ALGORITHMS FOR COMPUTATIONS IN LOCAL SYMMETRIC SPACES

JENNIFER R. FOWLER AND ALOYSIUS G. HELMINCK*

ABSTRACT. In the last two decades much of the algebraic/combinatorial structure of Lie groups, Lie algebras, and their representations has been implemented in several excellent computer algebra packages, including LiE, GAP4, Chevie, Magma and Maple. The structure of reductive symmetric spaces or more generally symmetric k-varieties is very similar to that of the underlying Lie group, with a few additional complications. A computer algebra package enabling one to do computations related to these symmetric spaces would be an important tool for researchers in many areas of mathematics, including representation theory, Harish Chandra modules, singularity theory, differential and algebraic geometry, mathematical physics, character sheaves, Lie theory, etc. However, until a few years ago only very few algorithms existed for computations in these symmetric spaces. This is in part due to the fact that the algebraic/combinatorial structure of these symmetric spaces is much more complicated and also much of this structure was traditionally proved using analytic and geometric methods, which do not lend themselves to designing algorithms. About 15 years ago the concept of real symmetric spaces was generalized to symmetric spaces over general fields of characteristic not 2 and it was shown that these generalized symmetric spaces (called symmetric k-varieties) have many algebraic/combinatorial properties similar to those of the classical real symmetric spaces. Most of this work mainly used algebraic and combinatorial methods, which lend themselves excellently to designing algorithms; and based on these results some of the first algorithms for symmetric spaces were developed a few years ago (see [Hel96, Hel00]). In this paper we lay the groundwork for computing the fine structure of symmetric spaces over the real numbers and other base fields, give a complete set of algorithms for computing the fine structure of symmetric varieties and use this to compute nice bases for the local symmetric varieties.

1. INTRODUCTION

Real reductive symmetric spaces have been studied for almost a hundred years and they are of importance in many areas of mathematics and physics. They can be defined as the homogeneous spaces G/H, where G is a reductive Lie group of Harish Chandra class and H is an open subgroup of the fixed point group of an involution of G. Initially one mainly studied the *Riemannian symmetric spaces*, i.e. the symmetric spaces for which H is a maximal compact subgroup of G. In the last 25 years the emphasis has shifted to include the general reductive symmetric spaces. Symmetric spaces are best known for their role in representation theory and many have studied the representations associated with these real reductive symmetric spaces (see for example [HC84, FJ80, ŌS80, ŌM84, BD92, vdBS97, Del98]).

Similar spaces over other fields occur throughout mathematics. For any field k of characteristic not 2 a *symmetric k-variety* is defined as the homogeneous space G_k/H_k, where G is a reductive algebraic group defined over k and $H = G^\sigma$ the fixed point group of a k-involution σ of G. Here we have used the notation K_k for the set of k-rational points of an affine algebraic group K defined over k. Similarly the p-adic symmetric k-varieties are also called reductive p-adic symmetric spaces or simply p-adic symmetric spaces. The symmetric k-varieties over algebraically closed fields are also called symmetric varieties. Instead of these global symmetric spaces it often suffices to study the corresponding subspaces in the Lie algebra. If $\mathfrak{g}$ is the Lie algebra of G, then σ also induces an involution of $\mathfrak{g}$ which we will also denote by σ. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{h}$ resp. $\mathfrak{q}$ are the $+1$ and -1-eigenspaces of σ in $\mathfrak{g}$. The subalgebra $\mathfrak{h}$ is the Lie algebra of H and the subspace $\mathfrak{q}$ is the tangent space in the identity of the subvariety $Q = \{x\sigma(x)^{-1} \mid x \in G\}$ which is isomorphic to G/H. Similarly

1991 *Mathematics Subject Classification.* 53C35, 17B45.
Key words and phrases. symmetric spaces, Lie algebras of linear algebraic groups.
*Author is partially supported by N.S.F. Grant DMS-9977392.

we have a decomposition $\mathfrak{g}_k = \mathfrak{h}_k \oplus \mathfrak{q}_k$ of the Lie algebra of G_k in eigenspaces of σ. The subspace $\mathfrak{q}_k$ is called a *local symmetric k-variety* or also a local symmetric space.

Symmetric k-varieties over base fields, other then the real numbers, occur in many problems in representation theory (see [BB81] and [Vog83, Vog82]), geometry (see [DCP83, DCP85] and [Abe88]), singularity theory (see [LV83] and [HS90]), the study of character sheaves (see for example [Lus90, Gro92]) and at the study of cohomology of arithmetic subgroups (see [TW89]). The p-adic symmetric spaces are also of importance for the study of automorphic forms.

Reductive groups over algebraically closed fields have a natural fine structure of a root system with its Weyl group coming from a maximal torus. If the group is defined over a field k which is not algebraically closed then there is a second root system and Weyl group which characterizes the k-structure of the group. This additional fine structure comes from the root system of a maximal k-split torus A of G together with its Weyl group and the multiplicities of the roots. The maximal k-split torus A is contained in a maximal k-torus T and the root system and Weyl group of the maximal k-split torus A can be identified with the projection of the root system of T to A and similarly the Weyl groups can be identified with the quotient of two subgroups of the Weyl group of T. This fine structure of the two root systems with their Weyl groups and multiplicities of the roots plays a fundamental role in all studies of reductive k-groups and their applications (representations). Symmetric k-varieties (or symmetric spaces) have a similar fine structure, which plays an equally important role in the study of these symmetric spaces and their applications as their counterpart in the groups case. However the fine structure of these symmetric spaces is much more complicated since it involves the integrate relations of 5 root systems and their Weyl groups instead of just one or two. Before we describe these we need a bit more notation. For a subgroup K of G and a torus S in K let $\Phi(S,K)$ denote the set of roots of S in K and let $W(S,K) = N_K(S)/Z_K(S)$ denote the Weyl group of S in K, where $N_K(S)$ is the normalizer in K of S and $Z_K(S)$ is the centralizer in K of S. In the case that $K = G$ we will also write $\Phi(S)$ for $\Phi(S,G)$ and $W(S)$ for $W(S,G)$. If S is a k-split torus, then $\Phi(S,G_k) = \Phi(S,G)$. The 5 root systems of a symmetric space are the following. First there is the natural root system associated with these symmetric k-varieties. This root system is the set of roots of a maximal k-split torus of G contained in the symmetric variety Q. The k-split tori in Q are also called (σ,k)-split tori and in [HW93] it was shown that the set of roots of a maximal (σ,k)-split torus A is actually a root system with Weyl group $W(A) = N_{G_k}(A)/Z_{G_k}(A)$. The second root system is that of a maximal k-torus A_1 contained in Q. Since this torus is split over the algebraic closure $\bar{k}$ it follows similar as for the maximal (σ,k)-split tori that $\Phi(A_1)$ is a root system with Weyl group $W(A_1) = N_G(A_1)/Z_G(A_1)$. The torus A_1 is also called σ-split and it can be chosen to contain A. The third root system comes from a maximal k-split torus A_2 of G, which again we can choose containing A. In [HW93] it was also shown that there exists a maximal k-torus T of G which contains A, A_1 and A_2. This maximal torus gives us our fourth root system. Finally the fifth root system is the subsystem Φ_0 of $\Phi(A,G_k) = \Phi(A,G)$ consisting of those roots $\alpha \in \Phi(A,G))$ for which the corresponding reflection in the Weyl group has a representative in H_k. The Weyl group of Φ_0 is precisely the subgroup $W(A,H_k) = N_{H_k}(A)/Z_{H_k}(A)$ of $W(A,G_k)$. The root systems $\Phi(A,G_k)$, $\Phi(A_1)$ and $\Phi(A_2,G_k)$ can be identified with the projections of $\Phi(T)$ to A, A_1 and A_2 and similarly the Weyl groups can be identified with the quotient of two subgroups of the Weyl group $W(T)$ of T. The root system $\Phi(A,G_k)$ can also be identified with the projections of $\Phi(A_1)$ or $\Phi(A_2,G_k)$ to A and similarly the Weyl group $W(A,G_k)$ can be identified with the quotient of two subgroups of the Weyl group $W(A_1)$ of A_1 or of the Weyl group $W(A_2,G_k)$ of A_2. The integrate relations between all these root systems and their Weyl group plays a fundamental role in the study of these symmetric spaces. An example of how one can use this fine structure to derive properties of the symmetric space is illustrated by Theorem 1, where we show that a real symmetric space is Riemannian if and only if $\Phi_0 = \Phi(A,G_k) = \Phi(A_1,G_k) = \Phi(A_2,G_k)$.

In the group case the integrate fine structure related to the root systems with their Weyl groups has been implemented in several symbolic computation packages, like LiE, Maple, GAP4, Chevie, and Magma. These packages have become an indispensable tool for scientists in many areas of mathematics and physics. For symmetric spaces none of this fine structure has been implemented yet in an computer algebra package, although such a package would be extremely useful for many scientists as well. There are several reasons for this. Not only is the fine structure of these symmetric spaces a lot more complicated than that of the Lie groups, because instead of just 1 root system, there are 5 root systems which are closely entangled. Also a lot of their structure was traditionally proved using analytic and geometric methods, which do not lend themselves very well for designing algorithms.

In this paper we make the first step towards building a computer algebra package with which one can compute the fine structure of these symmetric spaces by considering first the case of symmetric varieties. In section 2 we will show that the fine structure of these is the same as that of the Riemannian symmetric spaces (see Corollary 1), which is the same as the fine structure of real semisimple Lie algebras. In a forthcoming paper we will describe algorithms for the fine structure of general real symmetric spaces. All this fine structure of these symmetric spaces can also be computed in the Lie algebra setting what simplifies some of the computations. To compute the fine structure of these symmetric spaces it does not suffice to compute all the root systems involved together with their Weyl groups. In many problems about these symmetric spaces one needs to know which roots project down to a root in a restricted root system and also one often needs representatives for elements of the Weyl group of one of the restricted root systems in terms of representatives of the Weyl group of the related maximal torus. For example to compute nice bases for $\mathfrak{h}$ and $\mathfrak{q}$ one needs to decompose all root subspaces for roots of a maximal toral subalgebra $\mathfrak{a} \subset \mathfrak{q}$ as a sum of root subspaces of a maximal toral subalgebra containing $\mathfrak{a}$. From the σ-diagram of the involution one can easily do this for the roots in a basis, but for all other roots we need the Weyl group and its action on the root subspaces of the maximal toral subalgebra to compute the decomposition of the root subspace of an arbitrary root of $\mathfrak{a}$. So computing the fine structure of these symmetric spaces will include computing representatives for the restricted Weyl groups in terms of Weyl group elements of the maximal toral subalgebra and also computing all the roots that project down to a root in a restricted root system.

A classification of the involutions of semisimple Lie algebras defined over an algebraically closed field was given in [Hel88]. In this paper it is shown that the isomorphy classes of these involutions can be represented by a σ-diagram (see section 3.4) which essentially represents the action of the involution on the root system. For $\mathfrak{g}$ simple there are 24 types of local symmetric spaces and the corresponding σ-diagrams are called absolutely irreducible. For each of these 24 types of local symmetric spaces we give an algorithm to compute their fine structure, which is roughly as follows:

(i) For each case, recover the action of the involution on the original root system from the σ-diagram as given in [Hel88].
(ii) Determine all the positive roots that project down to each root in the base for the restricted root system.
(iii) Find representatives in the original Weyl group for each element in the restricted Weyl group.
(iv) Use the Weyl group representatives to find a complete list of positive roots in the restricted root system.
(v) Determine all the positive roots of $\mathfrak{g}$ that project down to each root in the base for the restricted root system using the restricted Weyl group and the representatives of its elements in the original Weyl group as in (iii).

In the latter part of this paper we show how the computation of this fine structure can be used to compute nice bases for the eigenspaces of σ: $\mathfrak{h}$ and $\mathfrak{q}$. The algorithm to compute these bases adds two steps:

(vi) Write the eigenspaces for each root in the restricted root system as the direct sum of eigenspaces from the original root system.
(vii) Determine basis for $\mathfrak{q}$ by using the projection map and eigenspace decomposition.

This algorithm can be implemented in any of the computer algebra packages in which the structure of Lie algebras and Lie groups has already been incorporated. We included tables for the data needed in the various steps of the algorithm and we worked out an example illustrating the algorithm (see section 9).

2. Riemannian symmetric spaces and symmetric varieties

In this section we introduce the notation and show that the fine structure of the real Riemannian symmetric spaces is the same as that of the symmetric varieties. For this we review first the relation between real symmetric spaces and pairs of commuting involutions and introduce the real Riemannian symmetric spaces.

2.1. **Relation between real symmetric spaces and pairs of commuting involutions.** Let G be a reductive group defined over a field k, let G_k denote the set of k-points of G and let $\mathfrak{g}$ denote the Lie algebra of G.

If the base field k is the real numbers, then the Galois group Γ has order 2. If $\delta \neq \mathrm{id} \in \Gamma$, then δ acts on the complex Lie algebra $\mathfrak{g}$ of G as a conjugation, i.e. it has order 2 and it is linear with respect to addition and conjugate linear with respect to scalar multiplication. We denote this conjugation by δ as well. The set of fixed points $\mathfrak{g}^\delta = \{X \in \mathfrak{g} \mid \delta(X) = X\}$ of the conjugation δ is exactly the (real) Lie algebra of the group $G_\mathbb{R}$. The Lie algebra $\mathfrak{g}^\delta$ is called a real form of $\mathfrak{g}$ and there is a one-to-one correspondence between real forms of $\mathfrak{g}$ and conjugations of $\mathfrak{g}$.

Let $\sigma \in \mathrm{Aut}(G_\mathbb{R})$ be a $\mathbb{R}$-involution. We denote the involution of $\mathfrak{g}^\delta$ induced by σ also by σ. By extending the base field we can lift our involution σ from $G_\mathbb{R}$ to G and similarly from $\mathfrak{g}^\delta$ to $\mathfrak{g}$. Since σ is an involution of $\mathfrak{g}^\delta$ we can write $\mathfrak{g}^\delta = \mathfrak{h} + \mathfrak{q}$ as a sum of eigenspaces, i.e $\mathfrak{h} = \{X \in \mathfrak{g}^\delta \mid \sigma(X) = X\}$ and $\mathfrak{q} = \{X \in \mathfrak{g}^\delta \mid \sigma(X) = -X\}$. The space $\mathfrak{q}$ is called a *local reductive symmetric space* or also a *local affine symmetric space*. From $\sigma(\mathfrak{g}^\delta) = \mathfrak{g}^\delta$ it follows that σ and δ commute. Instead of considering the action of σ on $\mathfrak{g}^\delta$ one can consider the action of the the pair (σ, δ) on the complex Lie algebra $\mathfrak{g}$. This can be simplified even further. Instead of considering the pair (σ, δ) of an involution and a conjugation commuting with it, we can instead switch to a pair of commuting involutions by replacing δ with the Cartan involution corresponding to $\mathfrak{g}^\delta$. For this we have to consider a compact real form. Up to isomorphy there exists a unique compact real form $\mathfrak{u}$ defined by the fact that the Killing form is negative definite on $\mathfrak{u}$. In particular it was shown in [Hel88, 10.3] that there exists a compact real form which is δ and σ-stable. Let τ be the conjugation of this compact real form $\mathfrak{u}$. The condition that $\mathfrak{u}$ is δ and σ-stable is equivalent to $\sigma\tau = \tau\sigma$ and $\delta\tau = \tau\delta$. Let $\theta = \delta\tau = \tau\delta$. Since both δ and τ are conjugations it follows that θ is an involution of $\mathfrak{g}^\delta$ and $\mathfrak{g}$ and $\theta|\mathfrak{g}^\delta$ is called a *Cartan Involution of* $\mathfrak{g}^\delta$. This involution is characterized by the fact that $\mathfrak{k} = \mathfrak{g}^\delta \cap \mathfrak{g}^\theta = \mathfrak{g}^\delta \cap \mathfrak{u}$ is a maximal compact subalgebra of $\mathfrak{g}^\delta$. In this case we write $\mathfrak{g}^\delta = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p} = \{X \in \mathfrak{g}^\delta \mid \theta(X) = -X\}$. The involutions σ and θ of $\mathfrak{g}$ commute and there is a one to one correspondence between the isomorphy classes of the pairs (σ, δ) of an involution and a conjugation commuting with it and the isomorphy classes of pairs of commuting involutions (σ, θ) with θ a Cartan involution (see [Hel88]). Conversely given any pair of commuting involutions (σ, θ) of $\mathfrak{g}$, then there exists a σ and θ-stable real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\theta|\mathfrak{g}_0$ is a Cartan involution. This can be accomplished as follows. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ which is σ and θ-stable and let τ be its conjugation. Then $\delta = \tau\theta$ is a conjugation and $\theta|\mathfrak{g}^\delta$ is a Cartan involution of $\mathfrak{g}^\delta$. This establishes a bijective correspondence between isomorphy classes of ordered pairs of commuting involutions of complex reductive Lie algebras, isomorphy classes of involutions of real reductive Lie algebras, and isomorphy classes of real reductive symmetric spaces.

In the case that $\sigma = \theta$ we call the symmetric space $\mathfrak{p} = \mathfrak{q} = \{X \in \mathfrak{g}^\delta \mid \theta(X) = -X\}$ a *local Riemannian symmetric space* and both the involution and the symmetric space are completely determined by choosing a maximal compact subalgebra of $\mathfrak{g}^\delta$. Since the Riemannian symmetric spaces correspond to the pairs (θ, θ) of $\mathfrak{g}$ we get a bijective correspondence between the Riemannian symmetric spaces and the complex symmetric varieties, where the complex symmetric varieties are complexifications of Riemannian symmetric spaces. To emphasize this relation between complex symmetric varieties and Riemannian symmetric spaces we will use $\theta \in \mathrm{Aut}(\mathfrak{g})$ as the involution defining a complex symmetric variety and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ as a sum of eigenspaces of θ.

In the remainder of this section we will show that the fine structure of the Riemannian symmetric spaces is the same as that of the complex local symmetric varieties coming from a complex semisimple Lie algebra with an involution.

2.2. **Fine structure of Riemannian symmetric spaces.** In this subsection we show that the fine structure of the reduced root systems with Weyl groups and multiplicities for a Riemannian symmetric space is the same as that of the symmetric varieties. First we will give a criterium to check whether a real symmetric space is Riemannian or not. This criterium only depends on the structure of the 5 root systems with their Weyl groups of these symmetric spaces. We use the same notation as in the introduction.

Theorem 1. *Let G be a real reductive Lie group and σ an involution of G with $H = G^\sigma$. Let $A \subset G$ be a maximal $(\sigma, \mathbb{R})$-split torus, $A_1 \supset A$ a maximal σ-split $\mathbb{R}$-torus, $A_2 \supset A$ a maximal $\mathbb{R}$-split torus of G and T a maximal $\mathbb{R}$-torus T of G which contains A, A_1 and A_2. Then the following are equivalent.*

(i) *G/H is a Riemannian symmetric space.*
(ii) *σ is a Cartan involution of G.*
(iii) *$\Phi_0 = \Phi(A, G_\mathbb{R}) = \Phi(A_1) = \Phi(A_2, G_\mathbb{R})$.*

Proof. The equivalence of (i) and (ii) is well known and can be found in [Hel78] and in slightly more generality in [HW93].

(ii) $\implies$ (iii). If σ is a Cartan involution of G, then any σ-split $\mathbb{R}$-torus is also $\mathbb{R}$-split (see [HW93]). Moreover any $\mathbb{R}$-split torus is also split for some Cartan involution of G and since all Cartan involutions are conjugate, it follows that $A = A_1 = A_2$. Finally since any maximal σ-split torus in G has representatives in H we get (iii).

(iii) $\implies$ (ii). From $\Phi(A, G_\mathbb{R}) = \Phi(A_1) = \Phi(A_2, G_\mathbb{R})$ it follows from [Hel88] that up to a quadratic element σ is a Cartan involution of G. Since $\Phi_0 = \Phi(A, G_\mathbb{R})$ the Weyl group $W(A)$ has representatives in $H_\mathbb{R}$ and hence (σ, σ) is a standard pair in the sense of [Hel88, Section 6], and hence by [Hel88] σ is a Cartan involution. □

Corollary 1. *Let $\bar{G}$ be a reductive Lie group defined over $\mathbb{R}$ and σ an involution of $\bar{G}$ with $\bar{H} = \bar{G}^\sigma$. Assume that σ is a Cartan involution of $G = \bar{G}_\mathbb{R}$. Then we have the following:*

(i) *the fine structure of the Riemannian symmetric space G/H can be identified with that of the corresponding symmetric variety $\bar{G}/\bar{H}$.*
(ii) *the fine structure of the Riemannian symmetric space G/H can be identified with that of the real group G.*

Proof. (i): Let $A \subset Q$ be a maximal $(\sigma, \mathbb{R})$-split torus, $A_1 \supset A$ a maximal σ-split $\mathbb{R}$-torus, $A_2 \supset A$ a maximal $\mathbb{R}$-split torus of G and T a maximal $\mathbb{R}$-torus T of G which contains A, A_1 and A_2. By Theorem 1 $\Phi_0 = \Phi(A_1) = \Phi(A_2, G_\mathbb{R}) = \Phi(A, G_\mathbb{R})$ and since A is $\mathbb{R}$-split we have $= \Phi(A, G_\mathbb{R}) = \Phi(A, G)$, what proves (i).

(ii) is well known. For a proof see [Hel78]. □

3. Preliminaries and Recollections

In the previous section we showed that the fine structure of the real Riemannian symmetric spaces is the same as that of the symmetric varieties over algebraically closed fields. In the remainder of this paper we will determine algorithms to compute their fine structure. For this it suffices to consider the local symmetric spaces.

3.1. **Symmetric spaces.** For the remainder of this paper G denotes a reductive algebraic group over an arbitrary field k of characteristic not 2, $\theta \in \mathrm{Aut}(G)$ is an involution, i.e. $\theta^2 = \mathrm{id}$ and write $K = \{A \in G \mid \theta(A) = A\}$ the fixed point group of θ and $P = \{A\theta(A)^{-1} \mid A \in G\}$. The variety P is called a *symmetric variety* or also a *reductive symmetric space*. If G is semisimple, P is also called a *semisimple symmetric space*.

Let $\mathfrak{g}$ denote the Lie algebra of G and denote the involution of $\mathfrak{g}$ induced by θ also by θ. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ is the Lie algebra of K and $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$ is the tangent space in the identity of P.

3.2. **Root space decomposition.** If V is a vector space, then we denote its dual by V^*. Let $\mathfrak{h}$ be a toral subalgebra of the Lie algebra $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ and let $\Phi(\mathfrak{h}) = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$. The elements of $\Phi(\mathfrak{h})$ are called roots and the subspaces $\mathfrak{g}_\alpha$ are called root-subspaces. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi(\mathfrak{h})} \mathfrak{g}_\alpha.$$

Here $\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{h})$ is the root space for $\alpha = 0$, which is exactly the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. For $\mathfrak{t}$ a maximal toral subalgebra in $\mathfrak{g}$ we have $\mathfrak{t} = \mathfrak{g}_0$ and $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi(\mathfrak{t})$.

Remark 1. For some toral subalgebras $\mathfrak{h}$ the set of roots $\Phi(\mathfrak{h})$ is a (reduced) root system in the sense of [Bou81]. Examples of this are $\mathfrak{h} = \mathfrak{t}$ a maximal total subalgebra and $\mathfrak{h} = \mathfrak{a} \subset \mathfrak{p}$ a maximal toral subalgebra of $\mathfrak{p}$ (see [Ric82]).

3.3. **Root space decomposition for a local Symmetric space.** In the following we recall a number of results from Richardson [Ric82] and Helminck [Hel88], which we will need for our analyses of the fine structure of the symmetric space. In the following let $\mathfrak{a}$ be a maximal toral subalgebra in $\mathfrak{p}$. We use the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi(\mathfrak{a})} \mathfrak{g}_\lambda$$

with respect to $\mathfrak{a}$ to find a basis for $\mathfrak{k}$ and $\mathfrak{p}$. Here $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x \text{ for all } t \in \mathfrak{a}\}$ and $\Phi(\mathfrak{a}) = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0 \text{ and } \mathfrak{g}_\lambda \neq 0\}$. We also have the root space decomposition of a maximal toral subalgebra containing $\mathfrak{a}$, which satisfies:

Lemma 1 ([Ric82]). *Let $\mathfrak{t} \supseteq \mathfrak{a}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then we have the following:*

(1) *$\mathfrak{t}$ is θ-stable.*
(2) *$\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$, where $\mathfrak{t}_\pm = \{x \in \mathfrak{t} \mid \theta(x) = \pm x\}$.*
(3) *$\mathfrak{t}_- = \mathfrak{a}$ and $\mathfrak{t}_+ \subset \mathfrak{k}$*

Notation 1. To avoid confusion between roots in $\Phi(\mathfrak{t})$ and roots in $\Phi(\mathfrak{a})$, we will reserve α for roots in $\Phi(\mathfrak{t})$ and λ for roots in $\Phi(\mathfrak{a})$.

3.3.1. *Nice basis.* Let $R(\mathfrak{t}) = \mathbb{Z}_{span}\{\Phi(\mathfrak{t})\}$ be the root lattice of $\Phi(\mathfrak{t})$ and $E \subset \mathfrak{t}^*$ be the Euclidean space spanned by $\Phi(\mathfrak{t})$, i.e. $E = R(\mathfrak{t}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ decompose $\mathfrak{g}$ orthogonally, we are looking at the projection of E onto $\mathfrak{a}^*$. The next natural question is how are the roots in $\Phi(\mathfrak{t})$ and the roots in $\Phi(\mathfrak{a})$ related?

Lemma 2. *Let $\alpha \in \Phi(\mathfrak{t})$ and $\lambda = \alpha|\mathfrak{a}$. Then $\lambda \in \Phi(\mathfrak{a}) \cup \{0\}$ and $\mathfrak{g}_\alpha \subset \mathfrak{g}_\lambda$.*

Let $\Delta(\mathfrak{t})$ be a basis for $\Phi(\mathfrak{t})$ and $\Delta(\mathfrak{a})$ be a basis for $\Phi(\mathfrak{a})$. For $\lambda_i \in \Phi(\mathfrak{a})$, we have that $\mathfrak{g}_{\lambda_i} = \mathfrak{g}_{\alpha_{i_1}} \oplus \mathfrak{g}_{\alpha_{i_2}} \oplus \ldots \oplus \mathfrak{g}_{\alpha_{i_r}}$. We need to find all roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}$ in $\Phi(\mathfrak{t})$ with $\alpha_{i_j}|\mathfrak{a} = \lambda_i$ and all roots that project to zero. Ideally we would like the natural projection map $\pi : \mathfrak{g}^* \to \mathfrak{p}^*$, defined by $\pi(\alpha) = \frac{1}{2}(\alpha - \theta^*(\alpha))$ to satisfy $\pi(\Delta(\mathfrak{t})) = \Delta(\mathfrak{a})$. This will only be the case with a special choice of basis for $\Phi(\mathfrak{t})$ (see 3.3.3).

Lemma 3. *If $\alpha \in \Phi(\mathfrak{t})$ with $\alpha|\mathfrak{a} = 0$, then $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$.*

Now denote $\Phi_0 = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|\mathfrak{a} = 0\}$ and $\mathfrak{m} = \mathfrak{t}_+ \oplus \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha$.

Lemma 4 ([Ric82]). *Let $\mathfrak{t} \supseteq \mathfrak{a}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then we have the following:*

(i) *Φ_0 is a closed subset of $\Phi(\mathfrak{t})$, i.e., Φ_0 is a root system.*
(ii) $\mathfrak{g}_0 = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha = \mathfrak{a} \oplus \mathfrak{t}_+ \oplus \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha = \mathfrak{a} \oplus \mathfrak{m}$.
(iii) $\mathfrak{m} \subset \mathfrak{k}$.

3.3.2. *Root systems and Weyl groups.* $\theta|\mathfrak{t}$ induces an involution on $\mathfrak{t}^*$ and hence on $\Phi(\mathfrak{t})$. By abuse of notation, we will denote the restricted involution also by θ. Conversely an involution $\tilde{\theta}$ of $(R(\mathfrak{t}), \Phi(\mathfrak{t}))$ will be called *admissible* if there is an involution θ of $(\mathfrak{g}, \mathfrak{t})$ such that θ induces $\tilde{\theta}$ on $(R(\mathfrak{t}), \Phi(\mathfrak{t}))$ and such that $\mathfrak{t}_\theta^- = \{X \in \mathfrak{t} \mid \theta(X) = -X\}$ is a maximal toral subalgebra contained in $\mathfrak{p}$.

Lemma 5 ([Hel88]). *Let $X_0(\theta) = \{\chi \in R(\mathfrak{t}) \mid \theta(\chi) = \chi\}$ and $\Phi_0(\theta) = \Phi(\mathfrak{t}) \cap X_0(\theta)$. Then*

(1) *$X_0(\theta)$ and $\Phi_0(\theta)$ are θ-stable.*
(2) *$\Phi_0(\theta)$ is a closed subsystem of $\Phi(\mathfrak{t})$.*
(3) $\Phi_0(\theta) = \Phi_0$.

Definition 1. For a subset $S \subset \Phi(\mathfrak{t})$, we let $W(S)$ denote the subgroup of $W(\Phi(\mathfrak{t}))$ generated by the reflections s_α with $\alpha \in S$.

Identify $W_0(\theta)$ with the subgroup $W(\Phi_0(\theta))$ of $W(\Phi(\mathfrak{t}))$. Let

$$W_1(\theta) = \{w \in W(\Phi(\mathfrak{t})) \mid w(X_0(\theta)) = X_0(\theta)\}.$$

Let $\overline{\Phi} = \pi(\Phi(\mathfrak{t}) - \Phi_0(\theta))$ denote the set of *restricted roots* of $\Phi(\mathfrak{t})$ relative to θ.

Denote $X_\theta = R(\mathfrak{t})/X_0(\theta)$ and define the projection: $\pi : X \to X_\theta$ by $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$. All $w \in W_1(\theta)$ induce a mapping $\pi(w) \in \mathrm{Aut}(X_\theta)$ such that $\pi(w(\chi)) = \pi(w)(\pi(\chi))$. Define $\overline{W} = \{\pi(w) \mid w \in W_1(\theta)\}$.

Notation 2. Let $W(\mathfrak{a})$ denote the Weyl group of the restricted root system $\Phi(\mathfrak{a})$.

Proposition 1 ([Hel88]). *Let $\overline{\Phi}$, $\overline{W}$, etc. be as above. Then*

(1) $\overline{\Phi} = \Phi(\mathfrak{a})$.
(2) $W(\mathfrak{a}) = \overline{W} \cong W_1(\theta)/W_0(\theta)$.

Definition 2. $W(\mathfrak{a})$ is called the *restricted Weyl group* with respect to the action of θ on $R(\mathfrak{t})$.

3.3.3. *θ-ordering.* Define a θ-order on $\Phi(\mathfrak{t})$ related to the action θ by

$$\text{if } \chi \in R(\mathfrak{t}), \chi > 0, \text{ and } \chi \notin X_0(\theta), \text{ then } \theta(\chi) < 0.$$

Remark 2. A basis $\Delta(\mathfrak{t})$ for $\Phi(\mathfrak{t})$ with respect to the θ-order will be called a θ-basis of $\Phi(\mathfrak{t})$.

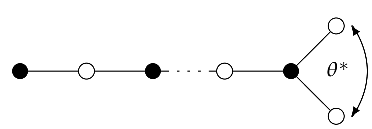
Proposition 2. $\pi(\Delta(\mathfrak{t}) - \Delta_0(\mathfrak{t})) = \Delta(\mathfrak{a})$, *where $\Delta(\mathfrak{a})$ is a basis for $\Phi(\mathfrak{a})$.*

Proposition 3 ([Hel88]). *Let Δ be a θ-basis of Φ, $w_0(\theta) \in W_0(\theta)$ the involution such that $w_0(\theta)(\Delta_0(\mathfrak{t})) = -\Delta_0(\mathfrak{t})$ and $\theta^* = -\mathrm{id} \circ \theta \circ w_0(\theta)$. Then*

(i) $w_0(\theta)$, θ^* *and* $-\mathrm{id}$ *commute.*
(ii) $\theta^* = \begin{cases} \mathrm{id} \\ \textit{Dynkin diagram automorphism of order 2} \end{cases}$

3.4. **θ-diagram.** If $\theta \in \mathrm{Aut}(R(\mathfrak{t}), \Phi)$ is an involution and Δ a θ-basis of Φ, then θ is determined by the quadruple $(R(\mathfrak{t}), \Delta, \Delta_0(\theta), \theta^*)$, because $\theta = -\theta^* w_0(\theta)$. We call such a quadruple $(R(\mathfrak{t}), \Delta, \Delta_0(\theta), \theta^*)$ a θ-diagram.

As in [Hel88] we make a diagrammatic representation of the θ-diagram of θ by coloring black those vertices of the ordinary Dynkin diagram of θ, which represent roots in $\Delta_0(\theta)$ and indicating the action of θ^* on $\Delta - \Delta_0(\theta)$ by arrows. We omit the action of θ^* on $\Delta_0(\theta)$ because $\theta^*|\Delta_0(\theta) = -w_0(\theta)$ is completely determined by the type of $\Phi_0(\theta)$. An example in type D_l is:



Lemma 6. *The involution of $(R(\mathfrak{t}), \Phi)$ can be recovered from the θ-diagram by*

$$\theta = -\mathrm{id} \circ \theta^* \circ w_0(\theta).$$

To use these θ-diagrams in the characterization of isomorphy classes of involutions, we need a notion of isomorphism between these diagrams. Two θ-diagrams $(X, \Delta, \Delta_0(\theta_1), \theta_1^*)$ and $(X, \Delta', \Delta_0'(\theta_2), \theta_2^*)$ are said to be *isomorphic* if there is a $w \in W(\Phi)$, which maps $(\Delta, \Delta_0(\theta_1))$ onto $(\Delta', \Delta_0'(\theta_2))$ and which satisfies $w\theta_1^*(\Delta)w^{-1} = \theta_2^*(\Delta')$.

Theorem 2 ([Hel88]). *Assume that $\mathfrak{g}$ is semisimple and $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$. Then there is a bijection between the set of isomorphy classes of involutorial automorphisms of $\mathfrak{g}$ and the isomorphism classes of θ-diagrams of admissible involutions of $(R(\mathfrak{t}), \Phi(\mathfrak{t}))$.*

A classification of involutions, reductive algebraic groups, and their associated Lie algebras using these θ-diagrams is given in [Hel88]. For the simple Lie algebras we list in Table 1 below the type of the Lie algebra and its involution, the θ-diagram, and the Dynkin diagram of the resulting root system $\Phi(\mathfrak{a})$. These diagrams, called absolutely irreducible θ-diagrams, will play a fundamental role in the algorithms to compute the structure of these symmetric spaces.

Table 1: θ-diagram

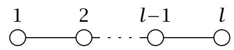
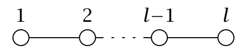
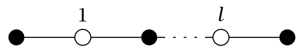
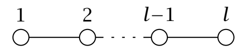
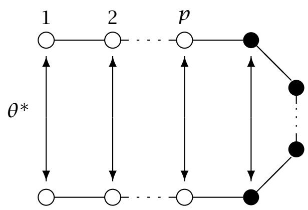
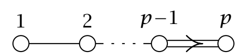
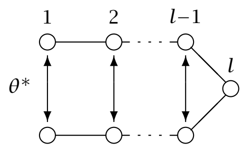
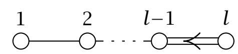
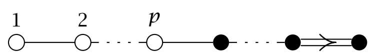
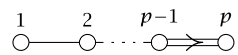
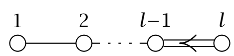
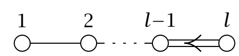
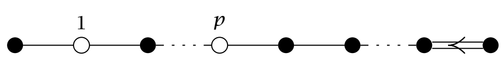
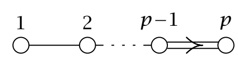
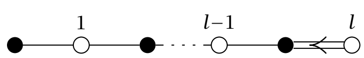
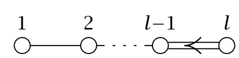
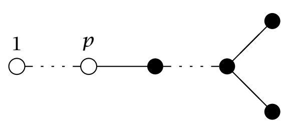
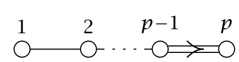
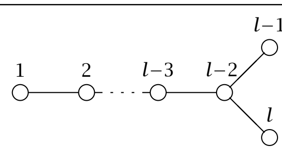
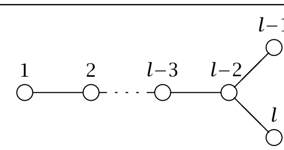
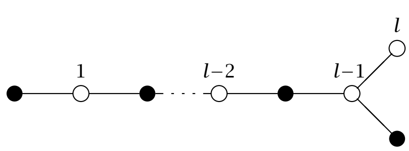
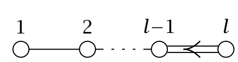
Type θ	θ-diagram	$\Delta(\mathfrak{a})$
AI		
AII		
AIII$_a$ (AIV ($p = 1$)) ($1 \leq 2p \leq l$)		
AIII$_b$ ($l \geq 2$)		
BI (BII ($p = 1$)) ($l \geq 2,\ 1 \leq p \leq l$)		
CI		
CII$_a$ ($l \geq 3$) ($1 \leq p \leq \frac{1}{2}(l-1)$)		
CII$_b$ ($l \geq 2$)		
DI$_a$ (DII ($p = 1$)) ($l \geq 4,\ 1 \leq p \leq l-1$)		
DI$_b$ ($l \geq 4$)		
DIII$_a$ ($l \geq 2$)		

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Table 1: *continued*

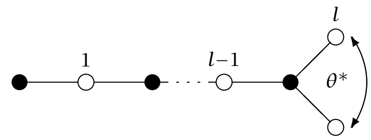
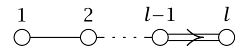
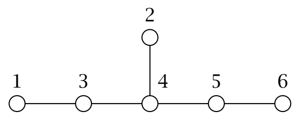
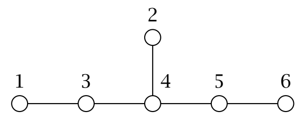
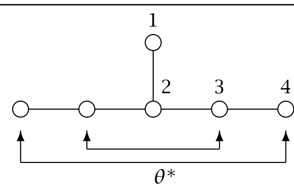
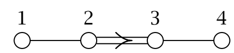
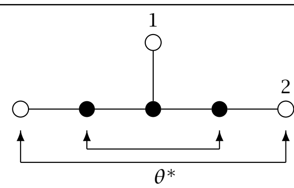
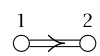
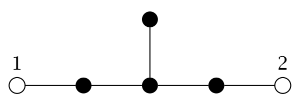
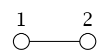
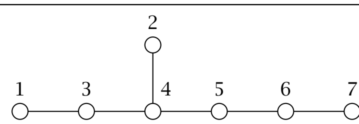
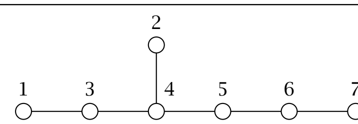
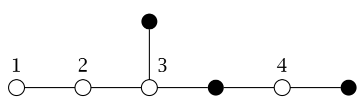
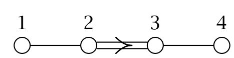
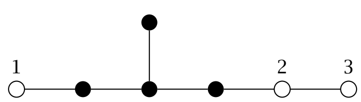
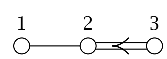
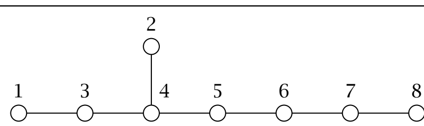
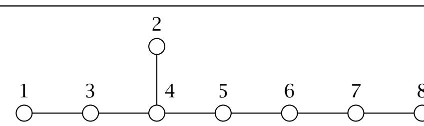
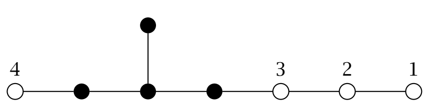
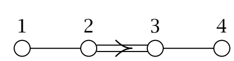
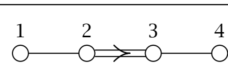
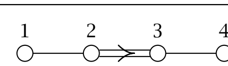
Type θ	θ-diagram	$\Delta(\mathfrak{a})$
$D\mathrm{III}_b$ ($l \geq 2$)		
$E\mathrm{I}$		
$E\mathrm{II}$		
$E\mathrm{III}$		
$E\mathrm{IV}$		
$E\mathrm{V}$		
$E\mathrm{VI}$		
$E\mathrm{VII}$		
$E\mathrm{VIII}$		
$E\mathrm{IX}$		
$F\mathrm{I}$		

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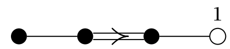
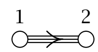
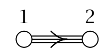
Table 1: *continued*

Type θ	θ-diagram	$\Delta(\mathfrak{a})$
FII		
G		

4. Algorithms

After recovering the action of θ on $\Phi(\mathfrak{t})$ and explicitly determining each λ, there is a beautiful seven step algorithm that we can follow to compute the fine structure of the symmetric varieties and also determine a “nice" basis for the local symmetric space $\mathfrak{p}$. We first need some more notation:

Definition 3. For all λ in $\Phi(\mathfrak{a})$, we define

$$\Phi(\lambda) = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda\}.$$

Main Algorithm.
Step 1. For each absolutely irreducible θ-diagram as given in Table 1, recover the action of the involution on the original root system.

Step 2. Compute $\Phi(\lambda)$ for all λ in $\Delta(\mathfrak{a})$. Essentially we are computing all roots $\alpha \in \Phi(\mathfrak{t})$ that project down to each λ. This also gives us the multiplicity of λ.

Step 3. Recall from Proposition 1 that

$$W(\mathfrak{a}) \simeq W_1(\theta)/W_0(\theta).$$

It becomes advantageous to compute representatives w_i in $W_1(\theta)$ for each s_{λ_i} in $W(\mathfrak{a})$.

Step 4. Compute the list of all positive roots in $\Phi(\mathfrak{a})$ by using the Weyl group representatives w_i applied to all λ in $\Delta(\mathfrak{a})$.

Step 5. The next step is to compute $\Phi(\lambda)$ and the multiplicity of all the roots λ in $\Phi(\mathfrak{a})$. Here we use the fact that

$$\Phi(s_{\lambda_i}(\lambda_j)) = \Phi(\lambda_j - \langle \alpha_j, \alpha_i \rangle \lambda_i) = w_i(\Phi(\lambda_j)).$$

Essentially, we use the same Weyl group representatives that we used in Step 4 to compute the list $\Phi(\mathfrak{a})^+$, but this time we apply them set-wise to the $\Phi(\lambda_i)$ from Step 2.

The above algorithm computes the fine structure of the symmetric space. In the remainder of this paper we work out these steps of the algorithm in detail. Using this fine structure we also obtain an algorithm to compute nice bases for $\mathfrak{k}$ and $\mathfrak{p}$ as follows:

Step 6. We have that $\mathfrak{g}_\lambda = \mathfrak{g}_{\alpha_{i_1}} \oplus \mathfrak{g}_{\alpha_{i_2}} \oplus \ldots \oplus \mathfrak{g}_{\alpha_{i_r}}$. All we need to do to compute each $\mathfrak{g}_\lambda$ is to note that each element in $\Phi(\lambda)$ will be in the direct sum decomposition, i.e. $\mathfrak{g}_\lambda = \sum_{\alpha \in \Phi(\lambda)} \mathfrak{g}_\alpha$.

Step 7. Since $x_\alpha \in \mathfrak{g}_\alpha$ implies $x_\alpha + \theta(x_\alpha) \in \mathfrak{k}$ and $x_\alpha - \theta(x_\alpha) \in \mathfrak{p}$ we have:

$$\text{a basis for } \mathfrak{k} = \{x_\alpha^i + \theta(x_\alpha^i)\}_{\alpha \in \Phi(\mathfrak{a})^+} + \{x_\alpha^i\}_{\alpha \in \Phi_0(\theta)} + \text{a basis for } \mathfrak{t}_0 \text{ and}$$
$$\text{a basis for } \mathfrak{p} = \{x_\alpha^i - \theta(x_\alpha^i)\}_{\alpha \in \Phi(\mathfrak{a})^+} + \text{a basis for } \mathfrak{a}.$$

Remark 3. These last 2 steps provide an easy algorithm to compute a basis for $\mathfrak{p}$ and $\mathfrak{k}$ in terms of $x_\alpha \in \mathfrak{g}_\alpha$ and $\theta(x_\alpha) \in \mathfrak{g}_{\theta(\alpha)}$. It does not compute the *structure constants for* θ, which are the set of $c_{\alpha,\theta} \in k$ such that $\theta(x_\alpha) = c_{\alpha,\theta} x_{\theta(\alpha)}$. As in [Hel88] one can prove that the vectors $x_\alpha \in \mathfrak{g}_\alpha$ can be normalized such that $c_{\alpha,\theta} = \pm 1$. The algorithm does not compute these structure constants for θ, since they do not only depend on the fine structure, but also on the involution and the Lie algebra. We hope to give an algorithm to compute these structure constants for θ in a future paper.

5. Computing the action of θ on $\Phi(\mathfrak{t})$

Before the restricted root system, $\Phi(\mathfrak{a})$, can be realized there are still some preliminary steps that must be taken. First, the longest Weyl group element $w_0(\theta)$ must be computed and consequently the action of the involution recovered. Then using the involution and the action of the natural projection map, $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$, the roots in the basis, $\Delta(\mathfrak{a})$, can explicitly be determined.

5.1. The involution $w_0(\theta)$ can easily be derived from the θ-diagram as follows.

Algorithm 1. (i) Write $\Phi_0(\theta) = \Phi_1 \cup \ldots \Phi_r$ as a union of irreducible components. Bases of the irreducible components Φ_i are those subsets Δ_i of roots of $\Delta_0(\theta)$ which correspond to connected components of the black dots in the θ-diagram.

(ii) Let $w_i \in W(\Phi_i)$ be the longest involution of $W(\Phi_i)$ with respect to the basis Δ_i. Identify the group $W(\Phi_i)$ with the canonical subgroup of $W(\Phi(\mathfrak{t}))$. Then $w_0(\theta) = w_1 \ldots w_r$.

Note that the involutions $w_1, \ldots, w_r$ commute, since they correspond to different irreducible components of $\Phi_0(\theta)$. So the involution $w_0(\theta)$ is independent of the order in which the involutions w_i are computed and multiplied. The involutions $w_0(\theta)$ derived from the absolutely irreducible θ-diagrams in Table 1 are listed in Table 2.

Table 2: $w_0(\theta)$

Type θ	$w_0(\theta)$
AI	id
AII	$s_{\alpha_1}s_{\alpha_3}\ldots s_{\alpha_{2l+1}}$
AIII$_a$	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}}$
AIII$_b$	id
BI	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m-1}}s_{\alpha_{p+m}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m}} \cdots s_{\alpha_{p+3}}s_{\alpha_{p+2}} \cdots s_{\alpha_{p+m-2}}s_{\alpha_{p+m-1}}s_{\alpha_{p+m}} s_{\alpha_{p+m-1}}s_{\alpha_{p+m-2}}s_{\alpha_{p+m-1}}s_{\alpha_{p+m}}s_{\alpha_{p+m-1}}s_{\alpha_{p+m}}$
CI	id
CII$_a$	$s_{\alpha_1}s_{\alpha_3} \cdots s_{\alpha_{2p-1}}s_{\alpha_{2p+1}}s_{\alpha_{2p+2}}s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m}}s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+3}}s_{\alpha_{2p+2}} s_{\alpha_{2p+1}}s_{\alpha_{2p+2}}s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m}} \cdots s_{\alpha_{2p+3}} s_{\alpha_{2p+2}} \cdots s_{\alpha_{2p+m-2}}s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m-2}}s_{\alpha_{2p+m}}s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m}}$
CII$_b$	$s_{\alpha_1}s_{\alpha_3}\ldots s_{\alpha_{2l-3}}s_{\alpha_{2l-1}}$
DI$_a$	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+4}}s_{\alpha_{p+3}} s_{\alpha_{p+2}}s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+2}} s_{\alpha_{p+m}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+3}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+4}} s_{\alpha_{p+m}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+5}} \cdots$
DIII$_a$	$s_{\alpha_1}s_{\alpha_3}\ldots s_{\alpha_{2l-1}}$
DIII$_b$	$s_{\alpha_1}s_{\alpha_3}\ldots s_{\alpha_{2l-1}}$
EI	id
EII	id
EIII	$s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}$
EIV	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$
EV	id
EVI	$s_{\alpha_2}s_{\alpha_5}s_{\alpha_7}$

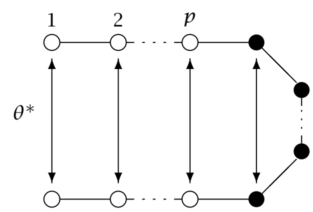
continued on next page

Table 2: *continued*

Type θ	$w_0(\theta)$
EVII	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$
EVIII	id
EIX	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$
FI	id
FII	$s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}$
G	id

5.2. **Roots of $\Delta(\mathfrak{a})$ in terms of roots in $\Delta(\mathfrak{t})$.** The roots in $\Delta(\mathfrak{a})$ are projections of the roots in $\Delta(\mathfrak{t})$ and hence can be expressed as a linear combination of the roots in $\Delta(\mathfrak{t})$. These projections can be computed from the θ-diagram as well as is illustrated in the following example:

Example 1. Type AIII$_a$



Here we start with the Lie algebra $\mathfrak{g} = A_{2p+m}$. The root α_{p+i} is fixed for $i = 1, \cdots, m$. Therefore,

$$w_0(\theta) = s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}}$$

and $\theta = -w_0(\theta)\theta^*$.

For λ_i in $\Delta(\mathfrak{a})$ with $i = 1, \cdots, p-1$, notice that there are two base roots that project down to λ_i. Basically we find that

$$\pi(\alpha_i) = \pi(\alpha_{2p+m+1-i}) = \lambda_i \text{ where } \lambda_i = \frac{1}{2}(\alpha_i + \alpha_{2p+m+1-i}) \text{ for } i = 1, \cdots, p-1 \text{ and}$$

$$\pi(\alpha_p) = \pi(\alpha_{p+m+1}) = \lambda_p = \frac{1}{2}(\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m} + \alpha_{p+m+1}).$$

5.3. The above example illustrates the following general procedure for computing the roots of $\Delta(\mathfrak{a})$ as the projections of the roots in $\Delta(\mathfrak{t}) - \Delta_0(\theta)$:

Algorithm 2. (i) If the white dot in the θ-diagram, representing $\alpha \in \Delta(\mathfrak{t}) - \Delta_0(\theta)$ is not adjacent to a black dot, then $\lambda = \pi(\alpha) = \frac{1}{2}(\alpha + \theta^*(\alpha)) \in \Delta(\mathfrak{a})$.

(ii) If the white dot in the θ-diagram, representing $\alpha \in \Delta(\mathfrak{t}) - \Delta_0(\theta)$ is adjacent to a black dot, then $\lambda = \pi(\alpha) = \frac{1}{2}(\alpha - w_0(\theta)\theta^*(\alpha)) \in \Delta(\mathfrak{a})$, where $w_0(\theta)$ is as in table 2.

Since a white dot in the θ-diagram is adjacent to at most 2 black dots, i.e. 2 irreducible components Φ_1 resp. Φ_2 of $\Phi_0(\theta)$, it suffices for this second step to compute $\pi(\alpha) = \frac{1}{2}(\alpha - w_1w_2\theta^*(\alpha))$ where w_1 resp. w_2 are the longest involutions in $W(\Phi_1)$ resp. $W(\Phi_2)$ with respect to the bases $\Phi_1 \cap \Delta_0(\theta)$ resp. $\Phi_2 \cap \Delta_0(\theta)$.

Using this procedure we obtain the expression of the roots of $\Delta(\mathfrak{a})$ in terms of roots in $\Delta(\mathfrak{t})$ for each of the θ-diagrams in Table 1. The results are given in Table 3 below.

Table 3: Basis of $\Phi(\mathfrak{a})$ in terms of basis of $\Phi(\mathfrak{t})$

Type θ	λ
AI	$\lambda_i = \alpha_i \ \forall i$
AII	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) \ \forall i$

continued on next page

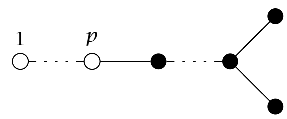
Table 3: *continued*

Type θ	λ
$A\mathrm{III}_a$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{l+1-i})$ for $i = 1, \cdots, p-1$ $\lambda_p = \frac{1}{2}(\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m} + \alpha_{p+m+1})$
$A\mathrm{III}_b$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{l+1-i})$ for $i = 1, \cdots, l-1$ $\lambda_l = \alpha_l$
$B\mathrm{I}$	$\lambda_i = \alpha_i$ for $i = 1, \cdots, p-1$ $\lambda_p = \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1} + \alpha_{p+m}$
$C\mathrm{I}$	$\lambda_i = \alpha_i \ \forall i$
$C\mathrm{II}_a$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, p-1$ $\lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m})$
$C\mathrm{II}_b$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, l-1$ $\lambda_l = \alpha_{2l-1} + \alpha_{2l}$
$D\mathrm{I}_a$	$\lambda_i = \alpha_i$ for $i = 1 \cdots p-1$ $\lambda_p = \frac{1}{2}(2\alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m})$
$D\mathrm{I}_b$	$\lambda_i = \alpha_i \ \forall i$
$D\mathrm{III}_a$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, l-1$ $\lambda_l = \alpha_{2l}$
$D\mathrm{III}_b$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, l-1$ $\lambda_l = \frac{1}{2}(\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1})$
$E\mathrm{I}$	$\lambda_i = \alpha_i \ \forall i$
$E\mathrm{II}$	$\lambda_1 = \alpha_2$ $\lambda_2 = \alpha_4$ $\lambda_3 = \frac{1}{2}(\alpha_3 + \alpha_5)$ $\lambda_4 = \frac{1}{2}(\alpha_1 + \alpha_6)$
$E\mathrm{III}$	$\lambda_1 = \frac{1}{2}(2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$ $\lambda_2 = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
$E\mathrm{IV}$	$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$ $\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
$E\mathrm{V}$	$\lambda_i = \alpha_i \ \forall i$
$E\mathrm{VI}$	$\lambda_1 = \alpha_1$ $\lambda_2 = \alpha_3$ $\lambda_3 = \frac{1}{2}(\alpha_2 + 2\alpha_4 + \alpha_5)$ $\lambda_4 = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7)$
$E\mathrm{VII}$	$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$ $\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$ $\lambda_3 = \alpha_7$
$E\mathrm{VIII}$	$\lambda_i = \alpha_i \ \forall i$
$E\mathrm{IX}$	$\lambda_1 = \alpha_8$ $\lambda_2 = \alpha_7$ $\lambda_3 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$ $\lambda_4 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
$F\mathrm{I}$	$\lambda_i = \alpha_i \ \forall i$
$F\mathrm{II}$	$\lambda_1 = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$
G	$\lambda_i = \alpha_i \ \forall i$

6. Computing $\Phi(\lambda)$ for $\lambda \in \Delta(\mathfrak{a})$

Now that we have explicitly computed each λ in $\Delta(\mathfrak{a})$ we will explore the structure of $\Phi(\mathfrak{a})$. We start by computing all the roots that project down to a root λ in $\Delta(\mathfrak{a})$. We denoted the set of these roots by $\Phi(\lambda)$. In finding this set, we are also computing the multiplicity of λ. We first illustrate the procedure with an example:

Example 2. Type DI_a



Here the multiplicity of each λ_i is

$$m_{\lambda_i} = 1 \quad \text{for } i = 1, \cdots, p-1$$
$$m_{\lambda_p} = 2m$$

We have that

$$\Phi(\lambda_i) = \{\alpha_i\} \text{ for } i = 1, \cdots, p-1 \text{ and}$$

$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$.

6.1. The above example illustrates the following general procedure for computing the set $\Phi(\lambda)$ for $\lambda \in \Delta(\mathfrak{a})$. Write $\Phi_0(\theta) = \Phi_1 \cup \ldots \Phi_r$ as a union of irreducible components.

Algorithm 3.
- (i) If $\lambda = \pi(\alpha)$ where α corresponds to a white dot in the θ-diagram, which is not adjacent to a black dot, then
  - (1) $\Phi(\lambda) = \{\alpha, \theta^*(\alpha)\}$ if $\theta^*(\alpha) \neq \alpha$.
  - (2) $\Phi(\lambda) = \{\alpha\}$ if $\theta^*(\alpha) = \alpha$.
- (ii) If $\lambda = \pi(\alpha)$ where α corresponds to a white dot in the θ-diagram, which is adjacent to one black dot which corresponds to a root in $\Phi_i \subset \Phi_0(\theta)$, then $\Phi(\lambda) = \{\alpha + \beta \in \Phi(\mathfrak{t}) \mid \beta \in \Phi_i^+, \langle \alpha, \beta \rangle \geq 0\}$.
- (iii) If $\lambda_i = \pi(\alpha)$ where α corresponds to a white dot in the θ-diagram, which is adjacent to two black dots which correspond to a root in Φ_i and Φ_j, then $\Phi(\lambda) = \{\alpha + \beta + \gamma \in \Phi(\mathfrak{t}) \mid \beta \in \Phi_i^+, \langle \alpha, \beta \rangle \geq 0, \gamma \in \Phi_j^+, \langle \alpha, \gamma \rangle \geq 0\}$.

Since the computation of the Cartan integers $\langle \alpha, \beta \rangle$ is a standard procedure the above gives an easily algorithm to compute the sets $\Phi(\lambda)$ for the simple roots $\lambda \in \Delta(\mathfrak{a})$. The results are given in Table 4 below.

Table 4: $\Phi(\lambda_i)$

Type θ	$\Phi(\lambda_i)$	m_{λ_i}
AI	$\Phi(\lambda_i) = \{\alpha_i\}$	1
AII	$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i}\}$	4
AIII_a	$\Phi(\lambda_i) = \{\alpha_i, \alpha_{l+1-i}\}$ for $i = 1, \cdots, p-1$	2
	$\Phi(\lambda_p) = \{\alpha_p, \alpha_{p+m+1}, \alpha_p + \alpha_{p+1}, \alpha_{p+m} + \alpha_{p+m+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m}, \alpha_{p+1} + \cdots + \alpha_{p+m+1}\}$	2(m+1)
	$\Phi(2\lambda_p) = \{\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1}\}$	1
AIII_b	$\Phi(\lambda_i) = \{\alpha_i, \alpha_{2l-i}\}$ for $i = 1, \cdots, l-1$	2
	$\Phi(\lambda_l) = \{\alpha_l\}$	1
BI	$\Phi(\lambda_i) = \{\alpha_i\}$ for $i = 1, \cdots, p-1$	1
	$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \cdots + \alpha_{p+m}, \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$	2m+1

continued on next page

Table 4: *continued*

Type θ	$\Phi(\lambda_i)$	m_{λ_i}
CI	$\Phi(\lambda_i) = \{\alpha_i\}$	1
CII$_a$	$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, p-1$	4
	$\Phi(\lambda_p) = \{\alpha_{2p}, \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-1} + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$	4m
	$\Phi(2\lambda_p) = \{2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$	3
CII$_b$	$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, l-1$	4
	$\Phi(\lambda_l) = \{\alpha_{2l}, \alpha_{2l-1} + \alpha_{2l}, 2\alpha_{2l-1} + \alpha_{2l}\}$	3
DI$_a$	$\Phi(\lambda_i) = \{\alpha_i\}$ for $i = 1, \cdots, p-1$	1
	$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$	2m
DI$_b$	$\Phi(\lambda_i) = \{\alpha_i\}$	1
DIII$_a$	$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, l-1$	4
	$\Phi(\lambda_l) = \{\alpha_{2l}\}$	1
DIII$_b$	$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, l-1$	4
	$\Phi(\lambda_l) = \{\alpha_{2l}, \alpha_{2l+1}, \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-1} + \alpha_{2l+1}\}$	4
	$\Phi(2\lambda_l) = \{\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}\}$	1
EI	$\Phi(\lambda_i) = \{\alpha_i\}$	1
EII	$\Phi(\lambda_1) = \{\alpha_2\}$	1
	$\Phi(\lambda_2) = \{\alpha_4\}$	1
	$\Phi(\lambda_3) = \{\alpha_3, \alpha_5\}$	2
	$\Phi(\lambda_4) = \{\alpha_1, \alpha_6\}$	2
EIII	$\Phi(\lambda_1) = \{\alpha_2, \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$	6
	$\Phi(\lambda_2) = \{\alpha_1, \alpha_6, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$	8
	$\Phi(2\lambda_2) = \{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$	1
EIV	$\Phi(\lambda_1) = \{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5,\}$	8
	$\Phi(\lambda_2) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,\}$	8
EV	$\Phi(\lambda_i) = \{\alpha_i\}$	1
EVI	$\Phi(\lambda_1) = \{\alpha_1\}$	1
	$\Phi(\lambda_2) = \{\alpha_3\}$	1
	$\Phi(\lambda_3) = \{\alpha_4, \alpha_2 + \alpha_4, \alpha_4 + \alpha_5, \alpha_2 + \alpha_4 + \alpha_5\}$	4
	$\Phi(\lambda_4) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7\}$	4
EVII	$\Phi(\lambda_1) = \{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$	8
	$\Phi(\lambda_2) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$	8
	$\Phi(\lambda_3) = \{\alpha_7\}$	1
EVIII	$\Phi(\lambda_i) = \{\alpha_i\}$	1
EIX	$\Phi(\lambda_1) = \{\alpha_8\}$	1
	$\Phi(\lambda_2) = \{\alpha_7\}$	1
	$\Phi(\lambda_3) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$	8

continued on next page

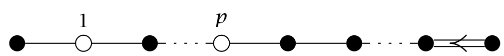
Table 4: *continued*

Type θ	$\Phi(\lambda_i)$	m_{λ_i}
	$\Phi(\lambda_4) = \{\alpha_1, \alpha_1+\alpha_3, \alpha_1+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5\}$	8
FI	$\Phi(\lambda_i) = \{\alpha_i\}$	1
FII	$\Phi(\lambda_1) = \{\alpha_4, \alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4, \alpha_2+2\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+2\alpha_3+\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+\alpha_4, \alpha_1+2\alpha_2+3\alpha_3+\alpha_4\}$	8
	$\Phi(2\lambda_1) = \{\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4, 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\}$	7
G	$\Phi(\lambda_i) = \{\alpha_i\}$	1

7. Computing The Weyl Group

To compute $\Phi(\lambda)$ for all $\lambda \in \Phi(\mathfrak{a})$ we use an element w in the Weyl group $W(\mathfrak{a})$ to map to λ from a root λ_i in the basis $\Delta(\mathfrak{a})$. If $\tilde{w}$ is a Weyl group representative in $W_1(\theta) \subset W(\mathfrak{t})$ for w, then $\Phi(\lambda) = \Phi(w(\lambda_i)) = \tilde{w}(\Phi(\lambda_i))$. So we need to compute representatives in $W_1(\theta) \subset W(\mathfrak{t})$ for the Weyl group elements of $W(\mathfrak{a})$. Since $W(\mathfrak{a})$ is generated by the reflections in the simple roots, it suffices to compute representatives $w_i \in W_1(\theta)$ for each s_{λ_i}, λ_i in $\Delta(\mathfrak{a})$. We give an algorithm to compute these in this section. In the next section we give algorithms how to compute the positive roots from a basis using the Weyl group representatives of the reflections in the simple roots. First an example.

Example 3. Type CII_a



We start with the Lie algebra $\mathfrak{g} = C_{2p+m}$. We have

$$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) \text{ for } i = 1, \cdots, p-1$$

$$\text{and } \lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m})$$

When $i = 1, \cdots, p-1$, the Weyl group representatives are only

$$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}.$$

The representative for λ_p is

$$w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}.$$

7.1. The above example illustrates that computing representatives $w_i \in W_1(\theta)$ for each s_{λ_i} with λ_i in $\Delta(\mathfrak{a})$ will depend strongly on the type of the θ-diagram. Most of the cases can be derived by the following procedure, but a few cases will need to be dealt with separately. Write $\Phi_0(\theta) = \Phi_1 \cup \ldots \Phi_t$ as a union of irreducible components.

Algorithm 4. (i) If $\lambda_i = \pi(\alpha)$ where α corresponds to a white dot in the θ-diagram, which is not adjacent to a black dot, then

(1) $w_i = s_\alpha$ if $\theta^* = \mathrm{id}$.
(2) $w_i = s_\alpha s_{\theta^*(\alpha)}$ if $\theta^* \neq \mathrm{id}$.

(ii) If $\lambda_i = \pi(\alpha)$ where α corresponds to a white dot in the θ-diagram, which is adjacent to two black dots which corresponds to a root β in Φ_r and γ in Φ_s, then

(1) if Φ_r and Φ_s are both of type A_1, then $\theta^* = \mathrm{id}$ and $w_i = s_\alpha s_\beta s_\gamma s_\alpha$.
(2) if Φ_r is of type A_1 and Φ_s is of type C_m, then $\theta^* = \mathrm{id}$ and

$$w_i = s_\alpha s_{\gamma_1} \cdots s_{\gamma_{m-1}} s_{\gamma_m} s_{\gamma_{m-1}} \cdots s_\alpha s_\beta s_\alpha \cdots s_{\gamma_{m-1}} s_{\gamma_m} s_{\gamma_{m-1}} \cdots s_{\gamma_1} s_\alpha,$$

where $\gamma_1, \ldots, \gamma_m$ are an ordered basis for Φ_s with $\gamma_1 = \gamma$.

(iii) If $\lambda = \pi(\alpha)$ where α corresponds to a white dot in the θ-diagram, which is adjacent to one black dot which corresponds to a root β in Φ_r, which has an ordered basis $\beta_1, \dots, \beta_m$, then
  (1) if Φ_r is of type A_m and $\theta^* = \text{id}$, then $m = 1$ and $w_i = s_\alpha s_\beta s_\alpha$.
  (2) if Φ_r is of type A_m, $\theta^*(\alpha) \neq \alpha$, then $w_i = s_{\theta^*(\alpha)} s_{\beta_m} \cdots s_{\beta_1} s_\alpha s_{\beta_1} \cdots s_{\beta_m} s_{\theta^*(\alpha)}$, where $\beta_1, \dots, \beta_m$ are an ordered basis for Φ_r with $\beta_1 = \beta$.
  (3) if Φ_r is of type A_m, $\theta^*(\alpha) = \alpha$ and $\theta^* \neq \text{id}$, then $m = 3$ and $w_i = s_\alpha s_{\beta_2} s_{\beta_3} s_{\beta_1} s_{\beta_2} s_\alpha$, where $\beta_1, \beta_2, \beta_3$ are an ordered basis for Φ_r with $\beta_2 = \beta$.
  (4) if Φ_r is of type B_m and β is a long root, then $w_i = s_{\beta_m} s_{\beta_{m-1}} \cdots s_{\beta_1} s_\alpha s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\beta_m}$, where $\beta_1, \dots, \beta_m$ are an ordered basis for Φ_r with $\beta_1 = \beta$.
  (5) if Φ_r is of type B_m and β is a short root, then $m = 3$ and
$$w_i = s_\alpha s_{\beta_3} s_{\beta_2} s_{\beta_1} s_{\beta_3} s_{\beta_2} s_\alpha s_{\beta_3} s_\alpha s_{\beta_2} s_{\beta_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_\alpha,$$
  where $\beta_1, \beta_2, \beta_3$ are an ordered basis for Φ_r with $\beta_3 = \beta$.
  (6) if Φ_r is of type D_m, then $\theta^*(\alpha) = \alpha$ and $w_i = s_\alpha s_{\beta_1} \cdots s_{\beta_{m-2}} s_{\beta_{m-1}} s_{\beta_m} s_{\beta_{m-2}} \cdots s_{\beta_1} s_\alpha$, where $\beta_1, \dots, \beta_m$ are an ordered basis for Φ_r with $\beta_1 = \beta$.

The results are given in Table 5 below.

Table 5: $W(\mathfrak{a})$

Type θ	s_{λ_i}-representative
AI	$w_i = s_{\alpha_i}\ \forall i$
AII	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}\ \forall i$
AIII$_a$	$w_i = s_{\alpha_i} s_{\alpha_{l+1-i}}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{p+m+1}} s_{\alpha_{p+m}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m+1}}$
AIII$_b$	$w_i = s_{\alpha_i} s_{\alpha_{l+1-i}}$ for $i = 1, \cdots, l-1$ $w_l = s_{\alpha_l}$
BI	$w_i = s_{\alpha_i}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$.
CI	$w_i = s_{\alpha_i}\ \forall i$
CII$_a$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} \cdots$ $s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}$
CII$_b$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, p-1$ $w_l = s_{\alpha_{2l}} s_{\alpha_{2l-1}} s_{\alpha_{2l}}$
DI$_a$	$w_i = \alpha_i$ for $i = 1 \cdots p-1$ $w_p = s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}$
DI$_b$	$w_i = s_{\alpha_i}\ \forall i$
DIII$_a$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, l-1$ $w_l = s_{\alpha_{2l}}$
DIII$_b$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, l-1$ $w_l = s_{\alpha_{2l+1}} s_{\alpha_{2l}} s_{\alpha_{2l-1}} s_{\alpha_{2l}} s_{\alpha_{2l+1}}$
EI	$w_i = s_{\alpha_i}\ \forall i$
EII	$w_1 = s_{\alpha_2}$ $w_2 = s_{\alpha_4}$ $w_3 = s_{\alpha_3} s_{\alpha_5}$ $w_4 = s_{\alpha_1} s_{\alpha_6}$
EIII	$w_1 = s_{\alpha_2} s_{\alpha_4} s_{\alpha_5} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2}$ $w_2 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$
EIV	$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$ $w_2 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}$

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Table 5: *continued*

Type θ	s_{λ_i}-representative
EV	$w_i = s_{\alpha_i}\ \forall i$
EVI	$w_1 = s_{\alpha_1}$ $w_2 = s_{\alpha_3}$ $w_3 = s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}$ $w_4 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_7}s_{\alpha_6}$
EVII	$w_1 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$ $w_2 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $w_3 = s_{\alpha_7}$
EVIII	$w_i = s_{\alpha_i}\ \forall i$
EIX	$w_1 = s_{\alpha_8}$ $w_2 = s_{\alpha_7}$ $w_3 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $w_4 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$
FI	$w_i = s_{\alpha_i}\ \forall i$
FII	$w_1 = s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}$
G	$w_i = s_{\alpha_i}\ \forall i$

Using the above Weyl group representatives for the reflections in the simple roots, one can easily compute a Weyl group representative in $W_1(\theta)$ for any $w \in W(\mathfrak{a})$ using any of the symbolic packages in which Weyl group algorithms have been implemented.

8. Algorithms to compute $\Phi(\mathfrak{a})^+$ and $\Phi(s_{\lambda_i}(\lambda_j))$

To compute the sets $\Phi(\lambda)$ for $\lambda \notin \Delta(\mathfrak{a})$ we need to generate the roots using the Weyl group. It suffices to generate the positive roots, since $\Phi(-\lambda) = -\Phi(\lambda)$. The usual algorithms to compute the positive roots of a root system do not use the Weyl group to compute them. This section contains algorithms for computing both the positive roots of $\Phi(\mathfrak{a})$ and the spaces $\Phi(w(\lambda_j))$. In all of these constructions $|\Delta(\mathfrak{a})| = n$. We note that it suffices to compute $\Phi(w(\lambda))$ for the symmetric varieties for which $\Phi(\mathfrak{a}) \neq \Phi(\mathfrak{t})$ (i.e. the involution θ is not split.) For these only the root systems of type A_n, B_n, C_n, BC_n and F_4. In the following we give algorithms to generate the positive roots in these cases.

8.1. **Type A construction.**

Algorithm 5. A root system of type A_n can be found by letting the weyl group elements $s_\lambda \in W(\mathfrak{a})$ act on the roots in $\Delta(\mathfrak{a})$. Roots of length l can be computed by

$$s_{\lambda_{i+l-1}} \cdots s_{\lambda_{i+1}}(\lambda_i)$$

where $i \in \{1, \cdots, n\}$ and $l \in \{2, \cdots, n+1-i\}$.

8.2. **Type B construction.**

Algorithm 6. The root system of type B_n can be computed in two steps

B.1 Roots of length l with coefficients in $\{0, 1\}$ can be computed by

$$s_{\lambda_{i-l+1}} \cdots s_{\lambda_{i-1}}(\lambda_i)$$

where $i \in \{1, \cdots, n\}$ and $l \in \{2, \cdots, i\}$.

B.2 A root of length 2 with coefficients in $\{0, 1, 2\}$ can be computed by

$$s_{\lambda_n}(\lambda_{n-1}),$$

while roots of length l with k coefficients equal to 2 can be computed by

$$s_{\lambda_{n-k+1}} \cdots s_{\lambda_n} s_{\lambda_{n-l+1}} \cdots s_{\lambda_{n-2}}(\lambda_{n-1}).$$

8.3. **Type C construction.**

Algorithm 7. The root system of type C_n can be computed in three steps

C.1 Roots of length l with coefficients in $\{0,1\}$ can be computed by using the A construction.

C.2 Roots of length l with the leading coefficient equal to 2 can be computed by

$$s_{\lambda_{n-l+1}} \cdots s_{\lambda_{n-1}}(\lambda_n)$$

where $l \in \{2,\cdots,n\}$. Note that roots of length l computed in this way have $l-1$ coefficients equal to 2.

C.3 A root of length 3 with one coefficient equal to 2 can be computed by

$$s_{\lambda_{n-1}} s_{\lambda_n} s_{\lambda_{n-2}}(\lambda_{n-1}),$$

while roots of length l with k coefficients equal to 2 can be computed by

$$s_{\lambda_{n-k}} \cdots s_{\lambda_{n-1}} s_{\lambda_n} s_{\lambda_{n-1}} \cdots s_{\lambda_{n-l+2}}(\lambda_{n-l+1})$$

where $l \in \{4,\cdots,n\}$.

8.4. **Type BC construction.**

Algorithm 8. The root system of type BC_n can be computed in three steps

BC.1 Roots of length l with coefficients in $\{0,1\}$ can be computed by using the B.1 construction.

BC.2 Roots of length l with the k coefficients equal to 2 and $k < l$ can be computed by using the B.2 construction.

BC.3 Roots of length l with all the coefficients equal to 2 can be computed by

$$s_{\lambda_{n-l+1}} \cdots s_{\lambda_{n-1}}(2\lambda_n),$$

where $l \in \{2,\cdots,n\}$.

8.5. **Type F construction.**

Algorithm 9. A root system of type F_4 can be computed in these steps.

Roots of length 2:

$$s_{\lambda_1}(\lambda_2) = \lambda_1 + \lambda_2$$
$$s_{\lambda_2}(\lambda_3) = \lambda_2 + \lambda_3$$
$$s_{\lambda_3}(\lambda_4) = \lambda_3 + \lambda_4$$
$$s_{\lambda_3}(\lambda_2) = \lambda_2 + 2\lambda_3$$

Roots of length 3:

$$s_{\lambda_1} s_{\lambda_2}(\lambda_3) = \lambda_1 + \lambda_2 + \lambda_3$$
$$s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_2 + \lambda_3 + \lambda_4$$
$$s_{\lambda_1} s_{\lambda_3}(\lambda_2) = \lambda_1 + \lambda_2 + 2\lambda_3$$
$$s_{\lambda_3} s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_2 + 2\lambda_3 + \lambda_4$$
$$s_{\lambda_2} s_{\lambda_1} s_{\lambda_3}(\lambda_2) = \lambda_1 + 2\lambda_2 + 2\lambda_3$$
$$s_{\lambda_4} s_{\lambda_3}(\lambda_2) = \lambda_2 + 2\lambda_3 + 2\lambda_4$$

Roots of length 4:

$$s_{\lambda_1} s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$
$$s_{\lambda_3} s_{\lambda_1} s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4$$
$$s_{\lambda_2} s_{\lambda_3} s_{\lambda_1} s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4$$
$$s_{\lambda_4} s_{\lambda_1} s_{\lambda_3}(\lambda_2) = \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4$$
$$s_{\lambda_3} s_{\lambda_2} s_{\lambda_3} s_{\lambda_1} s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4$$
$$s_{\lambda_4} s_{\lambda_2} s_{\lambda_1} s_{\lambda_3}(\lambda_2) = \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4$$
$$s_{\lambda_4} s_{\lambda_3} s_{\lambda_2} s_{\lambda_3} s_{\lambda_1} s_{\lambda_2} s_{\lambda_3}(\lambda_4) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4$$
$$s_{\lambda_3} s_{\lambda_4} s_{\lambda_2} s_{\lambda_1} s_{\lambda_3}(\lambda_2) = \lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4$$
$$s_{\lambda_2} s_{\lambda_3} s_{\lambda_4} s_{\lambda_2} s_{\lambda_1} s_{\lambda_3}(\lambda_2) = \lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4$$
$$s_{\lambda_1} s_{\lambda_2} s_{\lambda_3} s_{\lambda_4} s_{\lambda_2} s_{\lambda_1} s_{\lambda_3}(\lambda_2) = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4$$

8.6. **Computing $\Phi(\lambda)$ for all $\lambda \in \Phi(\mathfrak{a})^+$.** Now we apply the w used above in Algorithms 5, 6, 7, 8, and 9 setwise to the $\Phi(\lambda)$, $\lambda \in \Delta(\mathfrak{a})$.

For $w \in W(\mathfrak{a})$, we found the Weyl group representative $\tilde{w}$ in $W_1(\theta)$. So we compute the $\Phi(\lambda)$ in this way

$$\Phi(\lambda) = \Phi(w(\lambda_i)) = \tilde{w}(\Phi(\lambda_i)).$$

Example 4. Restricted Root System of type A
We compute the $\Phi(\lambda)$ in this way

$$\Phi(s_{\lambda_{i+l-1}} \cdots s_{\lambda_{i+1}}(\lambda_i)) = w_{i+l-1} \cdots w_{i+1}(\Phi(\lambda_i)).$$

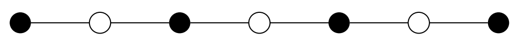
9. Example A_7^3(II)

Recall from [Hel88] that the notation A_7^3(II) means that the original root system, $\Phi(\mathfrak{t})$ is of type A_7, the involution θ is $\theta(A) = J(-A)^T J^{-1}$ where

$$J = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$$

which is a type II involution in [Hel78], and that the resulting restricted root system satisfies $|\Delta(\mathfrak{a})| = 3$.

9.1. **Step 1:** First we need to recover the action of the involution on $\Phi(\mathfrak{t})$ and find the formulation for the λ in $\Phi(\mathfrak{a})$ in terms of the α in $\Phi(\mathfrak{t})$. The θ-diagram in this case is:



We notice that

$$\Delta_0(\theta) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 = \{\alpha_1\} \cup \{\alpha_3\} \cup \{\alpha_5\} \cup \{\alpha_7\}.$$

We have $w_0(\theta) = s_{\alpha_1} s_{\alpha_3} s_{\alpha_5} s_{\alpha_7}$ and the involution θ on $\Delta(\mathfrak{a})$ is

$$\theta = -s_{\alpha_1} s_{\alpha_3} s_{\alpha_5} s_{\alpha_7}.$$

We have that $\alpha_1, \alpha_3, \alpha_5,$ and, α_7 are fixed by θ while

$$\begin{aligned} \theta(\alpha_2) &= -(\alpha_1 + \alpha_2 + \alpha_3), \\ \theta(\alpha_4) &= -(\alpha_3 + \alpha_4 + \alpha_5), \text{ and} \\ \theta(\alpha_6) &= -(\alpha_5 + \alpha_6 + \alpha_7). \end{aligned}$$

Now using the natural projection map $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$, we find that

$$\begin{aligned} \pi(\alpha_2) &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) = \lambda_1, \\ \pi(\alpha_4) &= \frac{1}{2}(\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_2, \\ \pi(\alpha_6) &= \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_3, \text{ and} \\ \pi(\alpha_1) &= \pi(\alpha_3) = \pi(\alpha_5) = \pi(\alpha_7) = 0. \end{aligned}$$

9.2. **Step 2:** We defined $\Phi(\lambda_i) = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda_i\}$. For this example, we have

$$\begin{aligned} \Phi(\lambda_1) &= \{\alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}, \\ \Phi(\lambda_2) &= \{\alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5\}, \text{ and} \\ \Phi(\lambda_3) &= \{\alpha_6, \alpha_5 + \alpha_6, \alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7\}. \end{aligned}$$

So the multiplicities are $m_{\lambda_1} = m_{\lambda_2} = m_{\lambda_3} = 4$.

9.3. **Step 3:** The representatives in $W_1(\theta)$ for s_{λ_1}, s_{λ_2} and s_{λ_3} are

$$\begin{aligned} w_1 &= s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2} \\ w_2 &= s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4} \text{ and} \\ w_3 &= s_{\alpha_6}s_{\alpha_5}s_{\alpha_7}s_{\alpha_6}. \end{aligned}$$

9.4. **Step 4:** Next we find the list of positive roots, $\Phi(\mathfrak{a})^+$, by using s_{λ_1}, s_{λ_2}, and s_{λ_3} applied to each λ. Here we have

$$\begin{aligned} s_{\lambda_2}(\lambda_1) &= \lambda_1 + \lambda_2 \\ s_{\lambda_3}(\lambda_2) &= \lambda_2 + \lambda_3 \\ s_{\lambda_3}s_{\lambda_2}(\lambda_1) &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

Therefore, $\Phi(\mathfrak{a})^+ = \{\lambda_1, \lambda_2, \lambda_3, \lambda_1+\lambda_2, \lambda_2+\lambda_3, \lambda_1+\lambda_2+\lambda_3\}$.

9.5. **Step 5:** Now we need to compute $\Phi(\lambda_i)$ for all λ_i in $\Phi(\mathfrak{a})^+$. This is easily accomplished by looking at the previous step. We must compute

$$\begin{aligned} \Phi(\lambda_1+\lambda_2) &= \Phi(s_{\lambda_2}(\lambda_1)) = w_2(\Phi(\lambda_1)), \\ \Phi(\lambda_2+\lambda_3) &= \Phi(s_{\lambda_3}(\lambda_2)) = w_3(\Phi(\lambda_2)), \text{ and} \\ \Phi(\lambda_1+\lambda_2+\lambda_3) &= \Phi(s_{\lambda_3}s_{\lambda_2}(\lambda_1)) = w_3w_2(\Phi(\lambda_1)), \end{aligned}$$

We arise with

$$\begin{aligned} \Phi(s_{\lambda_2}(\lambda_1)) =& \Phi(\lambda_1+\lambda_2) = \{\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\} \\ \Phi(s_{\lambda_3}(\lambda_2)) =& \Phi(\lambda_2+\lambda_3) = \{\alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\} \\ \Phi(s_{\lambda_3}s_{\lambda_2}(\lambda_1)) =& \Phi(\lambda_1+\lambda_2+\lambda_3) = \{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \\ & \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\} \end{aligned}$$

9.6. **Step 6:** The eigenspaces are

$$\begin{aligned} \mathfrak{g}_{\lambda_1} &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \\ \mathfrak{g}_{\lambda_2} &= \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5} \\ \mathfrak{g}_{\lambda_3} &= \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_7} \\ \mathfrak{g}_{\lambda_1+\lambda_2} &= \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} \\ \mathfrak{g}_{\lambda_2+\lambda_3} &= \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \\ \mathfrak{g}_{\lambda_1+\lambda_2+\lambda_3} &= \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \end{aligned}$$

9.7. **Step 7:** For the basis of A_7, we will choose $\{E_{ij}, i \neq j\}$ and the maximal torus

$$\begin{aligned} \mathfrak{t} =& \langle E_{11}-E_{55}, -E_{11}+E_{22}, -E_{22}+E_{66}, -E_{66}+E_{77}, E_{33}-E_{77}, -E_{33}+E_{44}, -E_{44}+E_{88}\rangle \\ =& \langle e_1-e_5, -e_1+e_2, -e_2+e_6, -e_6+e_7, e_3-e_7, -e_3+e_4, -e_4+e_8\rangle. \end{aligned}$$

The corresponding θ-basis of the root space is

$$\Delta(\mathfrak{t}) = \{\epsilon_1-\epsilon_5, -\epsilon_1+\epsilon_2, -\epsilon_2+\epsilon_6, -\epsilon_6+\epsilon_7, \epsilon_3-\epsilon_7, -\epsilon_3+\epsilon_4, -\epsilon_4+\epsilon_8\}.$$

We will denote these roots by:

$$\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}.$$

From

$$\mathfrak{g}_{\lambda_1} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3},$$

we consider $\mathfrak{g}_{\alpha_2} = \{E_{21}\}$. We know that $E_{21} - \theta(E_{21}) \in \mathfrak{p}$ and $E_{21} + \theta(E_{21}) \in \mathfrak{k}$. Therefore $E_{21} + E_{56} \in \mathfrak{p}$ and $E_{21} - E_{56} \in \mathfrak{k}$. Continuing on in this fashion we find that:

$$\begin{aligned} \mathfrak{p} &\supset \{E_{21}+E_{56}, E_{25}-E_{16}, E_{61}-E_{52}, E_{12}+E_{65}\} \text{ and} \\ \mathfrak{k} &\supset \{E_{21}-E_{56}, E_{25}+E_{16}, E_{61}+E_{52}, E_{12}-E_{65}\}. \end{aligned}$$

From

$$\mathfrak{g}_{\lambda_2} = \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5},$$

we have that

$$\mathfrak{p} \supset \{E_{23} + E_{76}, E_{36} - E_{27}, E_{72} - E_{63}, E_{32} + E_{67}\} \text{ and}$$
$$\mathfrak{k} \supset \{E_{23} - E_{76}, E_{36} + E_{27}, E_{72} + E_{63}, E_{32} - E_{67}\}.$$

From

$$\mathfrak{g}_{\lambda_3} = \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_7},$$

we have that

$$\mathfrak{p} \supset \{E_{43} + E_{78}, E_{47} - E_{38}, E_{83} - E_{74}, E_{34} + E_{87}\} \text{ and}$$
$$\mathfrak{k} \supset \{E_{43} - E_{78}, E_{47} + E_{38}, E_{83} + E_{74}, E_{34} - E_{87}\}.$$

From

$$\mathfrak{g}_{\lambda_1+\lambda_2} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}$$

we have that

$$\mathfrak{p} \supset \{E_{35} - E_{17}, E_{31} + E_{57}, E_{13} + E_{75}, E_{71} - E_{53}\} \text{ and}$$
$$\mathfrak{k} \supset \{E_{35} + E_{17}, E_{31} - E_{57}, E_{13} - E_{75}, E_{71} + E_{53}\}.$$

From

$$\mathfrak{g}_{\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$$

we have that

$$\mathfrak{p} \supset \{E_{46} - E_{28}, E_{42} + E_{68}, E_{24} + E_{86}, E_{82} - E_{64}\} \text{ and}$$
$$\mathfrak{k} \supset \{E_{46} + E_{28}, E_{42} - E_{68}, E_{24} - E_{86}, E_{82} + E_{64}\}.$$

From

$$\mathfrak{g}_{\lambda_1+\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$$

we have that

$$\mathfrak{p} \supset \{E_{41} + E_{58}, E_{45} - E_{18}, E_{81} - E_{54}, E_{14} + E_{85}\} \text{ and}$$
$$\mathfrak{k} \supset \{E_{41} - E_{58}, E_{45} + E_{18}, E_{81} + E_{54}, E_{14} - E_{85}\}.$$

From $\Delta_0(\theta) = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}$, we get

$$\mathfrak{k} \supset \{E_{15}, E_{51}, E_{26}, E_{62}, E_{37}, E_{73}, E_{48}, E_{84}\}.$$

A basis for $\mathfrak{a}$ is $\{-E_{11} + E_{22} + E_{55} - E_{66}, E_{22} - E_{33} - E_{66} + E_{77}, -E_{33} + E_{44} + E_{77} - E_{88}\}$ and a basis for $\mathfrak{t}_0$ is $\{E_{11} - E_{55}, -E_{22} + E_{66}, E_{33} - E_{77}, -E_{44} + E_{88}\}$.

Putting all this information together, we arise with the following "nice" bases for both $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathfrak{p} = \langle -E_{11}+E_{22}+E_{55}-E_{66}, E_{22}-E_{33}-E_{66}+E_{77}, -E_{33}+E_{44}+E_{77}-E_{88}, E_{21}+E_{56}, E_{25}-E_{16}, E_{61}-E_{52}, E_{12}+E_{65}, E_{23}+E_{76}, E_{36}-E_{27}, E_{72}-E_{63}, E_{32}+E_{67}, E_{43}+E_{78}, E_{47}-E_{38}, E_{83}-E_{74}, E_{34}+E_{87}, E_{35}-E_{17}, E_{31}+E_{57}, E_{13}+E_{75}, E_{71}-E_{53}, E_{46}-E_{28}, E_{42}+E_{68}, E_{24}+E_{86}, E_{82}-E_{64}, E_{41}+E_{58}, E_{45}-E_{18}, E_{81}-E_{54}, E_{14}+E_{85}\rangle$

$\mathfrak{k} = \langle E_{11}-E_{55}, -E_{22}+E_{66}, E_{33}-E_{77}, -E_{44}+E_{88}, E_{21}-E_{56}, E_{25}+E_{16}, E_{61}+E_{52}, E_{12}-E_{65}, E_{23}-E_{76}, E_{36}+E_{27}, E_{72}+E_{63}, E_{32}-E_{67}, E_{43}-E_{78}, E_{47}+E_{38}, E_{83}+E_{74}, E_{34}-E_{87}, E_{35}+E_{17}, E_{31}-E_{57}, E_{13}-E_{75}, E_{71}+E_{53}, E_{46}+E_{28}, E_{42}-E_{68}, E_{24}-E_{86}, E_{82}+E_{64}, E_{41}-E_{58}, E_{45}+E_{18}, E_{81}+E_{54}, E_{14}-E_{85}, E_{15}, E_{51}, E_{26}, E_{62}, E_{37}, E_{73}, E_{48}, E_{84}\rangle$

With our choice of θ-basis, we have structure constants $c_{\alpha_1,\theta} = c_{\alpha_3,\theta} = c_{\alpha_5,\theta} = c_{\alpha_7,\theta} = 1$ and $c_{\alpha_2,\theta} = c_{\alpha_4,\theta} = c_{\alpha_6,\theta} = -1$.

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Department of Mathematics, Lamar University, Beaumont, TX 77710
E-mail address: jrfowler@math.lamar.edu

Department of Mathematics, North Carolina State University, Raleigh, N.C., 27695
E-mail address: loek@math.ncsu.edu