-	-	+
$F_{4,1}^{21}(II)$		+	-	-	+
$F_{4,0}^{52}(I)$		+	-	-	+
$F_{4,0}^{52}(II)$		+	-	-	+
$G_{2,2}^0(I)$		+	+	+	+
$G_{2,0}^{14}(I)$		+	-	-	+

7. CONCLUDING REMARKS

The results in the previous chapters give some of the structure which is needed to study the harmonic analyses of symmetric k -varieties over fields other than the complex numbers and the real numbers. Undoubtly, many more results will be needed, but these results provide a first insight into the structure and geometry of these symmetric k -varieties.

Most promising seems to be the case of symmetric k -varieties over local fields. Until recently, there was little known about their representation theory. There were only a few results for some of the restricted rank one cases. An example of a recent more general result is [19]. This paper discusses the multiplicities in the Plancherel decomposition. Hopefully, these initial findings will provide a good foundation for the development of the representation theory of symmetric k -varieties.

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