Involutions of $\mathrm{SL}(n, k)$, $(n > 2)$

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Abstract. A first characterization of the isomorphism classes of k-involutions for any reductive algebraic groups defined over a perfect field was given in [7] using 3 invariants. In this paper we give a simple characterization of the isomorphism classes of involutions of $\mathrm{SL}(n, k)$ with k any field of characteristic not 2. We classify the isomorphism classes of involutions for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. We also determine in which cases the corresponding fixed point group H is k-anisotropic. In those cases the corresponding symmetric k-variety consists of semisimple elements.

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1. Introduction

Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k, H a k-open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. These varieties occur in many problems in representation theory, geometry and singularity theory. To study these symmetric k-varieties one needs first a classification of the related k-involutions. A characterization of the isomorphism classes of the k-involutions was given in [7] essentially using the following 3 invariants:

(1) classification of admissible (Γ, θ)-indices.

(2) classification of the G_k-isomorphism classes of k-involutions of the k-anisotropic kernel of G.

(3) classification of the G_k-isomorphism classes of k-inner elements of G.

For more details, see [7]. The admissible (Γ, θ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was

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included in [7] as well. For k algebraically closed or k the real numbers the full classification can be found in [6]. For other fields a classification of the remaining two invariants is still lacking. In particular the case of symmetric k-varieties over the $\mathfrak{p}$-adic numbers is of interest. We note that the above characterization only holds for k a perfect field.

In this paper we give a simple characterization of the isomorphism classes of k-involutions in the case that $G = \mathrm{SL}(n, k)$. This characterization does not depend on any of the results in [7] and also holds for any field of characteristic not 2, not only perfect fields. Using this characterization we classify the possible isomorphism classes for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. To classify the outer involutions of $\mathrm{SL}(n, k)$ we show that there exists a natural connection of these involutions with non degenerate symmetric or skew symmetric bilinear forms and that the isomorphism classes of these involutions correspond bijectively with semi-congruence classes of the bilinear forms. Here semi-congruence is defined as congruence up to a scalar multiple (see Definition 3).

For k the $\mathfrak{p}$-adic numbers the symmetric k-varieties G_k/H_k with H_k k-anisotropic have a similar structure as the Riemannian symmetric spaces and therefore these cases are of particular interest for studying their representations. In section 7 we compute the fixed point groups and for k the $\mathfrak{p}$-adic numbers we determine which fixed point groups are k-anisotropic. Finally in section 8 we identify the results in this paper with those in [7] by computing the k-inner elements. Since the group $\mathrm{SL}(n, k)$ is k-split the k-anisotropic kernel of G is just a torus.

Some of the results in this paper appeared in the PhD theses of Ling Wu [14] and Christopher Dometrius [5].

2. Preliminaries

Our basic reference for reductive groups will be the papers of Borel and Tits [3], [4] and also the books of Borel [2], Humphreys [10] and Springer [13]. We shall follow their notations and terminology. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups.

Throughout this paper we will use the following notation. Let k be a field of characteristic not 2, k_1 an extension field of k, $\bar{k}$ the algebraic closure of k and k^* the product group of all the nonzero elements, $(k^*)^2 = \{a^2 \mid a \in k^*\}$. Let $V = k^n$ a finite-dimensional vector space defined over k, $\bar{V} = \bar{k}^n$, $\mathrm{M}_n(k) = \mathrm{M}(n, k)$ the set of $n \times n$-matrices with entries in k, $\mathrm{Id} \in \mathrm{M}_n(k)$ the identity automorphism, $\mathrm{GL}_n(k) = \mathrm{GL}(n, k) = \{A \in \mathrm{M}_n(k) \mid \det(A) \neq 0\}$ and $\mathrm{SL}_n(k) = \mathrm{SL}(n, k) = \{A \in \mathrm{M}_n(k) \mid \det(A) = 1\}$. Let G be a connected reductive algebraic group defined over k, θ an involution of G defined over k,

H a k-open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. For k the real or p-adic numbers these are also called reductive symmetric spaces. One can also characterize the symmetric k-variety as the subvariety $X_k = \{x\theta(x)^{-1} \mid x \in G_k\}$ of G_k. Then $X_k \simeq G_k/H_k$.

Definition 1 (Isomorphic Involutions). Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G. For $A \in \mathrm{GL}(n, k)$ let Inn_A denote the inner automorphism defined by $\mathrm{Inn}_A(X) = A^{-1}XA$ for all $X \in \mathrm{GL}(n, k)$. Let $\mathrm{Inn}_k(G) = \{\mathrm{Inn}_A \mid A \in G\}$ denote the set of all *inner* automorphisms of G and let $\mathrm{Inn}(G)$ denote the set of automorphisms Inn_A of G with $A \in \overline{G}$ such that $\mathrm{Inn}_A(G) = G$.

(1) We say that θ and τ in $\mathrm{Aut}(G)$ are $\mathrm{Inn}(G)$-*isomorphic* if there is a ϕ in $\mathrm{Inn}(G)$ such that $\tau = \phi^{-1}\theta\phi$ and write $\theta \approx^{\mathrm{Inn}} \tau$.

(2) We say that θ and τ in $\mathrm{Aut}(G)$ are $\mathrm{Aut}(G)$-*isomorphic* if there exists a ϕ in $\mathrm{Aut}(G)$ such that $\tau = \phi^{-1}\theta\phi$ and write $\theta \approx^{\mathrm{Aut}} \tau$.

Remark 1. Notice that $\theta \approx^{\mathrm{Inn}} \tau \implies \theta \approx^{\mathrm{Aut}} \tau$. Thus, the set of $\mathrm{Inn}(G)$-isomorphism classes contains the set of all $\mathrm{Aut}(G)$-isomorphism classes. In this paper we will mainly characterize the $\mathrm{Inn}(G)$-isomorphism classes, and we will use the notation $\approx$ instead of $\approx^{\mathrm{Inn}}$.

2.1. BILINEAR FORMS AND THEIR INDUCED INVOLUTIONS

Let $\beta : V \times V \to k$ be a bilinear form on $V = k^n$, $\mathfrak{B} = \{e_i\}$ an ordered basis for V, and $M_{\mathfrak{B}} = (\beta(e_i, e_j))$ *the matrix of* β *with respect to* $\mathfrak{B}$. For $x \in V$ let $x_{\mathfrak{B}} = (x_1, \ldots, x_n)$ be the coordinates of x with respect to β. Then for all $x, y \in V$ we have $\beta(x, y) = x_{\mathfrak{B}}^T M_{\mathfrak{B}} y_{\mathfrak{B}}$. Recall that two matrices M_1 and M_2 are *congruent* over the field k if there exists a matrix $Q \in \mathrm{GL}(n, k)$ such that $M_2 = Q^T M_1 Q$. In this case we will write $M_1 \cong M_2$. The following theorem is well-known, and its proof can be found in [1].

Theorem 1. *Two matrices M_1 and M_2 represent the same bilinear form with respect to different bases if and only if they are congruent.*

Remark 2. If M_1 represents the matrix of the form β with respect to the basis $\mathfrak{B}_1$ and M_2 represents the matrix of the form β with respect to the basis $\mathfrak{B}_2$, then the matrix Q that gives the congruence relation is the change of basis matrix from $\mathfrak{B}_2$ to $\mathfrak{B}_1$.

2.1.1. *Adjoint of a Matrix*

Let M be the matrix of a non-degenerate bilinear form β over $V = k^n$ and let $A \in \mathrm{GL}(n, k)$. The *adjoint of A with respect to β* (denoted A') is defined as the matrix that satisfies the relation $\beta(Ax, y) = \beta(x, A'y)$. If it is clear from the context which bilinear form β we are considering, then we will also just call this the adjoint of A. It follows from the definition that

$$\beta(Ax, y) = (Ax)^T My = x^T A^T My = x^T MA'y = \beta(x, A'y)$$

which implies

$$A' = M^{-1}A^T M \tag{1}$$

2.1.2. *Using the Adjoint to Construct Involutions*

Definition 2. Given a bilinear form on V with matrix M, we define θ_M by $\theta_M(A) = (A')^{-1}$. We will omit the subscript M when the bilinear form involved is clear. Using Equation (1) we see that $\theta(A) = M^{-1}(A^T)^{-1}M$.

Proposition 1. *If β is either symmetric or skew-symmetric, then θ is an involution of* $\mathrm{GL}(n, k)$.

Proof. Note that from equation (1), it follows that for M either symmetric or skew-symmetric we have the following properties of the adjoint similar to those for the transpose operation: $(A')' = A$, $(A')^{-1} = (A^{-1})'$ and $(AB)' = B'A'$, where $A, B \in \mathrm{GL}(n, k)$. The result is immediate from this. □

3. Semi-Congruence and the Classification Theorem

A first question to pose is: Is there is a one-to-one correspondence between congruence classes of symmetric/skew-symmetric bilinear forms and isomorphism classes of their involutions? The answer is “almost."

Theorem 2. *Let M_1 and M_2 be the matrices of bilinear forms β_1 and β_2 over $V = k^n$, respectively, and let θ_1 and θ_2 be the corresponding involutions on* $\mathrm{GL}(n, k)$. *If $M_1 \cong M_2$, then $\theta_1 \approx \theta_2$.*

Proof. Suppose $M_1 \cong M_2$ over k with $M_2 = Q^T M_1 Q$ for some $Q \in \mathrm{GL}(n, k)$. Using the formulas for θ_1 and θ_2, we get that $\forall A \in \mathrm{GL}(n, k)$,

$$\begin{aligned}
\theta_2(A) &= \mathrm{Inn}_{M_2}(A^T)^{-1} = \mathrm{Inn}_{Q^T M_1 Q}(A^T)^{-1} \\
&= (Q^T M_1 Q)^{-1}(A^T)^{-1}(Q^T M_1 Q) \\
&= Q^{-1}(M_1^{-1}((Q^T)^{-1}(A^T)^{-1}Q^T)M_1)Q \\
&= Q^{-1}\,\mathrm{Inn}_{M_1}((\mathrm{Inn}_{Q^{-1}}(A))^T)^{-1}Q \\
&= Q^{-1}\theta_1(\mathrm{Inn}_{Q^{-1}}(A))Q \\
&= \mathrm{Inn}_Q\,\theta_1\,\mathrm{Inn}_{Q^{-1}}(A), \Longrightarrow \theta_2 = \mathrm{Inn}_Q\,\theta_1\,\mathrm{Inn}_{(Q)^{-1}}\,.
\end{aligned}$$

This means $\theta_2 = (\phi)^{-1}\theta_1\phi$ with $\phi = \mathrm{Inn}_{Q^{-1}}$, and $\theta_2 \approx \theta_1$ over $\mathrm{GL}(n, k)$. □

Conversely, we get an “almost" 1-to-1 correspondence between congruence classes of bilinear forms and isomorphism classes of involutions.

Theorem 3. *Suppose* θ_1 *and* θ_2 *are involutions on* $\mathrm{GL}(n,k)$ *which come from symmetric or skew-symmetric bilinear forms represented by the matrices* M_1 *and* M_2*, respectively, over* $V = k^n$*. If* $\theta_1 \approx \theta_2$*, then* $M_2 = \alpha Q^T M_1 Q$ *for some matrix* $Q \in \mathrm{GL}(n,k)$ *and scalar* $\alpha \in k$*.*

Proof. Assume $\exists\ \phi \in \mathrm{Inn}\,\mathrm{GL}(n,k)$ such that $\theta_2 = \phi^{-1}\theta_1\phi$. Let $P \in \mathrm{GL}(n,\bar{k})$ such that $\phi = \mathrm{Inn}_P$. Then $\forall A \in \mathrm{GL}(n,k)$,

$$\begin{aligned}\theta_2(A) = \phi^{-1}\theta_1\phi(A) &= \mathrm{Inn}_{P^{-1}}\,\mathrm{Inn}_{M_1}((\mathrm{Inn}_P(A))^T)^{-1}\\ &= PM_1^{-1}(P^T)(A^T)^{-1}(P^T)^{-1}M_1P^{-1}\\ &= M_2^{-1}(A^T)^{-1}M_2, \implies\\ M_2P(M_1)^{-1}P^T(A^T)^{-1}(P^T)^{-1}M_1P^{-1}M_2^{-1} &= (A^T)^{-1}\end{aligned}$$

This holds for all A^T, so it holds for all A. This implies:

$$((P^T)^{-1}M_1P^{-1}M_2^{-1})^{-1}A((P^T)^{-1}M_1P^{-1}M_2^{-1}) = A.$$

So $\mathrm{Inn}_{(P^T)^{-1}M_1P^{-1}M_2^{-1}} = \mathrm{Id}$, $(P^T)^{-1}M_1P^{-1}M_2^{-1} = \gamma I_{nxn}$ for some $\gamma \in k^*$, and $M_2 = 1/\gamma\,(P^T)^{-1}M_1P^{-1}$. The result follows by substituting $Q = P^{-1}, \alpha = 1/\gamma$. □

This scalar α that prevents an equivalence statement in the 2 theorems above motivates the following definition and theorem.

Definition 3. Two bilinear forms on $V = k^n$ with associated matrices M_1 and M_2 are *semi-congruent over k* ($M_1 \cong^s M_2$) if there exists a $Q \in \mathrm{GL}(n,k)$ and an $\alpha \in k$ such that $M_2 = \alpha Q^T M_1 Q$.

Theorem 4 (Classification Theorem). *If* θ_1 *and* θ_2 *are involutions on* $\mathrm{GL}(n,k)$ *which come from bilinear symmetric/skew-symmetric forms as above, then* $M_1 \cong^s M_2$ *over k* $\iff \theta_{M_1} \approx \theta_{M_2}$*.*

Proof. Since $M_1 \cong^s M_2$ there exists $\alpha \in k$ such that $M_1 \cong \alpha M_2$. Then by Theorem 2 $\theta_{M_1} \approx \theta_{\alpha M_2}$. Since $\theta_{\alpha M_2} = \theta_{M_2}$, it follows that $\theta_{M_1} \approx \theta_{M_2}$. The reverse statement follows from Theorem 3. □

3.1. RESTRICTING TO SL(n,k)

In this subsection we show that the above results for $G = \mathrm{GL}(n,k)$ imply similar results for $G = \mathrm{SL}(n,k)$. The following lemma is easy to verify.

Lemma 1. *An automorphism* θ *coming from a bilinear form is an involution on* $\mathrm{GL}(n,k) \iff$ *it is an involution on* $\mathrm{SL}(n,k)$*.*

Most importantly, the isomorphism classes will remain intact.

Theorem 5. *Two involutions* θ_1 *and* θ_2 *are isomorphic over* $\mathrm{GL}(n,k) \iff$ *they are isomorphic over* $\mathrm{SL}(n,k)$.

Proof. $\Longleftarrow$ is clear. If θ_1 and θ_2 are isomorphic via $P \in \mathrm{GL}(n,k)$, then we can replace P with $\hat{P} = \left(1/\sqrt[n]{\det(P)}\right) P$ and still retain $\operatorname{Inn} G$-isomorphism via $\operatorname{Inn}_{\hat{P}}$. Since $\det(\hat{P}) = 1$, the result follows. □

4. Outer Involutions of SL(n,k)

In this section we use the Classification Theorem 4 to classify the outer involutions of $\mathrm{SL}(n,k)$. In the next section we classify the inner involutions. Recall that an automorphism θ of G is said to be an *outer automorphism* if $\theta \notin \operatorname{Inn} G$. An outer automorphism that is an involution will be called an outer involution.

Lemma 2. *(1) If k is algebraically closed, then* $|\operatorname{Aut}(G)/\operatorname{Inn}(G)| = 2$.

(2) Any outer automorphism can be written as $\operatorname{Inn}_M \phi$*, where* ϕ *is a fixed outer automorphism.*

Remark 3. For our purposes we will take for ϕ the involution defined by $\phi(A) = (A^T)^{-1}$. See [2] for details of the above lemma.

Our construction of involutions as $\theta(A) = M^{-1}(A^T)^{-1}M = \operatorname{Inn}_M(\phi(A))$, $(M = M_\beta)$ indicates that the isomorphism classes of outer involutions will come from our semi-congruence classes of bilinear forms. We need first to check whether all outer involutions come from these forms.

Lemma 3. *(1)* $\operatorname{Inn}_M \phi$ *is an involution of* $G \iff \phi(M)M \in Z(G)$.

(2) $\phi(M)M \in Z(G) \iff M$ *is symmetric or skew-symmetric.*

(3) $\operatorname{Inn}_M \phi$ *is an involution of* $G \iff M$ *is symmetric or skew-symmetric, and* M *is only skew-symmetric if n is even.*

Proof. (1) We know that $\operatorname{Inn}_M \phi$ is an involution $\iff (\operatorname{Inn}_M \phi)^2 = \mathrm{Id}$ on G. Then the following are each equivalent:

(a) $\operatorname{Inn}_M \phi \operatorname{Inn}_M \phi(X) = X$ for all $X \in G$

(b) $\operatorname{Inn}_M \phi(M^{-1}\phi(X)M) = \operatorname{Inn}_M \phi(M^{-1})\phi(\phi(X))\phi(M) = X$

(c) $M^{-1}\phi(M^{-1})X\phi(M)M = X$

(d) $X\phi(M)M = \phi(M)MX$ for all $X \in G$

(e) $\phi(M)M \in Z(G)$

(2) $\Longleftarrow$: If M is symmetric, then $M = M^T$ and $\phi(M)M = (M^T)^{-1}M = M^{-1}M = I_{n\times n}$, which is clearly in $Z(G)$. If M is skew-symmetric, then $M = -M^T$ and $\phi(M)M = (M^T)^{-1}M = (-M)^{-1}M = -I_{n\times n} \in Z(G)$.
$\Longrightarrow$: If $\phi(M)M = (M^T)^{-1}M \in Z(G)$ then

$$(M^T)^{-1}MX = X(M^T)^{-1}M \quad \forall X \in G.$$

So $(M^T)^{-1}MXM^{-1}M^T = X \quad \forall X \in G$ and $\mathrm{Inn}_{M^{-1}M^T} = \mathrm{Id}$. Therefore $M^{-1}M^T = \alpha I_{n\times n}$ for some $\alpha \in k$, and $M^T = \alpha M$. Taking determinants of both sides we see that $\alpha^n = 1$. If n is odd, then $\alpha = 1$ and $M = M^T$. If n is even, then $\alpha = 1$ or -1, and $M = M^T$ or $M = -M^T$.
The third statement follows immediately from Proposition 1. □

This lemma tells us that *all* outer involutions are of the form θ_M for some symmetric or skew-symmetric matrix M, so we have the following application of the Classification Theorem 4.

Theorem 6 (Outer Classification Theorem). *If θ_1 and θ_2 are outer involutions on* SL(n,k)*, then they come from bilinear symmetric or skew-symmetric forms represented by M_1 and M_2 (respectively), and*

$$\theta_1 \approx \theta_2 \iff \mathrm{Inn}_{M_1}\phi \approx \mathrm{Inn}_{M_2}\phi \iff M_1 \cong^s M_2.$$

Proof. This result is immediate from the Classification Theorem 4, Lemma 2 and Lemma 3. □

4.1. CLASSIFICATION OF ISOMORPHISM CLASSES OF OUTER INVOLUTIONS

Before we classify the isomorphism classes of outer involutions we recall the following results about congruence classes of bilinear forms (see [12]).

Theorem 7. *(1) Symmetric matrices are congruent to diagonal matrices whose entries are representatives of the square-class group $k^*/(k^*)^2$.*

(2) Skew-Symmetric singular nonsingular matrices are congruent to the $(2m)\times(2m)$ matrix J_{2m}, where $n = 2m$ and

$$J_{2m} = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}.$$

Remark 4. Note that a symmetric bilinear form β is non degenerate iff M_β is congruent to a diagonal matrix with non-zero diagonal entries.

4.1.1. *Classification of outer involutions for $k = \bar{k}$*

For algebraically closed fields there is only one semi-congruence class of symmetric non-degenerate bilinear forms and only one semi-congruence class of skew-symmetric forms as well.

(a) ***n* odd**: There is only one isomorphism class of outer involutions, represented by ϕ.

(b) ***n* even**: There are two classes of outer involutions, represented by ϕ and $\mathrm{Inn}_{J_{2m}}\,\phi$.

4.1.2. *Classification of outer involutions for $k = \mathbb{R}$*

All congruence classes of symmetric forms are given by $M = I_{n-i,i}$, $i = 0, 1, \ldots, n$ where $I_{n-i,i} = \begin{pmatrix} I_{n-i\times n-i} & 0 \\ 0 & -I_{i\times i} \end{pmatrix}$. When n is even, we also have $M = J_{2m}$ representing skew-symmetric forms. Thus, the symmetric forms have $n+1$ congruence classes, but only $\frac{n+1}{2}$ semi-congruence classes since $I_{n-i,i} = -I_{i,n-i}$.

(a) ***n* odd**: There are $\frac{n+1}{2}$ isomorphism classes of involutions, represented by $\mathrm{Inn}_M\,\phi$, where $M = I_{n-i,i} \quad i = 0, 1, \ldots, \frac{n-1}{2}$.

(b) ***n* even**: There are $\frac{n}{2} + 2$ classes of involutions, represented by $\mathrm{Inn}_M\,\phi$, where $M = I_{n-i,i} \quad i = 0, 1, \ldots, \frac{n}{2}$ or J_{2m}.

4.1.3. *Some notation and recollections*

We now introduce some notation for matrices that will be important. The matrices $I_{n-i,i}$ and J_{2m} have already been introduced. Let

$$L_{n,x} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ x & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & x & 0 \end{pmatrix}.$$

Notice that $J_{2m} \cong^s L_{n,-1}$. Also, let

$$M_{n,x,y,z} = \begin{pmatrix} I_{n-3\times n-3} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix}.$$

We also need the following well known results (see e.g. [12]):

Lemma 4. *Assume $p \neq 2$. Then*

(1) $|\mathbb{F}_p^*/(\mathbb{F}_p^*)^2| = 2$ when $p \neq 2$.

(2) Every element of $k = \mathbb{F}_p$ can be written as the sum of 2 squares in $\mathbb{F}_p$.

(3) For $\alpha \in \mathbb{F}_p$ *we have* $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \cong I_{2\times 2}$.

4.1.4. *Classification of outer involutions for* $k = \mathbb{F}_p$, $p \neq 2$.
Let S_p denote the “smallest" a non-square element of $\mathbb{F}_p$. We can get either all 1’s on the diagonal or at most we have one S_p left in the (n, n) spot.
(a) ***n* odd**: We have 2 semi-congruence classes of symmetric bilinear forms, given by $M = I_{nxn}$ and $M = M_{n,1,1,S_p}$. Therefore we get 2 involution classes, represented by ϕ and $\mathrm{Inn}_{M_{n,1,1,S_p}} \phi$.
(b) ***n* even**: Now we add $\mathrm{Inn}_{J_{2m}} \phi$ to get 3 classes of involutions.

The results over the fields $k = \bar{k}$, $\mathbb{R}$ and $\mathbb{F}_p$ with $p \neq 2$ are summarized in Table I. We note that for $k = \mathbb{Q}$ there are infinitely many congruence classes and infinitely many isomorphism classes of involutions.

4.1.5. *Some results about bilinear forms over* $\mathbb{Q}_p$
Definition 4. Given a matrix M, let D_i be the determinant of the upper left $i \times i$ square submatrix of M. Define the *p-sign* (or Hilbert Symbol) of two non-zero p-adic numbers to be $(\alpha, \beta)_p = 1$ if $\alpha x^2 + \beta y^2 = 1$ has a solution in $\mathbb{Q}_p$, -1 otherwise. Then the *Hasse Symbol* $c_p(M)$ *of* M is defined by

$$c_p(M) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p.$$

Note that the Hasse Symbol is limited to 1 or -1.

Lemma 5. *If* M *is an* $n \times n$ *matrix and* $\alpha \in \mathbb{Q}_p$ *we have the following formula:*

$$c_p(\alpha M) = \begin{cases} (\alpha, -1)_p^{n/2} (\alpha, D_n)_p c_p(M) & \textit{if n is even} \\ (\alpha, -1)_p^{(n+1)/2} c_p(M) & \textit{if n is odd} \end{cases}$$

Proof. $(-1, -\alpha^n D_n)_p = (-1, x)_p^n (-1, -D_n)_p$, and

$$\begin{aligned} (\alpha^i D_i, -\alpha^{i+1} D_{i+1})_p &= (\alpha^i, \alpha^{i+1})_p (\alpha^i, -D_{i+1})_p (D_i, \alpha^{i+1})_p (D_i, -D_{i+1}) \\ &= (\alpha^i, -D_{i+1})_p (D_i, \alpha^{i+1})_p (D_i, -D_{i+1}) \\ &= \begin{cases} (\alpha, D_i)_p (D_i, -D_{i+1})_p & \text{if } n \text{ is even} \\ (\alpha, -D_{i+1})_p (D_i, -D_{i+1})_p & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

But then

$$\begin{aligned} c_p(\alpha M) &= (\alpha, -1)^n (\alpha, -D_2)(\alpha, D_2)(\alpha, -D_4)(\alpha, D_4) \ldots C_p(M) \\ &= \begin{cases} (\alpha, -1)_p^{n/2} (\alpha, D_n)_p c_p(M) & \text{if } n \text{ is even} \\ (\alpha, -1)_p^{(n+1)/2} c_p(M) & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

□

Table I. Outer Involutions of G over $k = \bar{k}$, $\mathbb{R}$ and $\mathbb{F}_p$ with $p \neq 2$.

Field	Semi-Congruence Class M	Involution Class on G
$k = \bar{k}$		
n odd	$M = I_{n\times n}$	$\theta(A) = (A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = J_{2m}\ \ (n = 2m)$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{R}$		
n odd	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \ldots, \frac{n-1}{2}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$
n even	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \ldots, \frac{n}{2}$ $M_{\frac{n}{2}+1} = J_{2m}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$ $\theta_{\frac{n}{2}+1}(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{F}_p\ \ p \neq 2$		
n odd	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$ $M_3 = J_{2m}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$ $\theta_3(A) = \mathrm{Inn}_{M_3}(A^T)^{-1}$

Theorem 8 ([11]). *Two symmetric matrices M_1 and M_2 with entries in $\mathbb{Q}_p$ are congruent if and only if*

$$\det(M_1) = \alpha^2 \det(M_2) \quad \textit{and} \quad c_p(M_1) = c_p(M_2).$$

4.1.6. *Classification of Outer Involutions of* $\mathrm{SL}(n, \mathbb{Q}_2)$

We are limited to at most 8 choices for a representative of the square class of the determinant $\{1, -1, 2, -2, 3, -3, 6, -6\}$, and the Hasse symbol has only the choices 1 and -1. So at most there will be 16 semi-congruence classes of symmetric forms over $\mathbb{Q}_2$. To represent these classes we will use the dual value (α, c), where α is a representative of the square class group and c is the

Hasse symbol. Let $d_{\alpha,c} = (\alpha, c)$, where α has the eight choices from above and c is 1 or -1.

(1) ***n* even**:

(a) ***n* = 4*k***: We get 10 semi-congruence classes. The first is represented by $M = I_{n\times n}$ and the second by $M = J_{2m}$. We also get $M = M_{n,2,3,6}$. The other 7 choices are $M = M_{n,1,1,x}$ where x is chosen from $\{-1, 2, -2, 3, -3, 6, -6\}$. These matrices represent the semi-congruence forms for the matrices with dual values

$$d_{1,1}, d_{1,-1}, d_{-1,1}, d_{2,1}, d_{-2,1}, d_{3,1}, d_{-3,1}, d_{6,1}, d_{-6,1}.$$

For each of these choices M, we get the involution class represented by $\theta_M = \mathrm{Inn}_M\, \phi$.

These results stem from the fact that the matrices with dual value $d_{1,1}$ aren't semi-congruent to those with dual value $d_{1,-1}$. The first give us ϕ and the second give $\mathrm{Inn}_{M_{n,2,3,6}}\, \phi$ identifying the involution classes. To show this we use Theorem 5 for $n = 4k$ and obtain

$$c_2(\alpha M) = (\alpha, D_n)_2 c_2(M). \tag{2}$$

When a matrix has determinant 1, then $\alpha x^2 + y^2 = 1$ having a solution in $\mathbb{Q}_2$ implies $(\alpha, D_n)_2 = 1$ and $c_2(\alpha M) = c_2(M)$ from the above equation. Thus, its Hasse symbol must remain the same under scalar multiplication. So $d_{1,1}$ and $d_{1,-1}$ represent different semi-congruence classes, while $(\gamma, 1)$ and $(\gamma, -1)$ are semi-congruent for $\gamma \neq 1$.

(b) ***n* = 4*k*+2**: Again we get 10, but instead of obtaining the semi-congruence class of $M_{n,2,3,6}$ we get one represented by $M = M_{n,2,3,-6}$. The formula from Theorem 5 now shows that $c_2(\alpha M)$ simplifies to:

$$(\alpha, -1)_2^{2k+1}(\alpha, D_n)_2 c_2(M) = (\alpha, -D_n)_2 c_2(M).$$

Since $\det(M) = -1$ gives us $(\alpha, -D_n)_2 = (\alpha, 1)_2 = 1$, we obtain $c_2(\alpha M) = c_2(M)$, which results in $d_{-1,1}$ and $d_{-1,-1}$ representing different classes of forms. However, now $(\gamma, 1)$ matrices and $(\gamma, -1)$ matrices are semi-congruent for $\gamma \neq -1$ since there is some α which will change the Hasse sign. Thus, we replace the class typified by $d_{1,-1}$ from the $n = 4k$ case by $d_{-1,-1}$ and get $\mathrm{Inn}_{M_{n,2,3,-6}}\, \phi$

(2) ***n* odd**: When n is odd, we can multiply a matrix M by the scalar $\gamma = \det(M)$ to get $\det(\gamma M) = (\gamma)^{2k}(\gamma)\det(M) = 1 \mod (\mathbb{Q}_2^*)^2$. So our only choices are matrices with dual value $d_{1,1}$ and $d_{1,-1}$.

(a) ***n* = 4*k* + 1**: Here we can replace a matrix M by some αM and change the Hasse symbol, so there is only one isomorphism class, represented by ϕ.

(b) ***n* = 4*k* + 3**: Now, there is no α that works as above, since Theorem 5 ensures that $c_2(\alpha M) = (\alpha, -1)_2^{2k+2} c_2(M) = c_2(M)$. Therefore, $d_{1,1}$ and $d_{1,-1}$ give different classes of forms, represented by ϕ and $\mathrm{Inn}_{M_{n,2,3,6}}$.

Table II. Outer Involutions of G over $k = \mathbb{Q}_2$

$\dim(V)$	Dual Value $d_{\alpha,c}$ of the Semi-Congruence Class	Matrix M such that $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$
$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{1,-1}$	$M_{n,2,3,6}$
	$d_{-1,1}$	$M_{n,1,1,-1}$
	$d_{2,1}$	$M_{n,1,1,2}$
	$d_{-2,1}$	$M_{n,1,1,-2}$
	$d_{3,1}$	$M_{n,1,1,3}$
	$d_{-3,1}$	$M_{n,1,1,-3}$
	$d_{6,1}$	$M_{n,1,1,6}$
	$d_{-6,1}$	$M_{n,1,1,-6}$
		$M = J_{2m}$
$n = 4k + 1$	$d_{1,1}$	$M = I_{n\times n}$
$n = 4k + 2$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{-1,1}$	$M_{n,1,1,-1}$
	$d_{-1,-1}$	$M_{n,2,3,-6}$
	$d_{2,1}$	$M_{n,1,1,2}$
	$d_{-2,1}$	$M_{n,1,1,-2}$
	$d_{3,1}$	$M_{n,1,1,3}$
	$d_{-3,1}$	$M_{n,1,1,-3}$
	$d_{6,1}$	$M_{n,1,1,6}$
	$d_{-6,1}$	$M_{n,1,1,-6}$
		$M = J_{2m}$
$n = 4k + 3$	$d_{1,1}$	$M = I_{n\times n}$
	$d_{1,-1}$	$M_{n,2,3,6}$

4.1.7. *Classification of Outer Involutions of* $\mathrm{SL}(n, \mathbb{Q}_p)$, $(p \neq 2)$

The process is similar to the $\mathbb{Q}_2$ case with the exception that now we have only 8 choices for the dual value $d_{\gamma,c}$ since $\gamma \in \{1, p, S_p, pS_p\}$.

(1) **n even:**

(a) **$n = 4k$**: There are 6 classes of involutions. We get 5 semi-congruence classes of symmetric forms, represented by matrices with dual values $d_{1,1}$, $d_{1,-1}$, $d_{p,1}$, $d_{S_p,1}$, $d_{pS_p,1}$, and we also have $M = J_{2m}$. Yet involutions coming from the same dual values can be different depending upon whether or not -1 is a square in $(\mathbb{Q}_p^*)^2$.

(i) $\boldsymbol{-1 \notin (\mathbb{Q}_p^*)^2}$: The involutions are ϕ,$\mathrm{Inn}_{J_{2m}}\,\phi$, and $\mathrm{Inn}_M\,\phi$, where $M = M_{n,1,1,\gamma}$ with $\gamma \in \{p, S_p, pS_p\}$, or $M = M_{n,p,S_p,pS_p}$. The matrices $M_{n,1,1,\gamma}$ correspond to the dual classes $d_{\gamma,1}$, and M_{n,p,S_p,pS_p} corresponds to the $d_{1,-1}$ class.

(ii) $\boldsymbol{-1 \in (\mathbb{Q}_p^*)^2}$: We still get 6 involutions, but here the involution coming from the dual class $d_{1,-1}$ is $\mathrm{Inn}_{M_{n,1,p,p}}\,\phi$.

(1)(b) $\boldsymbol{n = 4k+2}$: There are also 6 classes of involutions in this case.

(i) $\boldsymbol{-1 \notin (\mathbb{Q}_p^*)^2}$: In this case we get the formula $c_p(\alpha A) = (\alpha, -D_n)_p c_p(A)$. For $\det(A) \neq -1$ $(\alpha, -D_n)_p = (\alpha, -\gamma)_p$, and there will be an α such that $(\alpha, -\gamma)_p = -1$. So $c_p(\alpha A) = -c_p(A)$, and $d_{\gamma,1}$ represents a semi-congruent class of bilinear forms to $d_{\gamma,-1}$ for $\gamma \neq -1$. Yet for $\det(A) = -1$, the above equation will become $c_p(\alpha A) = (\alpha, 1)_p c_p(A) = c_p(A)$. Therefore, we get different classes of involutions for those coming from the 5 semi-congruence classes represented by $d_{1,1}$, $d_{p,1}$, $d_{S_p,1}$, $d_{S_p,-1}$, and $d_{pS_p,1}$, and we also have $M = J_{2m}$.

(ii) $\boldsymbol{-1 \in (\mathbb{Q}_p^*)^2}$: Here we have $c_p(\alpha A) = (\alpha, 1)_p c_p(A) = c_p(A)$. Hence, we get the same 5 dual classes as in the $n = 4k$ case. The 6 involutions obtained are identical to the case $n = 4k$, $-1 \in (\mathbb{Q}_p^*)^2$.

(2) $\boldsymbol{n}$ **odd:**

(a) $\boldsymbol{n = 4k+1}$: Again this depends upon whether or not -1 is a square in $(\mathbb{Q}_p^*)^2$.

(i) $\boldsymbol{-1 \notin (\mathbb{Q}_p^*)^2}$: As was the case for $\mathbb{Q}_2$, we get only one dual class $d_{1,1}$. The only involution class is that of ϕ.

(ii) $\boldsymbol{-1 \in (\mathbb{Q}_p^*)^2}$: Now $d_{1,1}$ and $d_{1,-1}$ represent different classes. There are 2 isomorphism classes of involutions: ϕ and $\mathrm{Inn}_{M_{n,p,S_p,pS_p}}\,\phi$.

(2)(b) $\boldsymbol{n = 4k+3}$:

(i) $\boldsymbol{-1 \notin (\mathbb{Q}_p^*)^2}$: We have 2 dual classes of symmetric forms, $d_{1,1}$ and $d_{1,-1}$. There are only the 2 involutions ϕ and $\mathrm{Inn}_{M_{n,p,S_p,pS_p}}\,\phi$.

(ii) $\boldsymbol{-1 \in (\mathbb{Q}_p^*)^2}$: We have the same 2 dual classes and the same 2 involution classes as when -1 is a square for $n = 4k+1$ above.

This completes the classification of isomorphism classes of outer involutions of $\mathrm{SL}(n, \mathbb{Q}_p)$. The results are summarized in Table III.

5. Inner automorphisms

In this section we consider involutions of inner type. By definition, for any automorphism θ of inner type, there exist a $n \times n$-matrix $Y \in \mathrm{GL}(n, \bar{k})$, such that $\theta = \mathrm{Inn}_Y|_G$.

Lemma 6. *Let* $Y \in \mathrm{GL}(n, \bar{k})$. *If* $\mathrm{Inn}_Y|_G = \mathrm{Id}$, *then* $Y = pI$ *for some* $p \in \bar{k}$, *i.e.* $\mathrm{Inn}_Y = \mathrm{Id}$ *over* $\mathrm{GL}(\bar{V})$.

Table III. Outer Involutions of $\mathrm{SL}(n, \mathbb{Q}_p)$ $(p \neq 2)$

	$\dim(V)$	Dual Value $d_{\alpha,c}$of the Semi-Congruence Class	Matrix Ms.t. $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$
$-1 \notin (\mathbb{Q}_p^*)^2$			
	$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	$n = 4k + 1$	$d_{1,1}$	$M = I_{n\times n}$
	$n = 4k + 2$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,p,pS_p}$
		$d_{S_p,-1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	$n = 4k + 3$	$d_{1,1}$	$M = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}
$-1 \in (\mathbb{Q}_p^*)^2$			
	$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{1,-1}$	$M_{n,1,p,p}$
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	$n = 4k + 2$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	n odd	$d_{1,1}$	$M = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}

Proof. Since $\mathrm{Inn}_Y\,|_G = \mathrm{Id}$, we have for all $A \in \mathrm{SL}(n,k)$, $\mathrm{Inn}_Y(A) = Y^{-1}AY = A$, i.e. $YA = AY$. Since A is arbitrary it follows that $Y = pI$ for some $p \in \bar{k}$. Furthermore $\mathrm{Inn}_Y = \mathrm{Inn}_{pI} = \mathrm{Inn}_I = \mathrm{Id}$. □

Lemma 7. *For any inner automorphism $\theta \in \mathrm{Inn}(G)$, suppose $Y \in \mathrm{GL}(n,\bar{k})$. Then $\theta = \mathrm{Inn}_Y \in \mathrm{Inn}(\bar{G})$ keeps G invariant if and only if $Y = pB$, for some $p \in \bar{k}$ and $B \in \mathrm{GL}(n,k)$. In other words, there is a matrix $B \in \mathrm{GL}(n,k)$ such that $\theta = \mathrm{Inn}_B\,|_G$.*

Proof. The proof of the if statement is obvious. Conversely assume

$$Y = \left(y_{ij}\right)_{n\times n} \in \mathrm{GL}(\bar{V})$$

and

$$A = (a_{ij})_{n\times n} \in G$$

where $y_{ij} \in \bar{k}$, $a_{ij} \in k$. We have Y^{-1} =det$(Y)^{-1}\left(Y_{ij}\right)_{n\times n}$, where Y_{ij} is the ij-th minor of Y. Then

$$\mathrm{Inn}_Y(A) = Y^{-1}AY = \det(Y)^{-1}\left(\sum_{m=1}^{n}\sum_{l=1}^{n} Y_{im}a_{ml}y_{lj}\right)_{n\times n}.$$

We will prove $\det(Y)^{-1}Y_{ij}y_{ml} \in k$. Without loss of generality, we prove the case for $\det(Y)^{-1}Y_{i1}y_{1l}$. Let $A = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \frac{1}{\delta} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$, and $Z = (z_{il}) = \mathrm{Inn}_Y(A)$. We have

$$z_{il} = \det(Y)^{-1}\left(\delta Y_{i1}y_{1l} + \frac{1}{\delta}Y_{i2}y_{2l} + Y_{i3}y_{3l} + \cdots + Y_{in}y_{nl}\right) \in k,$$

for all $i,l = 1,2,\ldots n$. Since $0 \neq \delta \in k$ is arbitrary it follows that $\det(Y)^{-1}Y_{i1}y_{1l} \in k$ and in general $\det(Y)^{-1}Y_{ij}y_{ml} \in k$. Hence $y_{ij}/y_{ml} \in k$, for all $i,j,m,l = 1,2,\ldots n$, provided $y_{ml} \neq 0$, i.e. $Y = pB$, for some $p \in \bar{k}$ and $B \in \mathrm{GL}(n,k)$. Since $\theta = \mathrm{Inn}_Y = \mathrm{Inn}_{pB} = \mathrm{Inn}_B$, where $B \in \mathrm{GL}(n,k)$, the result follows. □

Lemma 8. *Suppose $Y \in \mathrm{GL}(n,k)$ with $Y^2 = pI$. Then*

(1) If $p = c^2 \in k^{*2}$ then Y is conjugate to $cI_{n-i,i}$ for some $i = 0,1,\ldots,n$.

(2) If p is not in k^{*2}, then n is even and Y is conjugate to $L_{\frac{n}{2},p}$.

Proof. If there is a $c \in k$ such that $p = c^2$, then the characteristic polynomial of Y is $(x-c)^{n-i}(x+c)^i$, and the minimal polynomial is a factor of $(x+c)(x-c)$. So Y is conjugate to $cI_{n-i,i}$ for some $i = 0,1,\ldots,n$.

If p is not in k^{*2}, then the minimal polynomial is $(x^2 - p)$, which does not factor over k, therefore the characteristic polynomial is a power of the minimal polynomial. Hence n, which is the degree of the characteristic polynomial, is even. Furthermore, Y is conjugate to $L_{\frac{n}{2},p}$ since they have the same minimal and characteristic polynomials. $\square$

Lemma 9. *Suppose* $\theta \in \mathrm{Aut}(G)$ *is an involution of inner type. Then there is a matrix* $Y \in \mathrm{GL}(n, k)$*, such that* $\theta = \mathrm{Inn}_Y$*, where matrix* Y *is conjugate to* $cI_{n-i,i}$ *or* $L_{\frac{n}{2},p}$ *for some* $i \in \{0, 1, \ldots, n\}$*,* $c \in k^*$ *and* $p \in \bar{k}^*/\bar{k}^{*2}$.

Proof. By Lemma 6 we know that there is a matrix $Y \in \mathrm{GL}(n, k)$, such that $\theta = \mathrm{Inn}_Y$. Since θ is an involution we have $\theta^2 = \mathrm{Inn}_{Y^2} = \mathrm{Id}$. By Lemma 7 we have $Y^2 = pI$ for some $p \in \bar{k}$. It follows that Y is conjugate to one of the forms in Lemma 8. $\square$

We must determine which of these matrices lead to conjugate involutions.

Lemma 10. *The matrices* $I_{n-i,i}$ *and* $cI_{n-j,j}$ *are conjugate for some* $c \in k$ *if and only if* c *is one of the following:*

(1) $c = 1$ *and* $i = j$.

(2) $c = -1$ *and* $j = n - 1$.

Proof. Since the eigenvalues of both $I_{n-i,i}$ and $cI_{n-j,j}$ have to be exactly the same, that forces c to be 1 or -1. If $c = 1$, $I_{n-i,i}$ and $I_{n-j,j}$ are conjugate, therefore $i = j$. If $c = -1$, $cI_{n-j,j} = -I_{n-j,j}$ is conjugate to $I_{j,n-j}$, and $I_{n-i,i}$ will then be conjugate if $j = n - i$. $\square$

Lemma 11. *Let* $p, q \in \bar{k}^*/\bar{k}^{*2}$*. The matrix* $L_{\frac{n}{2},p}$ *is conjugate to* $cL_{\frac{n}{2},q}$ *for some* $c \in k$ *if and only if* $\frac{p}{q} \in k^{*2}$.

Proof. The minimal polynomial of $L_{\frac{n}{2},p}$ is $(x^2 - p)$ and that of $cL_{\frac{n}{2},q}$ is $(x^2 - c^2q)$. Additionally, the characteristic polynomial of $L_{\frac{n}{2},p}$ is $(x^2 - p)^n$ and that of $cL_{\frac{n}{2},q}$ is $(x^2 - c^2q)^n$. So $L_{\frac{n}{2},p}$ and $cL_{\frac{n}{2},q}$ are conjugate if and only $p = c^2q$, which forces $\frac{p}{q} = c^2 \in k^{*2}$. $\square$

Lemma 12. *The inner automorphisms* $\theta = \mathrm{Inn}_{Y_1}$ *and* $\phi = \mathrm{Inn}_{Y_2}$ *are conjugate if and only if the matrix* Y_1 *is conjugate to* cY_2 *for some* $c \in \bar{k}$.

Proof. The result follows from the following equivalent statements:

- $\theta = \mathrm{Inn}_{Y_1}$ is conjugate to $\phi = \mathrm{Inn}_{Y_2}$.
- there is a matrix $A \in \mathrm{GL}(n, k)$, such that $\mathrm{Inn}_{A^{-1}} \mathrm{Inn}_{Y_1} \mathrm{Inn}_A = \mathrm{Inn}_{Y_2}$.
- $\mathrm{Inn}_{A^{-1}Y_1AY_2{}^{-1}} = \mathrm{Id}$.

- there is $c \in k$, such that $A^{-1}Y_1AY_2{}^{-1} = c\,\mathrm{Id}$ for some $c \in \bar{k}$ (see Lemma 6).
- $A^{-1}Y_1A = cY_2$ for some $c \in \bar{k}$.
- Y_1 is conjugate to cY_2 for some $c \in \bar{k}$.

□

Lemma 13. *The matrix $L_{n,1}$ is conjugate to $I_{n,n}$.*

Proof. Their characteristic polynomials are both $(x^2 - 1)^n$, while their minimal polynomials are both $(x^2 - 1)$. □

Theorem 9. *Suppose the involution $\theta \in \mathrm{Aut}(G)$ is of inner type. Then up to isomorphism θ is one of the following:*

(1) $\mathrm{Inn}_Y|_G$*, where* $Y = I_{n-i,i} \in \mathrm{GL}(n, k)$ *where* $i \in \left\{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\right\}$.

(2) $\mathrm{Inn}_Y|_G$*, where* $Y = L_{\frac{n}{2},p} \in \mathrm{GL}(n, k)$ *where* $p \in k^*/k^{*2}$, $p \not\equiv 1 \mod k^{*2}$.

Note that (2) *can only occur when n is even.*

Proof. (1) By Lemma 9, the matrix $Y \in \mathrm{GL}(n, k)$ such that $\theta = \mathrm{Inn}_Y$ is conjugate to $cI_{n-i,i}$ for some $c \in k^*$ and $i = 1, 2, \ldots, n$ or $L_{\frac{n}{2},p}$ for some $p \in k^*/k^{*2}$. If Y is of the form $I_{n-i,i}$, then Y is conjugate to $(-1)I_{i,n-i}$. This limits i to $1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$, and by Lemma 12 each of these represents a unique isomorphism class.

(2) Since $p \in k^*/k^{*2}$, by Lemma 11, we can choose a representative q from k^*/k^{*2}, $q \not\equiv 1 \mod k^{*2}$. Since $\mathrm{Inn}_Y = \mathrm{Inn}_{cY}$, we can factor out the constant.

The above two classes are not conjugate to each other since those in (1) are split and those in (2) are not, thus it's impossible for them to have same eigenvalues. □

5.1. SUMMARY OF INVOLUTIONS OF INNER TYPE

Recall from [9, Section 2.1] that $k^*/(k^*)^2 \approx \{1\}$ if $k = \bar{k}$, $k^*/(k^*)^2 \approx \mathbb{Z}_2$ if $k = \mathbb{R}$ or $\mathbb{F}_p$ and $k^*/(k^*)^2 \approx \mathbb{Z}_2 \times \mathbb{Z}_2$ if $\mathbb{Q}_p$, $p \neq 2$. We will denote the nontrivial representative of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$ by S_p. Then 1, p, S_p are pS_p are representatives of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$. In the case of $k = \mathbb{Q}_2$ a set of representatives of $k^*/(k^*)^2$ are $\{1, -1, 2, -2, -3, 3, 6, -6\}$.

Corollary 1. *The number of involutions of inner type of G up to isomorphism is equal to* $\| k^*/(k^*)^2\| + \frac{n}{2} - 1$ *if n is even and* $\frac{n-1}{2}$ *if n is odd.*

For $k = \bar{k}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_p$ and $\mathbb{Q}_p$ we summarize the number of isomorphism classes of involutions of inner type in the following.

(1) $k = \bar{k}$ algebraically closed. There are $[\frac{n}{2}]$ isomorphism classes of involutions of inner type.

(2) $k = \mathbb{R}$ the real numbers. There are $\frac{n}{2} + 1$ isomorphism classes of involutions of inner type when n is even and $\frac{n-1}{2}$ when n is odd.

(3) $k = \mathbb{Q}$ the rational numbers. There are infinitely many isomorphism classes of involutions of inner type.

(4) $k = \mathbb{F}_p$ finite field ($p \neq 2$). There are $\frac{n}{2} + 1$ isomorphism classes of involutions of inner type when n is even and $\frac{n-1}{2}$ when n is odd.

(5) $k = \mathbb{Q}_p$ the $\mathfrak{p}$-adic numbers. For $p \neq 2$, there are $\frac{n}{2} + 3$ isomorphism classes of involutions of inner type for n is even and $\frac{n-1}{2}$ for n is odd. For $p = 2$ there are $\frac{n}{2} + 7$ isomorphism classes of involutions of inner type for n even and $\frac{n-1}{2}$ for n odd.

6. Summary of the classification on SL(n, k)

In this section we summarize the classification of involutions in the case that the field k is algebraically closed, $\mathbb{R}$, the $\mathfrak{p}$-adic numbers or a finite field $\mathbb{F}_p$ and we give representatives for each of the isomorphism classes.

6.1. $k = \bar{k}$: ALGEBRAICALLY CLOSED

(1) If n is odd, there are $\frac{n+1}{2}$ isomorphism classes of involutions. Representatives are Inn_Y with Y is one of the following $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ and θ.

(2) If n is even, there are $\frac{n}{2} + 2$ isomorphism classes of involutions. Representatives are Inn_Y where $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$, θ, and $\theta\,\mathrm{Inn}_{J_{2m}}$.

6.2. $k = \mathbb{R}$: THE REAL NUMBERS

(1) If n is odd, there are n isomorphism classes of involutions. Representatives are θ, Inn_Y and $\theta\,\mathrm{Inn}_Y$ where $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$.

(2) If n is even, there are $n + 3$ isomorphism classes of involutions. Representatives are $\mathrm{Inn}_{J_{2m}}$, θ, $\theta\,\mathrm{Inn}_{J_{2m}}$, Inn_Y and $\theta\,\mathrm{Inn}_Y$, $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$.

Table IV. Isomorphism Classes of Inner Involution of G

Field	Number of Inner Involutions	Representative Matrix Y such that $\theta = \mathrm{Inn}_Y$
n odd, $k =$ any field	$\frac{n-1}{2}$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n-1}{2}$
n even		
$k = \bar{k}$	$\frac{n}{2}$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$
$k = \mathbb{R}$	$\frac{n}{2} + 1$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,-1}$
$k = \mathbb{Q}$	∞	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \neq 1 \mod (\mathbb{Q}^*)^2$
$k = \mathbb{F}_p \quad p \neq 2$	$\frac{n}{2} + 1$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,S_p}$
$k = \mathbb{Q}_2$	$\frac{n}{2} + 7$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \in \{-1, \pm 2, \pm 3, \pm 6\}$
$k = \mathbb{Q}_p \quad p \neq 2$	$\frac{n}{2} + 3$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \in \{p, S_p, pS_p\}$

6.3. $k = \mathbb{F}_p$: FINITE FIELD, $p \neq 2$

Let S_p be a non trivial representative of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$.

(1) If n is odd, there are $\frac{n-1}{2} + 2$ isomorphism classes of involutions. Representatives are $\theta, \theta\, \mathrm{Inn}_{M_{n,S_p}}$ and Inn_Y where $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$.

(2) If n is even, there are $\frac{n}{2} + 4$ isomorphism classes of involutions. Representatives are $\mathrm{Inn}_{L_{n,S_p}}, \theta, \theta\, \mathrm{Inn}_{J_{2m}}, \theta\, \mathrm{Inn}_{M_{n,S_p}}$ and Inn_Y where $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$.

6.4. $k = \mathbb{Q}_p$: THE $\mathfrak{p}$-ADIC NUMBERS

If $p \neq 2$, then we take $1, p, S_p, pS_p$ as representatives of $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ and if $p = 2$, then we take $\{1, -1, 2, -2, -3, 3, 6, -6\}$ as representatives.

(1) If n is even, then there are $\frac{n}{2} + 9$ isomorphism classes of involutions for $p \neq 2$, $\frac{n}{2} + 17$ for $p = 2$. Representatives are

a) $p \neq 2$: $\mathrm{Inn}_{L_{n,x}}$ with $x = S_p,\ p$, or pS_p θ, $\theta\,\mathrm{Inn}_{J_{2m}}$, Inn_Y and $\theta\,\mathrm{Inn}_C$ and $\theta\,\mathrm{Inn}_D$. Here $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n}{2}$, and $C = M_{n,1,1,x}$ with $x = S_p,\ p$, or pS_p. For the matrix D we have the following cases:

$$D = \begin{cases} M_{n,p,S_p,pS_p} & \text{if } -1 \in \mathbb{Q}_p^2 \\ M_{n,1,p,p} & \text{if } -1 \notin \mathbb{Q}_p^2 \text{ and } n = 4k \\ M_{n,p,p,S_p} & \text{if } -1 \notin \mathbb{Q}_p^2 \text{ and } n = 4k+2 \end{cases}$$

b) $p = 2$: The same as $p \neq 2$, but x in $L_{n,x}$ and in C are chosen from 2, 3, 6, -1, -2, -3, -6. The matrix D is $I_{n-2,2}$ if $n = 4k$ and $M_{n,2,3,-6}$ if $n = 4k+2$.

(2) If $n = 4k+1$, there are $\frac{n-1}{2} + 2$ isomorphism classes of involutions if $-1 \in \mathbb{Q}_p^2$, otherwise $\frac{n-1}{2} + 1$. Representatives are Inn_Y, θ, and possibly Inn_D if $-1 \in \mathbb{Q}_p^2$, where $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n-1}{2}$ and D is M_{n,p,S_p,pS_p}.

(3) If $n = 4k+3$, there are $\frac{n-1}{2} + 2$ isomorphism classes of involutions. Representatives are Inn_Y, θ and $\theta\,\mathrm{Inn}_D$, where $Y = I_{n-i,i}$, $i = 1, 2, \ldots, \frac{n-1}{2}$ and D is

$$D = \begin{cases} M_{n,p,S_p,pS_p} & \text{if } -1 \in \mathbb{Q}_p^2 \\ M_{n,1,p,p} & \text{if } -1 \notin \mathbb{Q}_p^2 \\ I_{n-2,2} & \text{if } p = 2 \end{cases}$$

Note that $-1 \notin \mathbb{Q}_2$, and (2) and (3) hold for $p \neq 2$ and $p = 2$.

7. Fixed Point Groups and Symmetric *k*-Varieties

The fixed-point group $H = G^\delta$ for an involution δ over G is defined by

$$G^\delta = \{g \in G \mid \delta(g) = g\}.$$

The fixed point group determines much of the structure of the corresponding symmetric k-variety $X := \{g\delta(g)^{-1} \mid g \in G\}$. It is easy to see that $X \simeq G/G^\delta$. Moreover if G^δ is compact, so from [8] it follows that X consists of semisimple elements:

Proposition 2 ([8, Proposition 10. 8]). *Let G be a connected reductive algebraic k-group with* $\mathrm{char}(k) = 0$ *and $X = \{g\delta(g)^{-1} | g \in G\}$. Suppose that $H \cap [G, G]$ is anisotropic over k. Then X_k consists of semi-simple elements.*

In view of this result it is important to determine which involutions have an k-anisotropic fixed point group. For $k = \mathbb{R}$ or $\mathbb{Q}_p$ all k-anisotropic subgroups are compact. In this section, we'll study the compactness of these fixed point groups for $k = \mathbb{R}$ and $\mathbb{Q}_p$.

Table V. Summary: Number of Involution Classes of ***G***

Field	dim(V)	Number of Outer Involutions	Number of Inner Involutions	Total Number of Involutions
$k=\bar{k}$	n odd	1	$\frac{n-1}{2}$	$\frac{n+1}{2}$
	n even	2	$\frac{n}{2}$	$\frac{n}{2}+2$
$k=\mathbb{R}$	n odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$	n
	n even	$\frac{n}{2}+2$	$\frac{n}{2}+1$	$n+3$
$k=\mathbb{Q}$	n odd	∞	$\frac{n-1}{2}$	∞
	n even	∞	∞	∞
$k=\mathbb{F}_p$ $p\neq 2$	n odd	2	$\frac{n-1}{2}$	$\frac{n+3}{2}$
	n even	3	$\frac{n}{2}+1$	$\frac{n}{2}+4$
$k=\mathbb{Q}_2$	$n=4k$	10	$\frac{n}{2}+7$	$\frac{n}{2}+17$
	$n=4k+1$	1	$\frac{n-1}{2}$	$\frac{n+1}{2}$
	$n=4k+2$	10	$\frac{n}{2}+7$	$\frac{n}{2}+17$
	$n=4k+3$	2	$\frac{n-1}{2}$	$\frac{n+3}{2}$
$k=\mathbb{Q}_p$ $p\neq 2$	$n=4k$	6	$\frac{n}{2}+3$	$\frac{n}{2}+9$
	$n=4k+1$			
	$-1\notin(\mathbb{Q}_p^*)^2$	1	$\frac{n-1}{2}$	$\frac{n+1}{2}$
	$-1\in(\mathbb{Q}_p^*)^2$	2	$\frac{n-1}{2}$	$\frac{n+3}{2}$
	$n=4k+2$	6	$\frac{n}{2}+3$	$\frac{n}{2}+9$
	$n=4k+3$	2	$\frac{n-1}{2}$	$\frac{n+3}{2}$

Lemma 14. *For the matrix* $Y = I_{n-i,i}$, *the fixed point group* G^{Inn_Y} *consists of the matrices* $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ *where* $B \in \mathrm{GL}(n-i,k)$, $C \in \mathrm{GL}(i,k)$, $\det B \times \det C = 1$ *and the group* G^{Inn_Y} *is noncompact.*

Proof. For $Y = I_{n-i,i}$ and $\mathrm{Inn}_Y(A) = A$ write $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ with A_1 an $n-i \times n-i$ block and A_4 an $i \times i$ block. Then

$$\mathrm{Inn}_Y(A) = \begin{pmatrix} I_{n-i} & 0 \\ 0 & -I_i \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} I_{n-i} & 0 \\ 0 & -I_i \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

i.e. $\begin{pmatrix} A_1 & -A_2 \\ -A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$.

So $A_2 = 0$ and $A_3 = 0$. Since $A \in \mathrm{SL}(n,k)$ and $\det A = \det A_1 \times \det A_4$, the result follows. □

Lemma 15. *For the matrix* $Y = L_{n,p}$, *the noncompact fixed point group* G^{Inn_Y} *consists of the elements*

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix},$$

where $m = \frac{n}{2}$, *and* $A_{ij} = \begin{pmatrix} a_{ij} & y_{ij} \\ py_{ij} & a_{ij} \end{pmatrix}$.

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix}$ and assume $\mathrm{Inn}_Y(A) = A$. Then

$$A = -\begin{pmatrix} J_2 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix} \begin{pmatrix} J_2 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_2 \end{pmatrix}$$

That is $-J_2 A_{ij} J_2 = A_{ij}$ for all $i, j = 1, \dots, n$. Let $A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}$. Then $-J_2 A_{ij} J_2 = A_{ij}$ implies that $d_{ij} = a_{ij}$ and $c_{ij} = pb_{ij} = py_{ij}$, which proves the result. □

Lemma 16. *The solutions of* $y_1^2 + y_2^2 + y_3^2 + ay_4^2 = 1$ *over* $\mathbb{Q}_2$ *are all bounded if and only if* $a \in \mathbb{Q}_2^2$.

Proof. Assume first that $a=1$, i.e. $y_1^2+y_2^2+y_3^2+y_4^2=1$ and write $y_i=2^{s_i}+\sum_{j=s_i+1}^{\infty}\delta_{ij}2^j$. Then $y_i^2=2^{2s_i}+\sum_{j=2s_i+3}^{\infty}\pi_{ij}2^j$. Without loss of generality we may first assume that $s_1\le s_2\le s_3\le s_4$. If $s_1\ne s_2$, then

$$y_1^2+y_2^2+y_3^2+y_4^2=2^{2s_1}+\sum_{j=2s_i+2}^{\infty}\phi_{ij}2^j.$$

To make this equal to 1, we must have $s_1=0$ and y_1, y_2, y_3 and y_4 are units. If $s_1=s_2$ and $s_2\ne s_3$, then

$$y_1^2+y_2^2+y_3^2+y_4^2=2^{2s_1+1}+\sum_{j=2s_i+2}^{\infty}\phi_{ij}2^j$$

which cannot equal 1. If $s_1=s_2=s_3$ and $s_3\ne s_4$, then $y_1^2+y_2^2+y_3^2+y_4^2=2^{2s_1}+2^{2s_1+1}+\sum_{j=2s_i+2}^{\infty}\phi_{ij}2^j$, which is not equal to 1. If $s_1=s_2=s_3=s_4$ then

$$y_1^2+y_2^2+y_3^2+y_4^2=2^{2s_1+2}+\sum_{j=2s_i+3}^{\infty}\phi_{ij}2^j,$$

To make this equal to 1, we need to have $s_1=-1$, which forces the norms $|y_i|=2$. So $y_1^2+y_2^2+y_3^2+y_4^2=1$ forces $|y_i|\le 2$. Thus all solutions are in the ball with radius 2. On the other hand, if $a\notin\mathbb{Q}_2^2$ we can take a to be one of the representatives 2, 3, 6, -1, -2, -3 or -6 of $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, which can all be written as $\delta_0+2\delta_1+4\delta_2+\sum_{s=3}^{\infty}\phi_s 2^s$. Since δ_i and ϕ_s are 0 or 1 we get

$$ay_4=\delta_0 2^{2s_4}+\delta_1 2^{2s_4+1}+\delta_2 2^{2s_4+2}+\sum_{m=2s_4+3}^{\infty}\pi_s 2^s.$$

No matter the value of the δ_i, we can carefully choose y_1, y_2 and y_3 to make the coefficients all zero for those whose powers are less than $2s_4+3$. For the coefficients whose power are larger or equal to $2s_4+3$, we can choose y_1, y_2 and y_3 as we desire. In particular we can make the sum equal to 1. Since $s_4\le -2$ is an arbitrary integer, we can get as large a solution as we desire. □

Lemma 17. *The fixed point group of* θ *is the group* $\mathrm{SO}(n,k)=\{A\in G\mid A^TA=\mathrm{Id}\}$. *For* $k=\mathbb{R}$, *the group* $\mathrm{SO}(n,k)$ *is compact, for* $k=\mathbb{Q}_p$ *the* $\mathfrak{p}$*-adic numbers (*$p\ne 2$*), it is noncompact. For* $\mathbb{Q}_2$*, if the rank of* G *is* 3 *or* 4*, it is compact, while it is noncompact if the rank of* G *is larger than or equal to* 5.

Proof. For $k = \mathbb{R}$ and $\mathbb{Q}_p$ compactness is equivalent to being closed and bounded. It is easy to see that the fixed point group is closed. For $k = \mathbb{R}$, since the norm $\|A\| = n$, it follows that the fixed point group is bounded.

For $\mathbb{Q}_2$, consider the case of $\mathrm{rank}(G) = 4$ first. If $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$, then by Lemma 16 $|y_i| \leq 2$. Therefore for $\mathrm{rank}(G) = 4$ (thus 3 as well), the fixed point group is compact.

For rank of G of 5 or bigger, let $y_5 = y_4$, we have $y_1^2 + y_2^2 + y_3^2 + 2y_4^2 = 1$, also by Lemma 16 we know we can choose y_i to be as large as desired, so the fixed point group is noncompact.

For $\mathbb{Q}_p$, consider first the case $n = 3$. The matrices

$$A_3 = \begin{pmatrix} a & b & c \\ \frac{ac}{b-1} & c & 1 + \frac{c^2}{b-1} \\ b + \frac{c^2}{b-1} & -a & -\frac{ac}{b-1} \end{pmatrix}$$

are in the fixed point group as long as $a^2 + b^2 + c^2 = 1$. We know that when $n \geq 3$, we can choose a, b, c to be as large as we want. Hence the norms of the matrices are not all finite, and the set of these matrices is not bounded. For G of higher rank the matrices $\begin{pmatrix} A_3 & 0 \\ 0 & I_{n-3} \end{pmatrix}$ are in the fixed point group. □

Lemma 18. *The fixed point group of* $\theta\,\mathrm{Inn}_{J_{2m}}$ *is noncompact.*

Proof. The fixed point group is $H = \left\{ A \mid \theta(A) = \mathrm{Inn}_{J_{2m}}(A) \right\}$. Clearly

$$\begin{pmatrix} r_1 & 0 & \dots & 0 & 0 \\ 0 & r_1^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & r_m & 0 \\ 0 & 0 & \dots & 0 & r_m^{-1} \end{pmatrix}$$

is in H, which is not bounded. □

Lemma 19. *Let* $k = \mathbb{R}$ *or* $\mathbb{Q}_2$ *and* $Y = I_{n-i,i}$. *Then the fixed point group of* $\theta\,\mathrm{Inn}_Y$ *is noncompact.*

Proof. The matrix

$$\begin{pmatrix} I_{n-i-1} & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & I_{i-1} \end{pmatrix}$$

is in the fixed point group as long as $a^2 - b^2 = 1$. □

So far we proved that the only compact fixed point group is that of the involution θ for $k = \mathbb{R}$ and $k = \mathbb{Q}_2$ and $n = 3$ or 4.

Lemma 20. *Let $k=\mathbb{Q}_p$, $(p\neq 2$ and $p=2)$ and $C=M_{n,1,1,x}$. The fixed point group of $\theta\,\mathrm{Inn}_C$ is noncompact if the* $\mathrm{rank}(G)=n\geq 4$.

Proof. For $p\neq 2$ The matrices $\begin{pmatrix}B & 0\\ 0 & 1\end{pmatrix}$, where B is an $n-1\times n-1$ matrix such that $B^TB=\mathrm{Id}$, is in the fixed point group. By Lemma 17, if $n\geq 4$, it is unbounded. Therefore the fixed point group is unbounded and thus noncompact. If $p=2$, by Lemma 16, when the rank of G is larger than or equal to 4, the fixed point group is noncompact. □

We still need to consider the case when $\mathrm{rank}(G)=n=3$.

Lemma 21. *Let $k=\mathbb{Q}_p$ and $C=M_{3,1,1,x}$. The fixed point group of $\theta\,\mathrm{Inn}_C$ is noncompact if $-1\in\mathbb{Q}_p^2$.*

Proof. The matrices $\begin{pmatrix}a & b & 0\\ -b & \frac{1-b^2}{a} & 0\\ 0 & 0 & 1\end{pmatrix}$ are in the fixed point group as long as $a^2+b^2=1$. While $-1\in\mathbb{Q}_p^2$, the norm of a and b can be chosen to be arbitrarily large, and hence the fixed point group of $\theta\,\mathrm{Inn}_C$ is noncompact. □

Lemma 22. *Let $k=\mathbb{Q}_p$ and $C=M_{3,1,1,S_p}$. The fixed point group of $\theta\,\mathrm{Inn}_C$ is noncompact.*

Proof. The matrix $Y_3=\begin{pmatrix}a & b & c\\ \frac{y^2c-a^2b}{1-b^2} & a & \frac{ac}{b-1}\\ \frac{acy}{b-1} & yc & \frac{yc^2}{1-b}-1\end{pmatrix}$ is in the fixed point group as long as $a^2+b^2+yc^2=1$ has a solution. For $y=S_p$, we have infinitely many solutions, and we can choose the norms of the roots to be as large as we desire. □

Lemma 23. *Let $k=\mathbb{Q}_p$ and $C=M_{3,1,1,x}$, where x is p or pS_p. The fixed point group of $\theta\,\mathrm{Inn}_C$ is compact if $-1\notin\mathbb{Q}_p^2$.*

Proof. The fixed point group of $\theta\,\mathrm{Inn}_C$ consists of all $\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$, where

$$\begin{pmatrix}a_{11} & a_{12} & xa_{13}\\ a_{21} & a_{22} & xa_{23}\\ \frac{1}{x}a_{31} & \frac{1}{x}a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}a_{11} & a_{21} & a_{31}\\ a_{11} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}=I.$$

When x is S_p or pS_p and $-1\notin\mathbb{Q}_p^2$, all the solutions for $a_{i1}^2+a_{i2}^2+xa_{i3}^2=1$ or x are unit (norm less or equal 1). Therefore the fixed point group is bounded and hence compact. □

Lemma 24. *Let* $k = \mathbb{Q}_2$ *and* $C = M_{3,1,1,x}$*, where* $x = 2, 3, 6, -1, -2, -3,$ *or* -6*. The fixed point group of* $\theta\,\mathrm{Inn}_C$ *is compact if* $x = 2$, -3 *or* -6, *and noncompact otherwise.*

Proof. Whether the fixed point group is bounded or not depends upon whether the equation $a_1^2 + a_2^2 + xa_3^2 = 1$ has only bounded solutions or not, as follows from Lemma 22 and 23. Let $a_i = 2^{s_i} + \sum_{j=s_i+1}^{\infty} \delta_{ij} 2^j$. Without loss of generality we may assume $s_1 \leq s_2$. If $s_1 \geq s_3 + 1$, with $x \in \{-1, -2, -3, -6, 2, 3, 6\}$, then $a_1^2 + a_2^2 + xa_3^2 = x2^{2s_3} + x + \sum_{j=2s_3+2}^{\infty} \delta_{ij} 2^j$ is either not equal to 1 or $s_3 = 0$. This forces $|a_3| = 1$, thus $\max(|a_1|, |a_2|) \leq \frac{1}{2}$. So we can possibly only get a noncompact fixed group when $s_3 \geq s_1$. Therefore, we can write $a_2^2 = 2^{2s_1} \sum_{j=0}^{\infty} \delta_j 2^j$, and $xa_3^2 = 2^{2s_1} \sum_{j=0}^{\infty} \pi 2^j$. We also know that $a_1^2 = 2^{2s_1} + 2^{2s_1} \sum_{j=3}^{\infty} \phi_j 2^j$. If we want $s_1 \leq -1$ and $a_1^2 + a_2^2 + xa_3^2 = 1$, then we must have $1 + \delta_0 + \pi_0 = 2$, $\delta_1 + \pi_1 = 1$, $\delta_2 + \pi_2 = 1$. Furthermore, we know that if $\delta_0 = 1$, then $\delta_1 = \delta_2 = 0$, and if $\delta_0 = 0$, then $\delta_1 = 0$ (since $a_2^2 = 2^{2s_1} \sum_{j=0}^{\infty} \delta_j 2^j$). So if $\delta_0 = 1$, we have $\phi_0 = 0$, and $\phi_1 = \phi_2 = 1$, which means $x = 6$ or -2. If $\delta_0 = 0$, we have $\phi_0 = 1$ and $\phi_1 = 1$, which means $x = 3$ or -1. From the above discussion, we know that for x equal to -1, -2, 3 or 6, we can choose a_{ij} to be as large as possible. It follows that the fixed point group is noncompact, while for x equal to 2, -3 or -6 the a_{ij}'s can only be chosen from the unit ball. □

Lemma 25. *For the matrix* $M = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$*, the fixed point group of* $\theta\,\mathrm{Inn}_M$ *are the matrices*

$$\begin{pmatrix} a & b & c \\ \frac{\delta ybi - zacf}{xa^2+yb^2} & -\frac{\delta xai + zbcf}{xa^2+yb^2} & f \\ -\frac{z(aci+\delta bf)}{xa^2+yb^2} & \frac{z(\delta xaf - ybci)}{y\left(xa^2+yb^2\right)} & i \end{pmatrix}$$

where δ *is* 1 *or* -1 *and* a, b, c, f *and* i *satisfy the following equations:*

$$a^2 + \frac{y}{x}b^2 + \frac{z}{x}c^2 = 1$$

and

$$\frac{z}{x}c^2 + \frac{z}{y}f^2 + i^2 = 1.$$

Proof. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be in the fixed point group of $\theta\,\mathrm{Inn}_M$. Then $\theta(A) = \mathrm{Inn}_M(A)$. That is, $M^{-1}AMA^T = I$, i.e.

$$\begin{pmatrix} a & \frac{y}{x}b & \frac{z}{x}c \\ \frac{x}{y}d & e & \frac{z}{y}f \\ \frac{x}{z}g & \frac{y}{z}h & i \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = I.$$

From $\frac{x}{y}ad + be + \frac{z}{y}fc = 0$, it follows that $e = -\frac{(xad+zfc)}{yb}$. Therefore

$$\frac{x}{y}d^2 + e^2 + \frac{z}{y}f^2 = 1$$

$$\Longrightarrow \frac{x}{y}d^2 + \left(\frac{xa}{yb}\right)^2 d^2 + \left(\frac{2xzacf}{yb}\right) d + \frac{z^2f^2c^2}{y^2b^2} + \frac{z}{y}f^2 - 1 = 0$$

$$\Longrightarrow \left(xyb^2 + x^2a^2\right) d^2 + (2xzacf)\, d + z^2c^2f^2 + yzb^2f^2 - y^2b^2 = 0$$

Solve

$$d = \frac{\delta ybi - zacf}{xa^2 + yb^2}, \text{ where } \delta \text{ is } 1 \text{ or } -1.$$

Then

$$e = -\frac{xad + zcf}{yb} = -\frac{\delta xai + zbcf}{xa^2 + yb^2}$$

Furthermore, we have

$$ag + \frac{y}{x}bh = \frac{z}{x}ci = 0$$

and

$$\frac{x}{y}d + eh + \frac{z}{y}fi = 0$$

So

$$g = \frac{z\,(ce - bf)\,i}{y\,(bd - ae)}$$

$$h = \frac{z\,(af - cd)\,i}{y\,(bd - ae)}$$

Also, if we plug in d and e into $db - ae$, we have $db - ae = \delta i$. So

$$g = \delta z\,(ce - bf) = -\frac{z\,(aci + \delta bf)}{xa^2 + yb^2}$$

and

$$h = \frac{\delta z}{y}\,(af - cd) = \frac{z\,(\delta xaf - ybci)}{y\left(xa^2 + yb^2\right)}$$

□

Lemma 26. *For matrix* $C = M_{n,p,S_p,pS_p}$ *and* $-1 \in \mathbb{Q}_p^2$, *the fixed point group of* $\theta\,\mathrm{Inn}_C$ *is noncompact.*

Proof. The matrix $\begin{pmatrix} I_{n-3} & 0 \\ 0 & B_3 \end{pmatrix}$ is in the fixed point group if and only if B_3 is the form of Lemma 25. For $-1 \in \mathbb{Q}_p^2$, we can choose a, b and c as large as we want to satisfy Lemma 24. □

Lemma 27. *Assume* $-1 \notin \mathbb{Q}_p^2$. *For the matrix* $C = M_{n,1,p,p}$ *with* $n \geq 5$, *or* $C = M_{n,p,p,S_p}$ *with* $n \geq 6$, *the fixed point group of* $\theta\,\mathrm{Inn}_C$ *is noncompact.*

Proof. The result follows with a simular arguments as in Lemma 20, using Lemma 17. □

This proves that for each choice of the matrices C in Sections 4.1.6 and 4.1.7, the fixed point groups are noncompact except in the case that the rank of G is 4 and $-1 \notin \mathbb{Q}_p^2$, where it is compact from the following lemma.

Lemma 28. *For the matrix* $C = M_{4,1,p,p}$, *and* $-1 \notin \mathbb{Q}_p^2$, *the fixed point group of* $\theta\,\mathrm{Inn}_C$ *is compact.*

Proof. The fixed point group is the set of all $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, where

$$\begin{pmatrix} a_{11} & a_{12} & pa_{13} & pa_{14} \\ a_{21} & a_{22} & pa_{23} & pa_{24} \\ \frac{1}{p}a_{31} & \frac{1}{p}a_{32} & a_{33} & a_{34} \\ \frac{1}{p}a_{41} & \frac{1}{p}a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = I.$$

Since $-1 \notin \mathbb{Q}_p$, $x^2 + y^2$ begins with even power of p if we write it in the standard form. So consider the diagonal element in the above matrix identity. $a_{11}^2 + a_{12}^2 + pa_{13}^2 + pa_{14}^2 = 1$ can only occur inside the unit ball (with norm less or equal 1). A similar argument holds for the other rows. □

7.1. SUMMARY OF THE COMPACT FIXED GROUPS

For $\mathbb{R}$, we proved that the only compact fixed point group is the one for the involution θ, and that for $\mathbb{Q}_p$ with $p \neq 2$, the involutions with compact fixed point groups are

(1) $\mathrm{rank}(G) = n = 3$: $\theta\,\mathrm{Inn}_{M_{3,1,1,p}}$ and $\theta\,\mathrm{Inn}_{M_{3,1,1,pS_p}}$.

(2) $\mathrm{rank}(G) = n = 4$: $\theta\,\mathrm{Inn}_{M_{4,1,p,p}}$ if $-1 \in \mathbb{Q}_p^2$.

(3) $\mathrm{rank}(G) = n > 4$. None.

Finally for $\mathbb{Q}_2$ the involutions with compact fixed point groups are

(1) $\mathrm{rank}(G) = n = 3$: θ, $\theta\,\mathrm{Inn}_{M_{3,1,1,2}}$, $\theta\,\mathrm{Inn}_{M_{3,1,1,-3}}$and $\theta\,\mathrm{Inn}_{M_{3,1,1,-6}}$.

(2) $\mathrm{rank}(G) = n = 4$: θ.

(3) $\mathrm{rank}(G) = n > 4$. None.

8. Involutions and *k*-inner elements

In this section we discuss the connection between the classification of k-involutions of $\mathrm{SL}(n,k)$ in this paper and the characterization in [7] of the isomorphism classes of k-involutions of arbitrary reductive algebraic groups G defined over a field k of characteristic not 2. In [7] these isomorphism classes were characterized by essentially using the following 3 invariants:

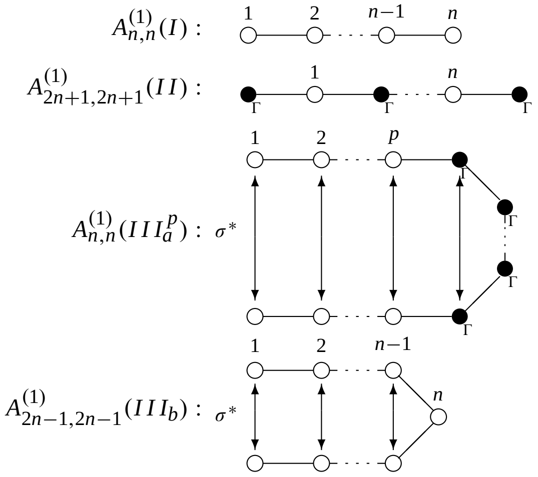
(1) classification of admissible (Γ,σ)-indices.

(2) classification of the G_k-isomorphism classes of k-involutions of the k-anisotropic kernel of G.

(3) classification of the G_k-isomorphism classes of k-inner elements of G.

For more details, see [7]. The admissible (Γ,σ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was included in [7] as well. To complete this classification it remains to classify the second and third invariant. As was shown in [7] a classification of the k-inner elements depend on the base field k and for general G a classification of this second and third invariant can be quite complicated. For $k = \mathbb{R}$ the k-inner elements were classified in [6] by using signatures as an invariant. For other fields additional invariants are needed. To get a good idea of the kind of invariant that might be needed we study first the case that G is k-split. In this case there is a maximal torus T which is k-split and hence there is no k-anisotropic kernel. So in this case we only need to classify the third invariant: the G_k-isomorphism classes of k-inner elements. The group $G_k = \mathrm{SL}(n,k)$ is k-split and in the previous sections we gave a characterization of the isomorphism classes of k-involutions and classified them for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers or a finite field $\mathbb{F}_p$. This classification was independent of the characterization in [7]. To be able to use these results as an indication of how to proceed with a general classification of the k-inner elements we need to translate the classification in this paper to fit the invariants/ characterization given in [7]. In particular we need to determine the

Table VI. (Γ, σ)-indices for $\mathrm{SL}(n, k)$



the maximal (σ, k)-split tori and the the G_k-isomorphism classes of k-inner elements. We discuss this in this section.

8.1. (σ, k)-SPLIT TORI AND k-INNER ELEMENTS

Since the group is k-split the (Γ, σ)-indices are exactly the σ-indices of the case that $k = \bar{k}$, only with an additional label Γ under all the black nodes in the σ-index. The latter were classified [6, Table II]. We recall that for $k = \bar{k}$ there is a bijective correspondence between the isomorphism classes of k-involutions and the congruence classes of σ-indices (see [6, Theorem 3.11]). The (Γ, σ)-indices for $\mathrm{SL}(n, k)$ are listed in Table VI.

For notations on the (Γ, σ)-indices we refer to [7, Section 5]. The involutions of $\bar{G} = \mathrm{SL}(n, \bar{k})$ corresponding to these (Γ, σ)-indices are respectively θ, $\theta\,\mathrm{Inn}_{J_{2m}}$ and Inn_Y with Y is one of $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$. The latter two (Γ, σ)-indices are both related to the involutions of inner type, but since the restricted root system for the related symmetric k-variety is of a different type both (Γ, σ)-indices occur in the list of (Γ, σ)-indices.

8.1.1. *k-inner elements*

Let T be a σ-stable maximal k-split torus of $\bar{G}$ containing a maximal (σ, k)-split torus A of G. Recall that a torus is σ-split if $\sigma(a) = a^{-1}$ for all $a \in A$ and (σ, k)-split if it is both k-split and σ-split. Since G is k-split T is a maximal torus of $\bar{G}$ as well. Moreover it follows from [7, Theorem 8.33] that we have the following characterization of the isomorphism classes:

Theorem 10 ([7, Theorem 8.33]). *Any k-involution of G is isomorphic to one of the form $\sigma\,\mathrm{Inn}_a$, where σ is a representative of a $\bar{G}$-isomorphism class of k-involutions, A a maximal (σ,k)-split torus and $a \in A$.*

The *set of k-inner elements of A* is defined as the set of those $a \in A$ such that $\sigma\,\mathrm{Inn}_a$ is a k-involution of G by $\mathrm{Inn}_k(A)$. We recall that from [7, Lemma 9.7] it follows that one can find a set of representatives for the isomorphism classes of the involutions $\sigma\,\mathrm{Inn}_A$ in the set $\mathrm{Inn}_k(A)/A_k^2$. Here A_k is the set a k-regular elements of A and $A_k^2 = \{a^2 \mid a \in A_k\}$.

In the remainder of this section we will rewrite the representatives for the isomorphism classes in the form $\sigma\,\mathrm{Inn}_a$ with σ one of the representatives from the algebraically closed case and in particular find a set of k-inner elements of A representing these isomorphism classes. This will lead to a set of invariants classifying these elements in the cases that $G = \mathrm{SL}(n,k)$.

8.2. COMPUTING THE MAXIMAL (σ,k)-SPLIT TORI

For each of the different types of involutions over the algebraically closed field $\bar{k}$ we will compute in the following first the maximal (σ,k)-split torus A and after that the k-inner elements representing the different isomorphism classes of involutions. In the following let T be the maximal k-split torus consisting of all the diagonal matrices.

(1) If $\sigma = \theta$, then the maximal (σ,k)-split torus is $S_1 = T_\sigma^- = \{t \in T \mid \sigma(t) = t^{-1}\} = T$.

(2) If $\sigma = \theta\,\mathrm{Inn}_{J_{2m}}$, then let $T' = B^{-1}TB$ with $B \in \mathrm{SL}(n,k)$. We need to choose B such that $S_2 = T_\sigma^- = \{B^{-1}tB \mid t \in T, \sigma(B^{-1}tB) = (B^{-1}tB)^{-1}\}^0$ has maximal dimension. Note that

$$\begin{aligned}\sigma(B^{-1}tB) = (B^{-1}tB)^{-1} &\Rightarrow \theta\,\mathrm{Inn}_{J_{2m}}(B^{-1}tB) = B^{-1}t^{-1}B\\ &\Rightarrow J_{2m}^{-1}(B^{-1}tB)J_{2m} = \theta(B^{-1}t^{-1}B) = B^Tt(B^T)^{-1}\\ &\Rightarrow BJ_{2m}B^Tt = tBJ_{2m}B^T.\end{aligned}$$

For $B = I$, the dimension of A_2 is maximal and equal to $\frac{n}{2}$. In particular the maximal (σ,k)-split torus is:

$$A_2 = \{\mathrm{diag}(a_1,a_2,\ldots,a_n) \mid a_1 = a_2, a_3 = a_4, \ldots a_{n-1} = a_n\}.$$

(3) If $\sigma = \mathrm{Inn}_A$ with A one of $I_{n-i,i}$, $i = 1,2,\ldots,\lceil\frac{n-1}{2}\rceil$, then let $T' = B^{-1}TB$ with $B \in \mathrm{SL}(n,k)$. We need to choose B such that $S_{n-i,i} = T_\sigma^- = \{B^{-1}tB \mid t \in T, \sigma(B^{-1}tB) = (B^{-1}tB)^{-1}\}^0$ has maximal dimension.

For the maximal (σ,k)-split torus and their dimensions, we have

Lemma 29. *The maximal (σ,k)-split torus for $I_{n-i,i}$, $i = 1,2,\ldots,\lceil\frac{n-1}{2}\rceil$ can be chosen as:*

$$A_{n-i,i} = \{B^{-1}\,\mathrm{diag}(a_1,\ldots,a_i,a_i^{-1},\ldots,a_1^{-1},1,\ldots,1)B\},$$

where B satisfies $BAB^{-1} = \begin{pmatrix} J & 0 \\ 0 & I_{n-2i} \end{pmatrix}$. *The dimension of the maximal (σ, k)-split torus is of course i.*

Proof. Note that

$$\begin{aligned} \sigma(B^{-1}tB) = (B^{-1}tB)^{-1} &\Rightarrow \mathrm{Inn}_A(B^{-1}tB) = B^{-1}t^{-1}B \\ &\Rightarrow A^{-1}(B^{-1}tB)A = B^{-1}t^{-1}B \\ &\Rightarrow tBAB^{-1} = BAB^{-1}t^{-1}. \end{aligned}$$

Since t is conjugate to t^{-1}, then highest possible dimension can only be less or equal to $\frac{n}{2}$. Also, if $t = \mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)$, and $tY = Yt^{-1}$, then we have $Y = \begin{pmatrix} J & 0 \\ 0 & Y_{n-2i} \end{pmatrix}$, therefore, if the (σ, k)-split tori has dimension of i, the corresponding $I_{n-j,j}$ has to be conjugate to $\begin{pmatrix} J & 0 \\ 0 & Y_{n-2i} \end{pmatrix}$. Hence a (σ, k)-split torus has dimension i if and only if the corresponding $I_{n-j,j}$ such that $j \geq i$, i.e. the maximal (σ, k)-split tori has dimension j for $I_{n-j,j}$. □

8.3. COMPUTING k-INNER ELEMENTS REPRESENTING ISOMORPHISM CLASSES

From [7, Theorem 8.33] we know now that any k-involution is conjugate to one of the following:

(1) $\theta\,\mathrm{Inn}(a),\ a \in S_1 = T,$,

(2) $\theta\,\mathrm{Inn}_{J_{2m}}\,\mathrm{Inn}(a),\ a \in S_2,$

(3) $\mathrm{Inn}_A\,\mathrm{Inn}(a),\ A = I_{n-i,i}, a \in S_{n-i,i}.$

Next we will compute the k-inner elements corresponding to the representatives of the isomorphism classes of involutions in Sections 4 and 5. Note that for $k = \mathbb{R}$ a classification of these k-inner elements can also be found in [6, Table II and IV], where they are called quadratic elements.

(1) $\boldsymbol{k = \mathbb{R}}$: the real numbers (see Sections 4 and 5)

a) θ is in case (1) with $a = I$.

b) Inn_A is in case (3) with $a = I$.

c) Inn_{J_n} is in case (3) with $a = B^{-1}tB \in A_{\frac{n}{2},\frac{n}{2}}$, where $t = \mathrm{diag}(i, \ldots, i, -i, \ldots, -i)$.

d) $\theta\,\mathrm{Inn}_{J_n}$ is in case (2) with $a = I$.

e) $\theta\,\mathrm{Inn}_A$ is in the case (1) with $a = A$.

(2) $\boldsymbol{k}=\mathbb{Q}_{\boldsymbol{p}}$: the $\mathfrak{p}$-adic numbers (see Sections 4.1.7 and 5)

a) θ is in case (1) with $a=I$.

b) Inn_A is in case (3) with $a=I$.

c) $\theta\,\mathrm{Inn}_A$ is in case (1) with $a=A$.

d) Inn_B is in case (3) with $a=B^{-1}tB\in A_{\frac{n}{2},\frac{n}{2}}$, where $t=(\sqrt{x})^{-1}\,\mathrm{diag}(x,\ldots,x,1\ldots,1)$, and x is p, S_p, pS_p for $p\neq 2$ and $-1,-2,-3,-6,2,3,6$ for $\mathbb{Q}_2$.

e) $\theta\,\mathrm{Inn}_{J_{2m}}$ is in case (2) with $a=I$.

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