Orbits and invariants associated with a pair of spherical varieties

Some examples

Aloysius G. Helminck (`loek@math.ncsu.edu`)*
Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205

Gerald W. Schwarz (`schwarz@brandeis.edu`)†
Department of Mathematics, Brandeis University, PO Box 549110, Waltham, MA 02454-9110

2000/04/29

Abstract. Let H and K be spherical subgroups of a reductive complex group G. In many cases, detailed knowledge of the double coset space $H\backslash G/K$ is of fundamental importance in group theory and representation theory. If H or K is parabolic, then $H\backslash G/K$ is finite, and we recall the classification of the double cosets in several important cases. If $H = K$ is a symmetric subgroup of G, then the double coset space $K\backslash G/K$ (and the corresponding invariant theoretic quotient) are no longer finite, but several nice properties hold, including an analogue of the Chevalley restriction theorem. In (Helminck and Schwarz, 2001) these properties were generalized to the case where H and K are fixed point groups of commuting involutions. We recall the main results of (Helminck and Schwarz, 2001). We also give examples to show the difficulty in extending these results if we allow $H = K$ to be a reductive spherical (non-symmetric) subgroup or if we have H symmetric and K spherical reductive.

Keywords: Symmetric variety, symmetric subgroup, spherical subgroup

20G15, 20G20, 22E15, 22E46:

1. Introduction

Let G be a connected complex reductive group and let $B \subseteq G$ be a Borel subgroup. A closed subgroup $K \subseteq G$ is called *spherical* if the number of K-orbits in the flag variety G/B is finite; equivalently, the set of (K, B)-double cosets $K\backslash G/B$ is finite. If H and K are spherical subgroups, we may also consider the double coset space $H\backslash G/K$. Results on double coset decompositions are important in representation theory (see, e.g., van den Ban and Schlichtkrull, 1997; Brylinski and Delorme, 1992; Delorme, 1998; Flensted-Jensen, 1980; Ōshima and Sekiguchi, 1980; Vogan, 1983) and for the orbit method of Kirillov and Kostant (Kirillov, 1993).

We have a natural action $(H \times K) \times G \to G$, where $((h, k), g) \mapsto hgk^{-1}$, $h \in H$, $k \in K$, $g \in G$. If H and K are reductive, let $G/\!\!/(H \times K)$ denote $\operatorname{Spec} \mathcal{O}(G)^{H\times K}$ (the invariant theoretical quotient).

---

[*] First author is partially supported by N.S.F. Grant DMS-9977392

[†] Second author is partially supported by N.S.F. Grant DMS-0070472

Let $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ denote the Lie algebras of G, K and H, respectively. The following problems arise naturally.

(1) Parametrize the (H, K)-double cosets in G and, if possible, single out the closed (H, K)-double cosets.

(2) In case H and K are reductive, determine the quotient $G/\!\!/(H \times K)$. In particular, does there exist a variant of the Chevalley restriction theorem? That is, is there a torus $A \subset G$ and an associated Weyl group W such that $A/W \xrightarrow{\sim} G/\!\!/(H \times K)$?

(3) What are the properties of the "slice" representation of $M = H \cap K$ on $\mathfrak{g}/(\mathfrak{h} + \mathfrak{k})$.

(4) (Weak version of (2)) Is there a torus $A \subset G$ and a $g \in G$ such that $H \times A \times K \to G$, $(h, a, k) \mapsto hagk$, is dominant?

In (Akhiezer, 1993) it was shown that (4) always holds (without any reductivity assumptions). For several large classes of spherical subgroups, such as symmetric subgroups and parabolic subgroups, much is known about problems (1)–(3). Unfortunately, for general spherical subgroups, little is known. In section 4 and 5 we will present a number of examples for which (2) fails and for which the slice representation (3) is "bad." Before that we will review what is known in the cases of symmetric subgroups, Borel subgroups and parabolic subgroups.

The authors thank M. Brion for helpful remarks.

2. Finite Double Coset Spaces

2.1. SOME NOTATION

For simplicity, we will work over the field $\mathbb{C}$ of complex numbers (but many results hold for algebraically closed fields k with $\operatorname{char} k \neq 2$). All groups, varieties, etc. will be defined over $\mathbb{C}$. All finite dimensional representations of algebraic groups that we consider will be assumed rational.

Recall that a subgroup $K \subset G$ is called symmetric if there is an involution $\theta \in \operatorname{Aut}(G)$ such that $K = G^{\theta} = \{x \in G \mid \theta(x) = x\}$. In the following we briefly discuss what is known about problems (1)–(3) in a number of special cases.

2.2. $H = P$ IS A PARABOLIC SUBGROUP AND $K = B$ IS A BOREL SUBGROUP

This case is well-known: by the Bruhat decomposition, each (B, P)-double coset intersects the Weyl group W in a unique coset of W_P, the Weyl group of P. Instead of (B, P)-double cosets one can also consider B-orbits in G/P. The unique closed B-orbit in G/P is the fixed point eP. So we have a solution of problem (1).

2.3. $H = B$ IS A BOREL SUBGROUP AND K IS A SYMMETRIC SUBGROUP

This case was studied extensively in (Springer, 1984) in which one finds several characterizations of the double cosets. For a maximal torus T of G, let $N_G(T)$ denote its normalizer, $Z_G(T)$ its centralizer, $W_G(T) = N_G(T)/Z_G(T)$ the Weyl group of T and $W_K(T) = N_K(T)/Z_K(T) = \{w \in W_G(T) \mid w \text{ has a representative in } N_K(T)\}$. Let $\mathcal{C}$ denote the set of pairs (B', T') where T' is a θ-stable maximal torus of a Borel subgroup B' of G and let $\mathcal{B}$ denote the variety of all Borel subgroups of G. The group G acts on $\mathcal{B}$ and $\mathcal{C}$ on the left by conjugation. Let $K\backslash\mathcal{B}$ (resp. $K\backslash\mathcal{C}$) denote the set of K-orbits in $\mathcal{B}$ (resp. $\mathcal{C}$).

Proposition 2.4 (Springer**,** 1984**).** *Let B be a Borel subgroup of G and let $\{T_i | i \in I\}$ be representatives of the K-conjugacy classes of θ-stable maximal tori in G. Then*

$$B\backslash G/K \cong K\backslash\mathcal{B} \cong \bigcup_{i\in I} W_G(T_i)/W_K(T_i) \cong K\backslash\mathcal{C}.$$

Obviously, there also exists a canonical bijection between the set of $(B \times K)$-orbits in G and the set of B-orbits in G/K. Geometrically, the orbits of K, B and $B \times K$ are completely different. For example, the B-orbits in G/K are always irreducible and connected, but the K-orbits in $\mathcal{B}$ are not necessarily connected since K need not be connected. All the K-orbits in $\mathcal{C}$ are closed; however, there is a bijection between the closures of the K-orbits in $\mathcal{B}$ and the closures of the $(B \times K)$-orbits in G. The closed $(B \times K)$-orbits are those corresponding to the K-conjugacy classes of pairs (B', T') where B' is θ-stable. If (B', T') is such a pair, then one can show that T' contains a maximal torus of K. Since all maximal tori containing a maximal torus of K are K-conjugate it follows that there is a bijection between the closed K-orbits in G/B and $W_K(T')\backslash W_G(T')^\theta$. So we have a solution of problem (1). See (Richardson and Springer, 1990) for more about the $(B \times K)$-orbits and for a combinatorial description of the Bruhat order on orbit closures.

2.5. $H = P$ IS A PARABOLIC SUBGROUP AND K IS A SYMMETRIC SUBGROUP

This case was studied extensively in (Brion and Helminck, 2000) and is very similar to the case that $H = B$ is a Borel subgroup.

Let $\mathcal{P}(G)$ denote the variety of all parabolic subgroups of G.

Proposition 2.6 (Brion and Helminck**,** 2000**).** *There is a bijective map from the set of* K*-orbits in* $\mathcal{P}(G)$ *onto the set of* K*-conjugacy classes of triples* (P, B, T) *where*

(1) P *is a parabolic subgroup of* G*,*

(2) B *is a Borel subgroup of* P *such that the product* $(P \cap K)B$ *is open in* P*, and*

(3) T *is a* θ*-stable maximal torus of* B*.*

The inverse maps the K*-conjugacy class of* (P, B, T) *to that of* P*.*

For the remainder of this section we fix a parabolic subgroup P of G. In the case that G has only one K-conjugacy class of maximal tori, then we get that $P\backslash G/K \cong W_P(T)\backslash W_G(T)/W_K(T)$, a result which essentially combines the orbit descriptions in 2.2 and 2.3. In the general case one easily verifies that one gets a union of double cosets.

Corollary 2.7. *Let* $x_1, \ldots, x_r \in G$ *such that* $T_1 = x_1 T x_1^{-1}, \ldots, T_r = x_r T x_r^{-1}$ *are representatives for the* K*-conjugacy classes of the* θ*-stable maximal tori occurring in the* K*-conjugacy classes of the triples* (P, B, T) *as above and* $P = P_1 = x_1 P x_1^{-1}, \ldots, P_r = x_r P x_r^{-1}$*. Then*

$$P\backslash G/K \cong \bigcup_{i=1}^{r} W_{P_i}(T_i)\backslash W_G(T_i)/W_K(T_i).$$

The characterization of the closed double cosets is somewhat more complicated in this case, since there does not need to exist a θ-stable conjugate of P. The closed K-orbits in G/P are characterized by the K-conjugacy classes of triples (P', B', T') as above where B' is θ-stable and $P' \supset B'$ is a conjugate of P satisfying $P' \cap \theta(P')$ is a parabolic subgroup of G. If (P', B', T') is such a triple, then similar to the case that $P = B$ one can show that there is a bijection between the closed K-orbits in G/P and $W_K(T')\backslash W_G(T')^\theta/W_{P'}(T')^\theta$. So again we have a solution of problem (1).

3. Symmetric Subgroups

In case H and K are reductive, the double coset space $H\backslash G/K$ is, in general, no longer finite, and we are led to study the invariant theoretic quotient of G by $H \times K$. For this we need some preliminaries on quotients:

If X is an affine G-variety, G-reductive, then $\mathcal{O}(X)^G$, the algebra of invariant functions on X, is finitely generated. Let $X/\!\!/G$ denote the affine variety corresponding to $\mathcal{O}(X)^G$ and let π (or π_X) denote the morphism $X \to X/\!\!/G$

dual to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$. For $x \in X$, let $T_x X$ denote the tangent space at x, Gx the G-orbit through x and G_x the isotropy group at x. We say that the G-action on X is *stable* if there is a nonempty open subset of X consisting of closed orbits. If all the G-orbits in X are closed (e.g., G is finite), then the quotient is called *geometric*, in which case the notation X/G is also used.

A G-module V is called *polar* if there exists a vector $v \in V$ such that $c := \{x \in V \mid T_x(Gx) \subset T_v(Gv)\}$ is a linear subspace of V of dimension $\dim V/\!\!/G$. The subspace c is called a *Cartan subspace of* V. The inclusion $c \to V$ induces an isomorphism $c/W(c) \to V/\!\!/G$, where $W(c)$ is the normalizer of c in G divided by the centralizer of c in G (Dadok and Kac, 1985). If G is connected, then $W(c)$ is generated by pseudoreflections, and $V/\!\!/G$ is smooth.

3.1. $H = K$ IS A SYMMETRIC SUBGROUP

This case was studied extensively in (Vust, 1974) and (Richardson, 1982). We assume that G is connected, and that we are given an involution θ of G with $K = G^\theta$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the (-1)-eigenspace of θ acting on the Lie algebra $\mathfrak{g}$ of G. There is an isomorphism $\beta\colon G/K \simeq P \subset G$, where $\beta(gK) = g\theta(g)^{-1}$ and $P := \operatorname{Im}\beta$. The left-action of K on G/K, under the isomorphism β, becomes conjugation. We say that a subset $S \subset G$ is θ*-split* if $\theta(s) = s^{-1}$ for all $s \in S$. Note that P consists of θ-split elements, and one can show that P is a connected component of the subvariety of θ-split elements of G. Let $A \subset P$ be a maximally θ-split torus, and let W denote the Weyl group of A relative to the action of K, i.e., the normalizer of A in K divided by the centralizer of A in K. Then Vust and Richardson show

Theorem 3.2. *Let G, θ, K, P, etc. be as above. Then*

1. *The inclusion $A \to P$ induces an isomorphism $A/W \simeq P/\!\!/K$.*
2. *The isotropy representation of K on $T_e(G/K) \simeq \mathfrak{p}$ is polar. In fact, a polar subspace is the Lie algebra $\mathfrak{a}$ of A and $W(\mathfrak{a}) = W$.*
3. *Each fiber of $P \to P/\!\!/K$ consists of finitely many orbits.*
4. *The closed K-orbits are exactly those which intersect A.*

Part (3) says that the map of $P/K \to P/\!\!/K$ is finite to one, so one has a complete solution to the problems (1)–(3), up to computing this finite to one map.

3.3. H AND K ARE SYMMETRIC SUBGROUPS

This case was studied extensively in (Helminck and Schwarz, 2001). One has analogous results to those of Vust and Richardson above. We assume that G is connected and that we have two commuting involutions σ and θ. Set $K = G^\theta$ and $H = G^\sigma$. Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{q}$ is the (-1)-eigenspace of σ. We have $G/K \simeq P$ as before, but the action of H on P is not by conjugation. The action is $(h, p) \mapsto h * p := hp\theta(h)^{-1}$, $h \in H$, $p \in P$. Let A denote a torus of G which is maximal among tori which are simultaneously σ and θ split. Let $W_H^*(A)$ denote the Weyl group for the $*$-action of H, i.e., $W_H^*(A) = N_H^*(A)/Z_H^*(A)$ where $N_H^*(A) = \{h \in H \mid h * A \subset A\}$ and $Z_H^*(A) = \{h \in H \mid h * a = a \text{ for all } a \in A\}$. Finally, we call the representation of $M := H_e$ on $T_eP/T_e(H * e)$ the "slice representation" of M. Note that $M = K \cap H$ and that $T_eP/T_e(H * e) \simeq \mathfrak{p} \cap \mathfrak{q}$. We have:

Theorem 3.4. *Let H, etc. be as above.*

(1) The inclusion $A \to P$ induces an isomorphism $A/W_H^*(A) \simeq P/\!\!/H$.

(2) The slice representation of M on $\mathfrak{p} \cap \mathfrak{q}$ is polar. In fact, a polar subspace is the Lie algebra $\mathfrak{a}$ of A and $W(\mathfrak{a}) \simeq \{w \in W_H^*(A) \mid w * e = e\}$.

(3) Each fiber of $P \to P/\!\!/H$ consists of finitely many orbits.

(4) The closed H-orbits are exactly those which intersect A.

Besides a solution to the problems (1)–(3), as outlined in the introduction, (Helminck and Schwarz, 2001) contains several other results. We review them in the remainder of this section.

3.5. SLICES

Fundamental in the proof of Theorem 3.4 is the existence of slices:

Theorem 3.6. *Let $a \in A$. Then there is a reductive subgroup $\hat{G} \subset G$ with an involution $\hat{\theta}$ and associated symmetric variety $\hat{P}$ such that*

(1) $(\hat{G})^{\hat{\theta}} = H_a^* = \{h \in H \mid h * a = a\}$.

(2) There is an étale slice $\mathcal{S}$ at a which is isomorphic to an open H_a^*-stable subvariety of $\hat{P}$.

(3) There is a bijection between the (finitely many) orbits in $\pi_{\hat{P}}^{-1}(\pi_{\hat{P}}(a))$ and those in $\pi_P^{-1}(\pi_P(a))$.

This theorem shows that, roughly, $\pi_P\colon P \to P/\!/H$ is locally the quotient map of a symmetric veriety.

3.7. THE WEYL GROUP $W_H^*(A)$

The Weyl group $W_H^*(A)$ is much more complicated than in the case $\sigma = \theta$, where one simply has $W_H^*(A) = W(A) = W_H(A) = N_H(A)/Z_H(A)$. In general, $W_H^*(A)$ is isomorphic to the semidirect product of $W_H(A)$ with a 2-group $F_0 \subset A$. In the following we will review this in some more detail.

If $h \in N_H^*(A)$, then $h * e = \beta(h) \in A$. But then $\beta(h)$ is σ-fixed and σ-split, so $\beta(h) \in A^{(2)}$, the elements of A of order 2. Hence $N_H^*(A) = \{h \in N_H(A) \mid \beta(h) \in A^{(2)}\}$. Let

$$\rho\colon W_H^*(A) \to W_H(A) \ltimes A^{(2)}, \quad [h] \mapsto (\mathrm{conj}(h), \beta(h)),$$

denote the canonical monomorphism. Here $[h]$ denote the class of $h \in N_H^*(A)$ in $W_H^*(A)$ and $\mathrm{conj}(h)$ denotes conjugation by h. The semidirect product $W_H(A) \ltimes A^{(2)}$ acts naturally on A by: $(w, a) * b = aw(b)$, $w \in W_H(A)$, $a \in A^{(2)}$, $b \in A$. The action of $W_H^*(A)$ factors through ρ: if $\rho([h]) = (w, a) \in W_H(A) \ltimes A^{(2)}$, then $[h] * b = aw(b)$, $b \in A$.

The twisted Weyl group $W_H^*(A)$ has a normal subgroup of "pure translations" W_0, where $W_0 = \rho^{-1}(\{e\} \times A^{(2)})$. We usually identify W_0 with its $W_H^*(A)$-stable image $F_0 := \beta(W_0) \subset A^{(2)}$.

By changing the involution θ by a *quadratic element* one can keep $W_H^*(A)$ fixed, while improving on the embedding into $W_H(A) \ltimes A^{(2)}$. Let $Z(G)$ denote the center of G. We say that $q \in A$ is *quadratic* if $q^2 \in Z(G)$ and we let $Q(A)$ denote the set of quadratic elements in A. If $q \in Q(A)$ and $\tilde{\theta} := \theta\,\mathrm{conj}(q)$, then $(\sigma, \tilde{\theta})$ is a pair of commuting involutions and the group $W_H^*(A)$ is the same for $(\sigma, \tilde{\theta})$ and (σ, θ). However, the embedding of $w = [h] \in W_H^*(A)$ into $W_H(A) \ltimes A^{(2)}$ switches from $(w, h\theta(h)^{-1})$ to $(w, h\tilde{\theta}(h)^{-1}) = (w, w(q^{-1})qh\theta(h)^{-1})$. Note that $w(q^{-1})q \in A^{(2)}$ for every $w \in W_H(A)$.

Theorem 3.8. *There is a $q \in Q(A)$ such that every $w \in W_H(A)$ has a representative $h \in N_H^*(A)$ such that $w(q^{-1})q\beta(h) = e$*

Corollary 3.9. $W_H^*(A) \simeq W_H(A) \ltimes F_0$.

Quadratic elements also play an essential role in the classification of pairs of involutions given in (Helminck, 1988).

3.10. CHARACTERIZATION OF F_0

We can characterize F_0 in terms of tori, as follows: Let $T \subset G$ be a (σ, θ)-stable maximal torus containing A. Set $\tau = \sigma\theta$. For $\delta = \sigma, \theta$ or τ we denote by T_-^δ the maximal δ-split torus in T which is also (σ, θ)-stable. We say that

T is *standard* if T^{σ}_{-} is a maximal σ-split torus of G and T^{θ}_{-} a maximal θ-split torus of G. By (Helminck, 1988, 5.13) standard maximal tori exist.

Theorem 3.11. *Let $T \supset A$ be a standard maximal torus. Then $F_0 = A \cap T^{\tau}_{-}$.*

3.12. REAL GROUPS

(Helminck and Schwarz, 2001) also contains a number of results about real groups. Suppose that G, σ and θ are defined over $\mathbb{R}$. For an affine variety X defined over $\mathbb{R}$ denote the real points of X by $X_{\mathbb{R}}$ or $X(\mathbb{R})$.

In the case that $G_{\mathbb{R}}$ is compact we have:

Theorem 3.13. *Suppose that $G_{\mathbb{R}}$ is compact. Then*

(1) All $H_{\mathbb{R}}$-orbits in $P_{\theta}(G_{\mathbb{R}})$ and H-orbits intersecting $P_{\theta}(G_{\mathbb{R}})$ are closed.

(2) $G_{\mathbb{R}} = H_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$.

(3) $\beta(A_{\mathbb{R}}) = A_{\mathbb{R}}$ and $W^*_H(A) = W^*_{H(\mathbb{R})}(A_{\mathbb{R}})$.

(4) $H_{\mathbb{R}} \backslash G_{\mathbb{R}} / K_{\mathbb{R}} \simeq A_{\mathbb{R}} / W^*_H(A)$.

In the case that $G_{\mathbb{R}}$ is noncompact and θ is a Cartan involution we have:

Theorem 3.14. *Suppose that θ is a Cartan involution of $G_{\mathbb{R}}$. Then*

(1) All $H_{\mathbb{R}}$-orbits in $P_{\theta}(G_{\mathbb{R}})$ and H-orbits intersecting $P_{\theta}(G_{\mathbb{R}})$ are closed.

(2) $G_{\mathbb{R}} = H^0_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$.

(3) $A_{\mathbb{R}}$ is isomorphic to $(\mathbb{R}^*)^l$ for some l. Hence $A^2_{\mathbb{R}} := \beta(A_{\mathbb{R}}) = \{a^2 \mid a \in A_{\mathbb{R}}\} \simeq (\mathbb{R}^+)^l$.

(4) $H_{\mathbb{R}} \backslash G_{\mathbb{R}} / K_{\mathbb{R}} \simeq A^2_{\mathbb{R}} / W_{M(\mathbb{R})}(A)$ where $M = H \cap K$.

4. Spherical Double Coset Spaces $K\backslash G/K$

We consider double coset spaces $K\backslash G/K$ where K is spherical and reductive. In case the isotropy representation of K on $V := \mathfrak{g}/\mathfrak{k}$ is not polar, then property (2) cannot hold. We give several examples where this occurs. Thus, in general, one cannot hope to have the same kind of Chevalley Restriction Theorem in the spherical case as in the symmetric case.

However, the isotropy representations come close to being polar, by results of (Knop, 1990) and (Panyushev, 1990). Let K and V be as above. Then there is a "generalized" Cartan subspace $c \subset V$ and a "generalized" Weyl group $W \subset \mathrm{GL}(c)$ as follows:

- The inclusion $c \to V$ induces an isomorphism $\mathbb{C}[V]^H \simeq \mathbb{C}[c]^W$.
- $H \cdot c$ is Zariski dense in V.
- There is a nonempty open subset $U \subset c$ such that H_x is H-conjugate to H_y for all $x, y \in U$.
- $\mathbb{C}[V]$ is a free $\mathbb{C}[V]^H$-module.

In general, c is not the Lie algebra of a torus (see equation 16 of (Panyushev, 1990)) and the Weyl group W is not the image of the normalizer of c in H (Remark 2 of Cor. 6 of (Panyushev, 1990)).

Definitions 4.1. Let V be an H-module, H reductive. We say that V is

(1) *coregular* if $\mathcal{O}(V)^H$ is a polynomial ring; equivalently, $V/\!/H$ is smooth;

(2) *cofree* if $\mathcal{O}(V)$ is a free $\mathcal{O}(V)^H$ module.

Thus our spherical isotropy representations are cofree, and this also holds for all polar representations (Dadok and Kac, 1985).

We will use the following (see Schwarz, 1978; Dadok and Kac, 1985)

Lemma 4.2. *Let H be reductive and V an H-module.*

(1) The module V is cofree if and only if V is coregular and all the fibers of $V \to V/\!/H$ have the same dimension.

(2) Suppose that $V = V_1 \oplus V_2$ where V_1 or V_2 is stable and V is polar. Then V_1 and V_2 are polar. Let c_i be a Cartan subspace of V_i, $i = 1, 2$. Then a Cartan subspace of V is $c_1 \oplus c_2$.

Corollary 4.3. *Suppose that $V = V_1 \oplus V_2$ where V_1 or V_2 is stable, and suppose that* $\dim V/\!/H > \dim V_1/\!/H + \dim V_2/\!/H$. *Then V is not polar.*

Below are some examples of spherical subgroups $K \subset G$ such that the isotropy representation of K at $T_e(G/K)$ is not polar. For a list of the “irreducible” spherical reductive subgroups of semisimple groups see (Brion, 1987).

If $\varphi\colon G \to \mathrm{GL}(V)$ is a representation, we will use both V and φ (when no confusion can arise) to denote the corresponding G-module. A trivial representation of G of dimension n is denoted Θ_n. For G a simple algebraic group of rank l, we use the notation $\varphi_1^{a_1} \dots \varphi_l^{a_l}$ to denote the module (or representation) corresponding to the highest weight $a_1\omega_1 + \cdots + a_l\omega_l$, where the ω_i are the fundamental highest weights (we use the numbering in (Bourbaki, 1968)). For example, the adjoint represention of SL_{n+1} is denoted $\varphi_1\varphi_n$. If $G = \mathbb{C}^*$ is a one-dimensional torus, then ν_j will denote the one-dimensional representation (or module) with weight $j \in \mathbb{Z}$.

Example 4.4. Let $G = \mathrm{SO}(n) \times \mathrm{SO}(n+1)$, $n \geq 7$, and let $K = \mathrm{SO}(n)$ embedded diagonally in G so that the restriction of $\varphi_1(\mathrm{SO}(n+1))$ to K is $\varphi_1(\mathrm{SO}(n)) + \Theta_1$. Then the isotropy representation of K is just the restriction to K of the adjoint representation of $\mathrm{SO}(n+1)$, which gives the K-representation $\rho := \varphi_2 \oplus \varphi_1$. Both φ_2 (the adjoint representation) and φ_1 (the standard action on $\mathbb{C}^n$) are stable. The dimension of $\rho /\!\!/ K$ is n, while the dimensions of $\varphi_1 /\!\!/ K$ and $\varphi_2 /\!\!/ K$ are 1 and $[\frac{n}{2}]$, respectively. Thus, by Corollary 4.3, the isotropy representation of K is not polar.

Example 4.5. Let $G = \mathrm{SL}_{2n+1}$, and let $K = \mathrm{Sp}_{2n} \subset G$ such that $(\varphi_1, \mathrm{SL}_{2n+1})$ restricts to $(\varphi_1 \oplus \Theta_1, K)$. The isotropy representation of K is $\varphi_2 + 2\varphi_1$ which is not polar by another application of Corollary 4.3.

Example 4.6. Let $K = \mathrm{SL}_n \times \mathbb{C}^*$ mapped diagonally to $G := \mathrm{SL}_n \times \mathrm{SL}_{n+1}$. Here $\varphi_1(\mathrm{SL}_{n+1})$ restricts to the representation $\varphi_1 \otimes \nu_1 \oplus \nu_{-n}$ of $\mathrm{SL}_n \times \mathbb{C}^*$. The isotropy representation of K is $\varphi_1\varphi_{n-1} \oplus \varphi_1 \otimes \nu_{n+1} \oplus \varphi_{n-1} \otimes \nu_{-n-1} \oplus \Theta_1$, which is not polar by 4.3.

5. Examples of Hybrid Coset Spaces $K\backslash G/H$

Here we consider the following situation. We have a reductive group G, an involution σ of G, and a spherical subgroup K of G which is σ-stable. We consider the categorical quotient of G by $H \times K$, where $H = G^\sigma$. Let M denote $H \cap K$. Then K acts on the left on G/H, the isotropy group at eH is M, and the representation of M on $T_e(G/H)/T_e(KH/H) \simeq \mathfrak{g}/(\mathfrak{k} + \mathfrak{h})$ is the slice representation. We give examples to show that this representation need not be cofree nor even coregular. When $K = G^\theta$ where θ and σ are commuting involutions, we always have a smooth quotient (and polar slice representation), as we saw in Theorem 3.4.

Example 5.1. Let $G = \mathrm{SL}_{2n+1}$ and $K = \mathrm{Sp}_{2n}$ be as in Example 4.5. Let θ be inverse transpose. Then K is θ-stable, and $M = K \cap G^\theta \simeq \mathrm{GL}_n$. Think of GL_n as a quotient of $\mathrm{SL}_n \times \mathbb{C}^*$. Then the slice representation of GL_n comes from the representation $\varphi_1\varphi_{n-1} \oplus \varphi_1 \otimes \nu_1 \oplus \varphi_n \otimes \nu_{-1}$ of $\mathrm{SL}_n \times \mathbb{C}^*$. One can compute that this representation is cofree, but it is not polar by an application of 4.3.

Example 5.2. Let $H = \mathrm{SL}_n \times \mathbb{C}^* \subset G := \mathrm{SL}_n \times \mathrm{SL}_{n+1}$ be as in 4.6. Then $H \subset G$ is stable under inverse transpose, and M is $\mathrm{SO}(n) \times \{\pm 1\}$. However, the orbits of the slice representation of M are the same as those of the index 2 subgroup $\mathrm{SO}(n)$, and the slice representation of M restricted to SO_n is $\varphi_1^2 \oplus \varphi_1$. (In terms of the standard representation on $\mathbb{C}^n$, this is $S^2(\mathbb{C}^n) \oplus \mathbb{C}^n$ minus a one-dimensional trivial factor). By (Schwarz, 1978), this representation is not coregular, hence neither cofree nor polar.

Example 5.3. Let $H = \mathrm{SL}_2 \subset G := (\mathrm{SL}_2)^3$ be diagonally embedded, and define θ to be simultaneous inverse transpose on the copies of SL_2 in G. Then M is a diagonal copy of $\mathrm{SO}_2 \simeq \mathbb{C}^* \subset G$, and the slice representation is two copies of $(\nu_2 \oplus \nu_{-2})$, which is not coregular.

References

Akhiezer, D.: 1993, 'On Lie algebra decompositions, related to spherical homogeneous spaces'. *Manuscripta Math.* **80**(1), 81–88.

Bourbaki, N.: 1968, *Groups et algébres de Lie, Chapitres 4,5,6*, Éléments de Mathématique. Paris: Hermann.

Brion, M.: 1987, 'Classification des espaces homogènes sphériques'. *Compositio Math.* **63**(2), 189–208.

Brion, M. and A. G. Helminck: 2000, 'On orbit closures of symmetric subgroups in flag varieties'. *Canad. J. Math.* **52**(2), 265–292.

Brylinski, J.-L. and P. Delorme: 1992, 'Vecteurs distributions H-invariants pour les séries principales géneralisées d'espaces symétriques réductifs et prolongement méromorphe d'integrales d'Eisenstein'. *Invent. Math.* **109**, 619–664.

Dadok, J. and V. Kac: 1985, 'Polar representations'. *J. Algebra* **92**(2), 504–524.

Delorme, P.: 1998, 'Formule de Plancherel pour les espaces symétrique réductifs'. *Annals of Math.* **147**(2), 417–452.

Flensted-Jensen, M.: 1980, 'Discrete series for semisimple symmetric spaces'. *Annals of Math.* **111**, 253–311.

Helminck, A. G.: 1988, 'Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces'. *Adv. in Math.* **71**, 21–91.

Helminck, A. G. and G. W. Schwarz: 2001, 'Orbits and invariants associated with a pair of commuting involutions'. *Duke Math. J.* **106**(2), 237–279.

Kirillov, A.: 1993, 'The orbit method. I. Geometric quantization'. In: *Representation theory of groups and algebras*, Vol. 145 of *Contemp. Math.* Providence, RI, pp. 1–32, Amer. Math. Soc.

Knop, F.: 1990, 'Weylgruppe und Momentabbildung'. *Invent. Math.* **99**(1), 1–23.

Ōshima, T. and J. Sekiguchi: 1980, 'Eigenspaces of invariant differential operators in an affine symmetric space'. *Invent. Math.* **57**, 1–81.

Panyushev, D. I.: 1990, 'Complexity and rank of homogeneous spaces'. *Geom. Dedicata* **34**(3), 249–269.

Richardson, R. W.: 1982, 'Orbits, invariants and representations associated to involutions of reductive groups'. *Invent. Math.* **66**, 287–312.

Richardson, R. W. and T. A. Springer: 1990, 'The Bruhat order on symmetric varieties'. *Geom. Dedicata* **35**(1-3), 389–436.

Schwarz, G. W.: 1978, 'Representations of simple Lie groups with regular rings of invariants'. *Invent. Math.* **49**(2), 167–191.

Springer, T. A.: 1984, 'Some results on algebraic groups with involutions'. In: *Algebraic groups and related topics*, Vol. 6 of *Adv. Stud. in Pure Math.* Orlando, FL: Academic Press, pp. 525–543.

van den Ban, E. P. and H. Schlichtkrull: 1997, 'The most continuous part of the Plancherel decomposition for a reductive symmetric space I'. *Ann. Math.* **145**, 267–364.

Vogan, D. A.: 1983, 'Irreducible characters of semi-simple Lie groups III. Proof of the Kazhdan-Lusztig conjectures in the integral case'. *Invent. Math.* **71**, 381–417.

Vust, T.: 1974, ‘Opération de groupes reductifs dans un type de cônes presque homogènes’. *Bull. Soc. Math. France* **102**, 317–334.

Address for Offprints:
Aloysius G. Helminck
Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205
loek@math.ncsu.edu

and

Gerald W. Schwarz
Department of Mathematics
Brandeis University
PO Box 549110
Waltham, MA 02454-9110
schwarz@brandeis.edu