ON RATIONALITY PROPERTIES OF INVOLUTIONS OF REDUCTIVE GROUPS

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Introduction.

Let k be a field of characteristic not two and G a connected linear reductive k-group. By a k-involution θ of G, we mean a k-automorphism θ of G of order two. For $k = \mathbb{R}$, $\mathbb{C}$ or an algebraically closed field, such involutions have been extensively studied emerging from different interests. As manifested in **[8, 18, 28]**, the interactions with the representation theory of reductive groups are most rewarding. The application of discrete series of affine symmetric spaces to the cohomology of arithmetic subgroups [27] invites the study of $\mathbb{Q}$-involutions. In the present paper, we give a treatment on rationality problems of general k-involutions. Here we generalize most of the earlier results **[15, 16, 23, 29]**, sharpen some and add new ones.

Let H be an open subgroup of the fixed point group G^θ of an involution θ of G. In §1, we show that H^0 characterizes θ when G is semi-simple. It follows that (1.6) θ is defined over k if and only if H^0 is a k-subgroup of G. In §2, we deal with θ-stable k-split tori of G for a k-involution θ of G. The key, unlocking the door to rationality discussion, is the simple existence result (2.4) that every minimal parabolic k-subgroup P of G contains a θ-stable maximal k-split torus.

In general, proper θ-stable parabolic k-subgroups of G do not exist. In §3, we present a simple criterion for their existence (3.4) and a structure theorem (3.5) for the minimal θ-stable parabolic k-subgroups of G. A parabolic subgroup Q of G is θ-split if Q and $\theta(Q)$ are opposite. In §4, we discuss θ-split parabolic k-subgroups of G following Vust [29]. It is known that G has proper θ-split parabolic k-subgroups if and only if the restriction of θ to the isotropic factor of G over k is nontrivial. The minimal θ-split parabolic k-subgroups of G are determined by the maximal (θ, k)-split tori of G (4.7) and are conjugate by elements of G_k. However in general they are not H_k-conjugate.

To each k-involution θ of G, there correspond two root systems. The discussion is carried out in §5. As a consequence, we have the conjugacy theorem (5.8) for minimal θ-stable parabolic k-subgroups.

Let P be a minimal parabolic k-subgroup and H a k-open subgroup of G^θ. Consider the double coset space $P_k \setminus G_k/H_k$. The geometry of these orbits is of importance in the study of Harish-Chandra modules [28] with $k = \mathbb{C}$ and of discrete series of affine symmetric spaces **[8, 18]** with $k = \mathbb{R}$. The §6 deals with the reduction theory of the double coset space for general k. Our main result is Propositon 6.8 following Springer, and a slightly different characterization of Rossmann is given in 6.10. We also show that (6.16) $P_k \setminus G_k/H_k$ is finite when k is a local field. For $k = \mathbb{R}$, this finiteness result is due to J. Wolf [30] (see also T. Matsuki [15]).

Let Φ be a root system in a finite dimensional real vector space V and θ an involution of V leaving Φ invariant. Then θ induces an automorphism, also denoted by θ, of the Weyl group W of Φ given by $\theta(w) = \theta \circ w \circ \theta$, $w \in W$. An element $w \in W$ is called a twisted involution if $\theta(w) = w^{-1}$. T. A. Springer initiated the study of the twisted involutions. The most elegant result is his decomposition theorem [23, Prop. 3.3]. Here we contribute certain uniqueness conditions of the decomposition ((iii) of 7.9). Inspired by the work of Matsuki [16], we present a new proof of constructive nature which yields also the classification of such decompositions (7.24). In §8, we establish some dimension formulas needed for our study on orbit closures.

The double coset space $P \setminus G/H$ has a unique open element, called the big cell. We characterize the big cell in 9.2. For $g \in G_k$, let $\mathrm{cl}(PgH)$ denote the Zariski closure of PgH in G. The structure theorem of orbit closure is given in 9.5 in terms of the decomposition of a twisted involution. When k is algebraically closed, the result is due to Springer [23]. If k is a local field, we consider the topological closure of $P_k g H_k$ in G_k. We obtain an analogous structure theorem in 9.9.

In §10, we study the set $(PH)_k$ for a minimal parabolic k-subgroup P of G and a k-open subgroup H of G^θ. As an application, we present a new proof of the Iwasawa decomposition of $G_{\mathbb{R}}$ (10.11). For a field k with $-1 \notin (k^\times)^2 = (k^\times)^4$, we generalize the notion of a Cartan involution over k. In §11, we give a systematic discussion on Cartan involutions over k. We succeed in extending almost all the results of $k = \mathbb{R}$. In §12, we have a reduction theorem of $P_k \setminus G_k / H_k$ in terms of conjugacy classes in a restricted Weyl group (12.10 and 12.15). In §13, we revisit orbit closure. Here we characterize orbits with minimal (resp. maximal) dimension when k is a local field.

Some of the results in this paper were announced in [12]

0. Notations.

Our basic reference for reductive groups is [3].

0.1. All algebraic groups considered are linear algebraic groups. Given an algebraic group G, the identity component is denoted by G^0. We use $L(G)$ (resp. $\mathfrak{g}$, the corresponding lower case German letter) for the Lie algebra of G.

If H is a subset of G, $N_G(H)$ (resp. $Z_G(H)$) is the normalizer (resp. centralizer) of H in G. We write $Z(G)$ for the center of G. The commutator subgroup of G is denoted by $D(G)$ or $[G, G]$.

0.2. Let k be a field. By an algebraic k-group, we mean an algebraic group defined over k. Let G be an algebraic k-group. For an extension K of k, the set of K-rational points of G is denoted by G_K or $G(K)$.

0.3. Let G be a reductive k-group and A a k-split torus of G. Denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of A (resp. one-parameter subgroups of A) and by $\Phi(G, A)$ the set of the roots of A in G.

Given a quasi-closed subset ψ of $\Phi(G, A)$, the group G_ψ (resp. G^*_ψ) is defined in [3, 3.8]. If G^*_ψ is unipotent, ψ is said to be unipotent and often one writes U_ψ for G^*_ψ.

0.4. Let G be an algebraic k-group. For any subset Y of G, the Zariski closure of Y in G is denoted by $cl(Y)$.

If k is a local field, G_k has a natural topology endowed by the topology of k. We call this topology of G_k the t-topology. Given $Y \subset G_k$, the closure of Y in G_k with respect to the t-topology is written by t-cl(Y).

0.5. Let G be a group. We use 1 (resp. -1) for the identity map (resp. the inverse map $g \mapsto g^{-1}$).

0.6 Lemma. *Let U denote a connected unipotent k-group and θ a semi-simple k-automorphism of U. Let U^θ be the set $\{u \in U | \theta(u) = u\}$, c_θ the map $u \mapsto u\theta(u)^{-1}$ of U into U and $M = c_\theta(U)$. Then we have the following conditions:*

(i) *The product map $M \times U^\theta \to U$ and $c_\theta | M : M \to M$ are k-isomorphisms of varieties.*

(ii) *If θ is an involution, $M = \{u \in U | \theta(u) = u^{-1}\}$ and $L(U^\theta) = L(U)^\theta$.*

Proof. (i) is [3, Lemma 11.1].
(ii) The assertion $L(U^\theta) = L(U)^\theta$ is immediate from (i). The assertion on M follows by an easy induction on the length of the central lower series of U.

0.7. For an involution $\theta \in Aut(G)$ and a θ-stable subgroup A of G we will also write A_θ for A^θ. Moreover we will write A_θ^+ for A_θ^0 and A_θ^- for the identity component of $\{a \in A | \theta(a) = a^{-1}\}$. If A is a torus of G, then we have $A = A_\theta^+ \cdot A_\theta^-$ (almost direct product).

0.8. In the following, k is a field with $ch(k) \neq 2$. Let G be a reductive algebraic k-group and θ an involution of G defined over k. The differential of θ will also be denoted by θ.

1. The fixed point group.

Let G be a connected reductive algebraic group and θ an involution of G. Let H denote the fixed point group of θ given by

$$H = G^\theta = \{g \in G | \theta(g) = g\}.$$

In this section, we show that H determines θ when G is semi-simple.

1.1 Lemma. *Let G_1 and G_2 be algebraic groups such that $R_u(G_1) \approx R_u(G_2) \approx G_a$. If $f : G_1 \to G_2$ is an isomorphism, then the restriction of f to $R_u(G_1)$ is determined by the restriction of its differential df to the Lie algebra $L(R_u(G_1))$ of $R_u(G_1)$.*

Proof. View $R_u(G_1)$ and $R_u(G_2)$ as one dimensional vector space. Then $f|R_u(G_1) : R_u(G_1) \to R_u(G_2)$ is a scalar multiplication which coincides with its differential.

1.2 Proposition. *Let G be a connected semi-simple algebraic group, θ_i involution of G and $H_i = G^{\theta_i}$, $(i = 1, 2)$. Let H_1^0 and H_2^0 be the identity components of H_1 and H_2 respectively. If $H_1^0 = H_2^0$, then $\theta_1 = \theta_2$.*

Proof. We show the assertion in several steps.

Step 1. Let S be a maximal torus of $H_1^0 = H_2^0$. By [21, Lemma 5.3], the centralizer $T = Z_G(S)$ of S in G is a maximal torus of G. Hence there exists $\lambda \in X_*(S)$ such that $\langle \lambda, \alpha \rangle \neq 0$ for $\alpha \in \Phi(G, T)$. Now let Φ^+ denote the system of positive roots consisting of $\alpha \in \Phi(G, T)$ with $\langle \lambda, \alpha \rangle > 0$. Then the Borel subgroup $B = TU_{\Phi^+}$ is stable under both θ_1 and θ_2.

Step 2. Let $\alpha \in \Phi^+$ with $\theta_1\alpha \neq \alpha$. Then $\theta_1\alpha = \theta_2\alpha$ and $\theta_1|U_\alpha = \theta_2|U_\alpha$.

Let $L(G) = L(T) \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$ be the decomposition of $L(G)$ into root subspaces. Choose $0 \neq X \in \mathfrak{g}_\alpha$. Then $X + \theta_1(X) \in L(H_1) = L(H_2)$. From $X + \theta_1(X) = \theta_2(X + \theta_1(X))$, it yields that $\{\alpha, \theta_1\alpha\}$ is θ_2-stable. Suppose that $\theta_2\alpha = \alpha$. Then $\theta_2(X) = X$ and as a consequence $\theta_1(X) = X$. Certainly this is a contradiction to $\theta_1\alpha \neq \alpha$. Hence $\theta_2\alpha = \theta_1\alpha$ and $\theta_2(X) = \theta_1(X)$. By Lemma 1.1, $\theta_1|U_\alpha = \theta_2|U_\alpha$.

Step 3. Let $\alpha \in \Phi^+$ with $\theta_1\alpha = \alpha$. If $\theta_2\alpha \neq \alpha$, then by Step 2, $\theta_1\alpha = \theta_2\alpha \neq \alpha$. Hence $\theta_2\alpha = \alpha$. In this case, $\theta_1|U_\alpha = \pm 1$ and $\theta_2|U_\alpha = \pm 1$. Since $H_1^0 = H_2^0$, we have that $\theta_1|U_\alpha = \theta_2|U_\alpha$.

Step 4. Let $U = U_{\Phi^+}$ and $U^- = U_{-\Phi^+}$. By Steps 3 and 4, $\theta_1|U = \theta_2|U$ and similarly $\theta_1|U^- = \theta_2|U^-$. Since U and U^- generate G, our assertion follows.

1.3 Corollary. *Let G be a connected reductive algebraic group, θ an involution of G and $g \in G$. The following conditions are equivalent:*

(i) $g\theta(g)^{-1} \in Z(G)$.
(ii) $g\theta(g)^{-1} \in Z_G(H)$.
(iii) $g \in N_G(H)$.
(iv) $g \in N_G(H^0)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii). Set $z = g\theta(g)^{-1}$. For $h \in H$, $\theta(g^{-1}hg) = g^{-1}(zhz^{-1})g = g^{-1}hg$. Hence $g \in N_G(H)$.

(iii) $\Rightarrow$ (iv) is trivial.
(iv) $\Rightarrow$ (i). Let $Int(g)$ be the inner automorphism $x \mapsto gxg^{-1}$ of G. Set $\tau = Int(g)^{-1} \circ \theta \circ Int(g)$. Clearly τ is an involution of G. Let $D(G) = [G, G]$. By the condition (iv), $D(G)^\theta$ and $D(G)^\tau$ have the same identity component. By the preceding proposition, $\theta|D(G) = \tau|D(G)$. Note also that $\theta|Z(G) = \tau|Z(G)$. Thus $\theta = \tau$ and condition (i) is immediate.

Remark. Another proof of the implication (iv)$\Rightarrow$(i) of 1.3, based on a result from [21] is given in a paper by De Concini and Springer in "Geometry Today", Progress in Mathematics vol. 60 (Birkhaüser 1985), p. 99–100.

1.4 An example. Let $G = SL(3)$, T the maximal torus of diagonal matrices and B the Borel subgroup of upper triangular matrices. Let τ be the involution of G given by

$$\tau(g) = J^t g^{-1} J, \qquad g \in G \tag{1}$$
$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then T and B are τ-stable. For $t \in T$, write $t = \mathrm{diag}(t_1, t_2, t_3)$. Then $\tau(t) = \mathrm{diag}(t_3^{-1}, t_2^{-1}, t_1^{-1})$. Let α, β and γ be the characters of T defined by

$$\alpha(t) = t_1 t_2^{-1}, \quad \beta(t) = t_2 t_3^{-1}, \quad \gamma(t) = t_1 t_3^{-1}.$$

Then $\Phi(B, T) = \{\alpha, \beta, \gamma\}$. It is easy to see that $\gamma = \alpha + \beta$, $\tau\alpha = \beta$ and $\tau\beta = \alpha$.

1.5 Lemma. *Let G, T, B, τ, α, β and γ be as in 1.4 and θ an involution of G such that T is θ-stable and $\theta\alpha = \beta$. Then we have the following conditions:*

(i) *There exists $t = diag(\lambda, 1, \lambda)$ such that $\theta = Int(t) \circ \tau$.*
(ii) *If $R_u(B)^\theta$ is defined over a field k, then $\lambda \in k$ and θ is defined over k.*

Proof. Observe that $\theta|T$ and $\tau|T$ induce the same automorphism of $X^*(T)$. It follows that $\theta|T = \tau|T$. Let $x_\alpha(a)$, $x_\beta(b)$, $x_\gamma(c)$ denote the upper triangular matrices given by

$$x_\alpha(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_\beta(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, x_\gamma(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the explicit formula (1.4.1) of τ, we have that

$$\tau(x_\alpha(a)) = x_\beta(-a), \quad \tau(x_\beta(b)) = x_\alpha(-b), \quad \tau(x_\gamma(c)) = x_\gamma(-c).$$

Since $\theta\alpha = \beta$, we can write $\theta(x_\alpha(a)) = x_\beta(-\lambda^{-1}a), \quad \theta(x_\beta(b)) = x_\alpha(-\lambda b)$. Now set $t = \operatorname{diag}(\lambda, 1, \lambda)$. Then, θ coincides with $Int(t) \circ \tau$ on B. Hence

$$\theta = Int(t) \circ \tau.$$

By a simple computation, we show that $R_u(B)^\theta$ consists of all the elements of the form

$$\begin{pmatrix} 1 & a & -\lambda^{-1}a^2/2 \\ 0 & 1 & -\lambda^{-1}a \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the assertion (ii) is immediate.

1.6 Proposition. *Let G be a connected semi-simple algebraic k-group and θ an involution of G. Then θ is defined over k if and only if H^0 is defined over k.*

Proof. $\Rightarrow$) is well known.
$\Leftarrow$) we establish the assertion in several steps.

Step 1. We may replace k by its separable closure. Let $\bar{k}$ be the algebraic closure of k and $Gal(\bar{k}/k)$ the Galois group. For $\sigma \in Gal(\bar{k}/k)$, the conjugate ${}^\sigma\theta$ of θ by σ is also an involution of G. Since H^0 is defined over k, H^0 is also the identity component of the fixed point group of ${}^\sigma\theta$. By Proposition 1.2, we have that ${}^\sigma\theta = \theta, \quad \sigma \in Gal(\bar{k}/k)$. Thus we may replace k by any separable extension of k.

Step 2. By Step 1, we may assume that G and H^0 are k-split. As in Step 1 of the proof of Proposition 1.2, there exist θ-stable maximal k-split torus T of G and θ-stable Borel k-subgroup B of G containing T. Let B^- denote the Borel subgroup of G with $B^- \cap B = T$, $U = R_u(B)$ and $U^- = R_u(B^-)$. The product map $U^- \times T \times U \to U^-TU$ is an isomorphism of k-varieties. It follows that $H^0 \cap U^-TU = (U^-)^\theta T_+ U^\theta$ where $T_+ = (T^\theta)^0$. As a consequence, $(U^-)^\theta$ and U^θ are defined over k.

Step 3. Let Δ denote the set of simple roots of $\Phi(B, T)$. Then Δ is θ-stable. Given $\alpha \in \Delta$, let ψ be the subsystem of $\Phi(G, T)$ consisting of integral combinations of α and $\theta\alpha$. Let $\xi = \Phi(B, T) \cap \psi$ and $\zeta = \Phi(B, T) - \psi$. Then the product map

$$U_\xi \times U_\zeta \to U$$

is an isomorphism of k-varieties. Note that U_ξ and U_ζ are θ-stable. Since U^θ is defined over k, it follows that U_ξ^θ and U_ζ^θ are defined over k.

Step 4. Let $\alpha \in \Delta$ with $\theta\alpha = \alpha$. Then $\theta|U_\alpha = \pm 1$ and $\theta|U_\alpha$ is defined over k.

Step 5. Let $\alpha \in \Delta$ with $\theta\alpha \neq \alpha$. Then ψ is of type $A_1 \times A_1$ or A_2.

Case 1. ψ is of type $A_1 \times A_1$. In this case, U_ξ is abelian and the product map

$$U_\alpha \times U_{\theta\alpha} \to U_\xi$$

is an isomorphism of k-groups. The group U_ξ^θ coincides with the image of the graph of $\theta|U_\alpha$ under the above product map. From Step 3, U_ξ^θ is defined over k. Hence $\theta|U_\alpha$ is defined over k.

Case 2. ψ is of type A_2. Consider the semi-simple group G_ψ^*. It is k-split and θ-stable. Note that $T \cap G_\psi^*$ is a maximal k-split torus of G_ψ^* and $B \cap G_\psi^*$ is a Borel k-subgroup of G_ψ^*. By [25, 9.16], involutions can be lifted to simply connected covering groups. By Lemma 1.5, $\theta|G_\psi^*$ is defined over k.

Step 6. For $\alpha \in \Delta$, $\theta|U_\alpha$ is defined over k and similarly so is $\theta|U_{-\alpha}$. Since U_α, $U_{-\alpha}$, $\alpha \in \Delta$ generate G, it follows readily that θ is defined over k.

1.7. Lemma. *Let P be a parabolic subgroup of G. Then PH is closed in G if and only if $P \cap \theta(P)$ is a parabolic subgroup of G.*

Proof. $\Rightarrow$) Let B be a Borel subgroup of P. Consider the action of B on PH/H. There exists a closed orbit. It follows that there is $x \in P$ such that BxH is closed in G. By [23, (i) of Cor. 6.6], $x^{-1}Bx$ is θ-stable. Clearly $P \cap \theta(P)$, containing $x^{-1}Bx$, is a parabolic subgroup of G.

$\Leftarrow$) By a result of Steinberg [25, §7], there exists a θ-stable Borel subgroup B of $P \cap \theta(P)$. Then by [23, (i) of Cor. 6.6] BH is closed in G. Since P/B is complete, the desired assertion follows.

1.8. Lemma. *Let P and P^- be opposite θ-stable parabolic subgroups of G. If $PH = G$, then $R_u(P)$ and $R_u(P^-)$ are contained in H.*

Proof. Set $M = P \cap P^-$, $U = R_u(P)$ and $U^- = R_u(P^-)$. Given $u' \in U^-$, there exist $m \in M$ and $u \in U$ such that $umu' \in H$. Since M, U and U^- are θ-stable, $u' \in H$. This shows that $U^- \subset H$. Now let T be a θ-stable maximal torus of M and $T_+ = (T \cap H)^0$. There exists $\lambda \in X_*(T_+)$ with $P = P(\lambda)$ and $P^- = P(-\lambda)$ (3.3). Let S be a maximal torus of H containing T_+. Let $P_0(\lambda)$ (resp. $P_0(-\lambda)$) denote the parabolic subgroup of H^0 containing S defined by λ (resp. $-\lambda$). We have the condition that $P_0(\lambda) = (M \cap H)^0(U \cap H)$ and $P_0(-\lambda) = (M \cap H)^0(U^- \cap H)$. It follows that $R_u(P_0(\lambda)) = U \cap H$ and $R_u(P_0(-\lambda)) = U^- \cap H$. Since H is reductive and $P_0(\lambda)$, $P_0(-\lambda)$ are opposite, $\dim(U \cap H) = \dim(U^- \cap H)$. From the condition that $U^- \subset H$ and $\dim(U) = \dim(U^-)$, we have readily that $U \subset H$.

1.9. Lemma. *Let T be a θ-stable maximal torus of G, $\Phi = \Phi(G, A)$ and $L(G) = L(T) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, be the decomposition of $L(G)$ into root subspaces of T. If $\mathfrak{g}_\alpha \subset L(H)$, then U_α, $U_{-\alpha} \subset H$.*

Proof. Since $\mathfrak{g}_\alpha \subset L(H)$, we have that $\theta\alpha = \alpha$. Since $L(U_\alpha \cap H) = L(U_\alpha)^\theta = L(U_\alpha)$, $U_\alpha \subset H$. We may assume that G is the group generated by U_α and $U_{-\alpha}$. Now let B^- denote the group $TU_{-\alpha}$. Clearly B^- is θ-stable and by Lemma 1.7, B^-H is closed in G. As $U_\alpha \subset H$, B^-H is also open in G. Consequently $G = B^-H$ and the desired assertion follows from Lemma 1.8.

1.10. Proposition. *Let G be a connected semi-simple algebraic group and θ_1, θ_2 involutions of G such that $H_1^0 \subset H_2^0$, where $H_i = G^{\theta_i}$, $(i = 1, 2)$. Then there exists an almost direct product $G = G'G''$ such that $\theta_1|G' = \theta_2|G'$ and $\theta_2|G''$ is trivial.*

Proof. We prove the proposition in several steps.

(1) $\theta_1\theta_2 = \theta_2\theta_1$ is an involution of G. Let $\tau = \theta_2\theta_1\theta_2$. The fixed point subgroup of τ has identity component H_1^0. Hence by Proposition 1.2, $\tau = \theta_1$. Then the assertion is obvious.

(2) There exist (θ_1, θ_2)-stable maximal torus T and (θ_1, θ_2)-stable Borel subgroup B of G containing T. Let T_1 be a maximal torus of H_1 and $T = Z_G(T_1)$. Clearly T is a (θ_1, θ_2)-stable maximal torus of G. Since $T = Z_G(T_1)$, there exists $\lambda \in X_*(T_1)$ such that the parabolic subgroup $P(\lambda)$ of G containing T defined by λ is a Borel subgroup of G. Obviously the group $B = P(\lambda)$ is (θ_1, θ_2)-stable.

(3) Let $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{q}_1$ and $\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{q}_2$ be the decompositions of $\mathfrak{g} = L(G)$ into eigen subspaces of θ_1 and θ_2 respectively where $\mathfrak{h}_1 = L(H_1)$ and $\mathfrak{h}_2 = L(H_2)$. Then $\mathfrak{q}_2 \subset \mathfrak{q}_1$. Given $Y \in \mathfrak{q}_2$, $Y + \theta_1 Y \in \mathfrak{h}_2 \cap \mathfrak{q}_2 = \{0\}$ for $\mathfrak{h}_1 \subset \mathfrak{h}_2$. Hence $Y \in \mathfrak{q}_1$.

(4) Let $\Phi = \Phi(G, T)$, $\Phi^+ = \Phi(B, T)$, Δ the set of simple roots of Φ^+ and $\mathfrak{g} = L(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ the decomposition of $L(G)$ into root subspaces of T. Given $\alpha \in \Phi^+$ with $\theta_1\alpha \neq \alpha$, then either $\theta_1\alpha = \theta_2\alpha$ and $\theta_1\theta_2|U_\alpha = 1$ or $\theta_2\alpha = \alpha$ and $\theta_2|U_\alpha = 1$. Choose $0 \neq X \in \mathfrak{g}_\alpha$. It follows that $X + \theta_1 X \in \mathfrak{h}_1 \subset \mathfrak{h}_2$. Hence $\theta_2 X = \theta_1 X$ or $\theta_2 X = X$. The assertion is immediate from Lemma 1.9.

(5) We may assume that G is almost simple. Let $G = G_1 \dots G_\ell$ be an almost direct product of almost simple groups. If $\theta_1(G_i) \neq G_i$, by (4) $\theta_1 = \theta_2$ on $G_i\theta_1(G_i)$ or $\theta_2 = 1$ on $G_i\theta_1(G_i)$. If $\theta_2(G_i) \neq G_i$, by (3) $\theta_1 = \theta_2$ on $G_i\theta_2(G_i)$. It follows that we may assume that G is almost simple.

(6) Let $\theta = \theta_2$ (resp. $\theta_1\theta_2$, or θ_1). If $\mathfrak{g}_\alpha \subset L(G^\theta)$, $\alpha \in \Delta$, then $\theta = 1$. By Lemma 1.9, U_α, $U_{-\alpha} \subset H$, $\alpha \in \Delta$. Since G is generated by U_α, $U_{-\alpha}$, $\alpha \in \Delta$, the assertion is obvious.

(7) We may assume that $\theta_2 \neq 1$. Set $\varphi = \mathfrak{q}_2 + [\mathfrak{q}_2, \mathfrak{q}_2]$. Then there exists $\alpha \in \Delta$ with $\mathfrak{g}_\alpha \subset \varphi$.

By (6), there exists $\alpha \in \Delta$ with $\mathfrak{g}_\alpha \not\subset \mathfrak{h}_2$. If $\theta_2\alpha = \alpha$, $\mathfrak{g}_\alpha \subset \mathfrak{q}_2$. Suppose that $\theta_2\alpha \neq \alpha$. Choose $0 \neq X \in \mathfrak{g}_\alpha$. We have that $X - \theta_2(X) \in \mathfrak{q}_2$. Let Φ_1 be the set of all integral combinations of α and $\theta_2\alpha$ contained in Φ and $G_1 = G^*_{\Phi_1}$. Then G_1 is of type $A_1 \times A_1$ or A_2. Note that φ is an ideal of $L(G)$ and $L(G_1) \cap \varphi$ is an ideal of $L(G_1)$. In this case, one checks easily that $X, \theta_2(X) \in L(G_1) \cap \varphi$.

(8) Let G be a semi-simple group of type A_2 or C_2. If φ is an ideal of $L(G)$ containing $\mathfrak{g}_\alpha$ for some $\alpha \in \Delta$, then $\mathfrak{g}_\alpha \subset \varphi$ for all $\alpha \in \Delta$. The assertion can be verified by a straight forward computation.

(9) Case 1: G is not of type G_2. By (7), (8) and an easy induction, $\mathfrak{g}_\alpha \subset \varphi$, $\alpha \in \Delta$. Note that $L(G^{\theta_1\theta_2}) = \mathfrak{h}_1 + \mathfrak{q}_2$. Since $[\mathfrak{q}_2, \mathfrak{q}_2] \subset [\mathfrak{q}_1, \mathfrak{q}_1] \subset \mathfrak{h}_1$ by (3), $L(G^{\theta_1\theta_2}) \supset \varphi$. By (6), $\theta_1\theta_2 = 1$.

Case 2: G is of type G_2.

By [6, §21], G is simply connected and $Z(G) = \{e\}$. It follows that θ_1 (resp. θ_2) is given by an inner automorphism $g \mapsto tgt^{-1}$, $g \in G$. Here t is an element of T with $t^2 = e$. Let $\alpha_1 \in \Delta$ (resp. $\alpha_2 \in \Delta$) be the short root (resp. long root). Then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1\}$. Assume that $t_1, t_2 \in T$ satisfy the conditions: (i) $\alpha_j(t_i) = \pm 1$, $i, j = 1, 2$ (ii) $\alpha(t_2) = 1$ for $\alpha \in \Phi^+$ with $\alpha(t_1) = 1$. It is easy to check that $t_1 = t_2$ or $t_2 = e$. Hence $\theta_1 = \theta_2$ or $\theta_2 = 1$.

2. θ-stable k torus.

2.1. Let G be a connected reductive algebraic k-group and θ an involution of G defined over k. Set

$$H = G^\theta = \{g \in G | \theta(g) = g\},$$
$$Q = \{g^{-1}\theta(g) | g \in G\}.$$

Then H (resp. H^0) is a reductive k-subgroup of G and Q a closed k-subvariety of G.

2.2 Lemma. *Let P_1 and P_2 be parabolic k-subgroups of G such that $L = P_1 \cap P_2$ is θ-stable. Then L is a connected k-subgroup of G, $R_u(L)$ is defined over k and L has θ-stable Levi k-subgroups.*

Proof. By [3, Proposition 4.7], L is a connected k-subgroup of G, $R_u(L)$ is defined over k and L has Levi k-subgroups. Let M be a Levi k-subgroup of L. Since L is θ-stable, $\theta(M)$ is also a Levi k-subgroup of L. By the same proposition, there exists $v \in R_u(L)_k$ such that

$$\theta(M) = vMv^{-1}.$$

Then $\theta(v)v$ lies in the normalizer of M in $R_u(L)$ which is $\{e\}$. It follows that $\theta(v) = v^{-1}$. Then by Lemma 0.6, there is $w \in R_u(L)_k$ with $\theta(w)^{-1}w = v$. It is easy to see that wMw^{-1} is a θ-stable Levi k-subgroup of L.

2.3 Proposition. *There exists a θ-stable maximal k-torus T of G such that its k-split part T_d is a maximal k-split torus of G.*

Proof. Without losing any generality, we may assume that G is semi-simple.

Case 1. G is anisotropic over k.

If H is finite, by [21, 5.2] G is a torus. Hence we may assume that there exists a non-trivial k-torus S of G contained in H. Then consider the group $G_1 = Z_G(S)$. Clearly G_1 is θ-stable, reductive and defined over k. Since G is semi-simple, $\dim(G_1) < \dim(G)$. By induction on $\dim(G)$, G_1 has a θ-stable maximal k-torus T.

Case 2. G is isotropic over k.

Let P be a proper parabolic k-subgroup of G. Consider the group $L = P \cap \theta(P)$. By Lemma 2.2, L has a θ-stable Levi k-subgroup M. By [3, Corollary 4.18], M contains the centralizer of a maximal k-split torus of G. Clearly $\dim(M) < \dim(G)$. By induction on $\dim(G)$, the assertion is true for M, hence so is for G.

The same argument yields also the following results.

2.4 Lemma. *Every minimal parabolic k-subgroup P of G contains a θ-stable maximal k-split torus of P, unique up to conjugation by an element from $(H \cap R_u(P))_k$.*

Proof. The first statement follows by the same argument as above. Suppose A_1 and A_2 are θ-stable, maximal k-split tori in P and write $U = R_u(P)$. Then $A_2 = uA_1u^{-1}$ with $u \in (U \cap \theta(U))_k$. Since $\theta(A_2) = A_2$ it follows that $u^{-1}\theta(u) \in N_{G_k}(A_1)$. From the uniqueness properties of the Bruhat decomposition, it follows as before that $u^{-1}\theta(u) = e$, hence $u \in (H \cap R_u(P))_k$.

The same argument yields also the following result.

2.5 Lemma. *Let P_1 and P_2 be θ-stable parabolic k-subgroups of G. Then there exists a θ-stable maximal k-split torus of G contained in $P_1 \cap P_2$.*

3. θ-stable parabolic k-subgroups.

In this section, we discuss θ-stable parabolic k-subgroups. In general, such proper subgroups may not exist. We present a simple criterion for their existence and establish some structure properties for such subgroups.

3.1. Let A be a maximal k-split torus of G, $\Phi(G, A)$ the set of roots of A in G and $X_*(A)$ the set of one parameter subgroups of A. By chambers, facets of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, we mean those with respect to the hyperplanes $\ker(\alpha)$, $\alpha \in \Phi(G, A)$. The k-parabolic subgroups of G containing A are in bijective correspondence with the facets of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. Given a facet F, the corresponding parabolic k-subgroup $P(F)$ of G containing A is determined by

$$\Phi(P(F), A) = \{\alpha \in \Phi(G, A) | \langle x, \alpha \rangle \geq 0, \quad x \in F\}.$$

For $\lambda \in X_*(A)$, let $F(\lambda)$ denote the facet containing λ. For simplicity, we write $P(\lambda)$ for the parabolic k-subgroup $P(F(\lambda))$ of G containing A.

3.2. Let A be a θ-stable torus. Let A_+ and A_- denote the maximal subtori of A such that

$$\theta | A_+ = 1, \qquad \theta | A_- = -1,$$

where 1 is the identity map and -1 the map $x \mapsto x^{-1}$. Then we have the decomposition $A = A_+ A_-$.

3.3 Lemma. *Let P be a θ-stable parabolic k-subgroup of G and M a θ-stable Levi k-subgroup of P. Let A be a θ-stable maximal k split torus of M and F the facet with $P = P(F)$. Then we have the following conditions:*

(i) $\theta(F) = F$.
(ii) *There is $\lambda \in X_*(A_+)$ such that $P = P(\lambda)$ and $M = Z_G(\lambda)$.*

Proof. (i) Note that $P(F) = \theta(P(F)) = P(\theta(F))$. Hence $\theta(F) = F$.
(ii) There exists $\tau \in X_*(A) \cap F$. By (i), $\theta(\tau) \in F$ and $\lambda = \tau + \theta(\tau) \in X_*(A_+) \cap F$. Then λ has the desired property.

3.4 Proposition. *G has a proper θ-stable parabolic k-subgroup if and only if $[G, G]^\theta$ is isotropic over k.*

Proof. $\Rightarrow$) $[G, G]$ has a proper θ-stable parabolic k-subgroup. From (ii) of Lemma 3.3, there exists a nontrivial k-split torus contained in $H \cap [G, G]$.

$\Leftarrow$) Let S be a nontrivial k-split torus contained in $H \cap [G, G]$. By Proposition 2.3, there exists a θ-stable maximal k-split torus A contained in $Z_G(S)$. Choose any $0 \neq \lambda \in X_*(S)$. Since $\theta(\lambda) = \lambda$, $P(\lambda)$ is a proper θ-stable parabolic k-subgroup of G containing A.

3.5 Proposition. *Let P be a θ-stable parabolic k-subgroup and M a θ-stable Levi k-subgroup of P. Let A be a θ-stable maximal k-split torus of M. Then P is a minimal θ-stable parabolic k-subgroup of G if and only if*

(i) $M = Z_G(A_+)$,
(ii) *A_+ is a maximal k-split torus of H.*

Proof. $\Rightarrow$) The minimality condition implies that M has no proper θ-stable parabolic k-subgroups. Hence by the preceding proposition, any k-split torus of $H \cap M$ is central in M. By (ii) of Lemma 3.3, one concludes that $M = Z_G(A_+)$.

Now let S be any k-split torus of H containing A_+. Then $S \subset M \cap H$. Since S is central in M, $S \subset A$. It follows readily that $S \subset A_+$. Therefore A_+ is a maximal k-split torus of H.

$\Leftarrow$) By Proposition 3.4, M has no proper θ-stable parabolic k-subgroups. The assertion is now obvious.

3.6 Examples. (i) $G = SL(n)$, $\theta(x) = {}^t x^{-1}$, $H = SO(n)$. Then H is anisotropic over $\mathbb{R}$, and G has no proper θ-stable parabolic $\mathbb{R}$-subgroup.
(ii) $G = SL(2)$, $\theta(x) = axa^{-1}$, $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then H is the subgroup of diagonal matrices. The Borel subgroup of upper triangular matrices is θ-stable.

3.7 Lemma. *Let $L = A \ltimes B$ be a semi-direct product of groups. If C is a subgroup of A, then the normalizer $N_L(C)$ of C in L is given by*

$$N_L(C) = N_A(C) \ltimes Z_B(C).$$

Proof. Given $x, y \in L$, let $[x, y]$ denote the element $xyx^{-1}y^{-1}$. Let $a \in A$ and $b \in B$ such that ab normalizes C. For $c \in C$, $[c, ab] = [c, a] \cdot {}^a[c, b] \in C$. It follows that $[c, b] = e$, $[c, a] \in C$.

3.8 Lemma. *Let P be a minimal θ-stable parabolic k-subgroup of G and A, A' two θ-stable maximal k-split tori of P. Then we have the following conditions:*

(i) $A_+ R_u(P) = A'_+ R_u(P)$.
(ii) There exists $v \in H_k \cap R_u(P)$ with $vA_+v^{-1} = A'_+$.

Proof. By Proposition 3.5, the image of A_+ (resp. A'_+) in $P/R_u(P)$ coincides with $((R(P)/R_u(P))_d)_+$. Hence (i) follows.

By the conjugacy theorem of maximal k-split tori, there exists $v \in R_u(P)_k$ with $vA_+v^{-1} = A'_+$. Since A_+ and A'_+ are both θ-stable,

$$\theta(v)^{-1}v \in N_{R_u(P)}(A_+) = Z_{R_u(P)}(A_+) = \{e\}.$$

Thus $v \in H_k \cap R_u(P)$.

3.9 Lemma. *Let P be a minimal θ-stable parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. If $H^0 \not\subset P$, then $N_{H^0}(A_+)_k \not\subset Z_G(A_+)$.*

Proof. Let P^- be the parabolic k-subgroup of G such that $P^- \cap P = Z_G(A_+)$. Clearly P^- is θ-stable. By Lemma 1.7, P^-H^0 is closed. It follows that $H^0 \cap P^-$ is a parabolic k-subgroup of H^0. If $H^0 \cap R_u(P^-) = \{e\}$, then $H^0 \cap P^- = H^0 \cap Z_G(A_+)$ is reductive and so $H^0 \cap P^- = H^0 \subset Z_G(A_+)$, a contradiction. Choose $e \neq x \in R_u(H^0 \cap P^-)_k \subset R_u(P^-)$. Then $xPx^{-1} \neq P$.

Clearly $P' = xPx^{-1}$ is a θ-stable minimal parabolic k-subgroup of G. By Lemma 2.5, there is a θ-stable maximal k-split torus A' of G contained in $P \cap P'$. We have that

$$P = Z_G(A'_+) \ltimes U, \quad P' = Z_G(A'_+) \ltimes U'.$$

Note that A' and $xA'x^{-1}$ are θ-stable maximal k-split tori of P'. By Lemma 3.8, there exists $v \in H_k \cap U'$ with $v(xA'x^{-1})_+v^{-1} = A'_+$. Set $n = vx$. Then $n \in H^0_k$ and

$$nA'_+n^{-1} = A'_+,$$
$$nPn^{-1} = P' \neq P.$$

Since $Z_G(A'_+) \subset P = N_G(P)$, $n \notin Z_G(A'_+)$. Thus we have the condition $N_{H^0}(A'_+)_k \not\subset Z_G(A'_+)$. However A_+ and A'_+ are maximal k-split tori of H^0. They are conjugate by elements of H^0_k. Therefore the assertion for A_+ follows from that for A'_+.

3.10 Proposition. *Let P be a minimal θ-stable parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Let $N_G(A_+;\theta) = \{g \in N_G(A_+) | \theta(g)^{-1}g \in Z_G(A_+)\}$. Then $N_G(A_+;\theta)_k \not\subset Z_G(A_+)$, if $[G,G] \cap H$ is isotropic over k.*

Proof. We show the assertion in several steps. Let P^- be the opposite parabolic k-subgroup of G with $P^- \cap P = Z_G(A_+)$.

Step 1. Without losing any generality, we may assume that G is semi-simple and has no anisotropic factors over k. Note that P^- is θ-stable. By Lemma 3.9, we may assume further that

$$H^0 \subset P \cap P^- = Z_G(A_+).$$

Step 2. Set $U = R_u(P)$. Then $\Phi(U, A)$ and $\Phi(Z_G(A_+), A)$ are orthogonal to each other.

Note that $P = Z_G(A_+) \ltimes U$ and $H^0 \subset Z_G(A_+)$. It follows that $U^\theta = \{e\}$. By Lemma 0.6,

$$\theta(x) = x^{-1}, \qquad x \in U.$$

This yields that $\theta | \Phi(U, A) = 1$. On the other hand, $A = A_+A_- \subset Z_G(A_+)$, so $\theta | \Phi(Z_G(A_+), A) = -1$. Thus $\Phi(U, A)$ and $\Phi(Z_G(A_+), A)$ are orthogonal to each other.

Step 3. Choose a minimal k-parabolic subgroup P_0 of P containing A. Let Δ be the set of simple roots of $\Phi(P_0, A)$. Let ψ be the subset of Δ given by

$$\psi = \{\alpha \in \Delta | \alpha | A_+ = 0\}.$$

By Step 2, ψ and $\Delta - \psi$ are orthogonal to each other. Now let Φ_1 and Φ_2 denote the subsystems of $\Phi(G, A)$ consisting of integral combinations of ψ and $\Delta - \psi$ respectively.

Then we have that

$$\Phi(G, A) = \Phi_1 \cup \Phi_2.$$

Clearly Φ_1 and Φ_2 are ideals of $\Phi(G, A)$. Let $G_1 = G^*_{\Phi_1}$ and $G_2 = G^*_{\Phi_2}$ be the subgroups ([3, 3.8]) of G corresponding Φ_1 and Φ_2 respectively. It follows that $G = G_1 {\cdot} G_2$ is an almost direct product.

Step 4. From the definition of Φ_1, $G_1 \subset Z_G(A_+)$. Then $P = G_1 \cdot (P \cap G_2)$ and $P^- = G_1 \cdot (P^- \cap G_2)$. Observe that $\theta | \Phi(G_2, A) = 1$ from Step 2. The minimality of P implies that $P \cap G_2$ and $P^- \cap G_2$ are opposite minimal parabolic k-subgroups of G_2. Thus there exists $g \in (G_2)_k$ such that

$$gPg^{-1} = P^-,$$
$$gAg^{-1} = A.$$

Since P, P^- and A are θ-stable, we have that $\theta(g)^{-1}g \in N_P(A)$. Note that $P = Z_G(A_+) \ltimes U$ and $Z_U(A) = \{e\}$. By Lemma 3.7, $\theta(g)^{-1}g \in N_{Z_G(A_+)}(A)$. It follows that $\theta(gag^{-1}) = gag^{-1}$, $a \in A_+$. By the definition of A_+, $gA_+g^{-1} = A_+$. Therefore

$$g \in N_G(A_+; \theta)_k.$$

Observe that $Z_G(A_+) \subset P$ and $gPg^{-1} = P^- \neq P$. We conclude readily that $g \notin Z_G(A_+)$.

4. θ-split parabolic k-subgroups.

Let G be a connected reductive algebraic k-group and θ an involution of G defined over k. Recall that a parabolic subgroup P of G is θ-split if P and $\theta(P)$ are opposite parabolic subgroups. Here we study θ-split parabolic k-subgroups. We follow the discussion given by Vust in [29].

4.1. Let $G = Z \cdot G_1 \cdot G_2$ denote the almost direct product of k-groups where Z is the maximal central torus, G_1 is semi-simple anistropic over k and G_2 has no anistropic factors over k. We call G_2 (resp. Z, G_1) the isotropic (resp. central, anistropic) factor of G. These factors are invariant under any k-automorphism of G.

4.2. Given a maximal k-split torus A of G, let A' (resp. A'') denote the identity component of $A \cap Z(G)$ (resp. $A \cap [G, G]$). Then A' is invariant under any k-automorphism of G.

4.3 Proposition. *Let θ be an involution of a connected algebraic reductive k-group G defined over k. The following conditions are equivalent:*

(i) *θ is trivial on the isotropic factor of G*
(ii) *For any θ-stable maximal k-split torus A of G, $A'' \subset H$.*
(iii) *Any parabolic k-subgroup of G is θ-stable.*
(iv) *Any maximal k-split torus of G is θ-stable.*

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii). Let P be a parabolic k-subgroup of G. By Lemma 2.4, there exists a θ-stable maximal k-split torus A of P. Since $\theta|A'' = 1$, it follows readily that P is θ-stable.
(iii) $\Rightarrow$ (iv). Let A be a maximal k-split torus of G. Let P and P^- denote a pair of opposite minimal parabolic k-subgroups of G such that $P \cap P^- = Z_G(A)$. As P and P^- are θ-stable, so is $Z_G(A)$. This implies easily that A is θ-stable.

(iv) $\Rightarrow$ (i) Let A be any maximal k-split torus of G. For $x \in G_k$, xAx^{-1} and A are θ-stable. It follows that $x^{-1}\theta(x) \in N_G(A)$, $x \in G_k$. Now let P and P^- be opposite minimal parabolic k-subgroups of G containing A, $U = R_u(P)$ and $U^- = R_u(P^-)$. Given $u \in U_k$ (resp. U_k^-), $u^{-1}\theta(u) \in N_G(A) \cap U\theta(U)$ (resp. $N_G(A) \cap U^-\theta(U^-)$). By [3, 5.15], $N_G(A) \cap U\theta(U) = \{e\}$. As a consequence $\theta|U_k = 1$ and $\theta|U_k^- = 1$. By [3, 3.23],

U and U^- are θ-stable. Let $\alpha_1, \ldots, \alpha_m$ denote the elements of $\Phi(P, A)$ arranged by an increasing order and set $\Phi_{m-i+1} = \{\alpha_i, \ldots, \alpha_m\}$. For each i, Φ_i is an ideal of $\Phi(P, A)$. Set $\psi_i = \Phi_i \cup \theta(\Phi_i)$. We have θ-stable central series $U_{\psi_1} \subset U_{\psi_2} \cdots \subset U_{\psi_m} = U$. Note that $U_{\psi_{i+1}}/U_{\psi_i}$ is abelain and k-split for each i. By Lemma 0.6 and an easy induction, $\theta|U_k = 1$ implies $\theta|U_{\psi_i} = 1$ for all i. Similarly we have $\theta|U^- = 1$. Thus assertion (i) follows.

4.4. In the sequel, we assume that θ is nontrivial on the isotropic factor of G over k. A torus S of G is called (θ, k)-split if S is k-split and $\theta(s) = s^{-1}$, $s \in S$. By Proposition 4.3, there exist nontrivial (θ, k)-split tori of $[G, G]$.

4.5 Lemma. *Let S be a maximal (θ, k)-split torus of G. Let C, M_1, M_2 denote the central, anisotropic and isotropic factors of $Z_G(S)$ over k respectively. Then we have the following conditions:*

(i) *S is the unique maximal (θ, k)-split torus of $Z_G(S)$.*
(ii) *$M_2 \subset H$.*
(iii) *If A is any maximal k-split torus of $Z_G(S)$, then A is θ-stable and moreover $CM_1 \subset Z_G(A)$.*

Proof. (i) is obvious. (ii) is immediate from Proposition 4.3. (iii) Let C_d be the splitting component of C. We have that $C_d \subset A \subset C_d M_2$. By (ii), A is θ-stable. Observe that $C\,M_1$ centralizes $C\,M_2$. Hence $C\,M_1$ centralizes A.

4.6 Lemma. *Let P be a θ-split parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Then there exists $\lambda \in X_*(A_-)$ such that $P = P(\lambda)$ and $P \cap \theta(P) = Z_G(\lambda)$.*

Proof. Since A is θ-stable, $A \subset \theta(P)$. Let F be the facet of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $P = P(F)$. Note that $\theta(P) = P(-F)$. Hence $\theta(F) = -F$. Now choose $\tau \in X_*(A) \cap F$. Since $\tau, -\theta(\tau) \in F$ and F is a convex cone, $\lambda = \tau - \theta(\tau) \in F$. Then λ has the desired property.

4.7 Propositon. *Let P be a θ-split parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Then the following conditions are equivalent:*

(i) *P is a minimal θ-split parabolic k-subgroup of G.*
(ii) *$P \cap \theta(P)$ has no proper θ-split parabolic k-subgroups.*
(iii) *θ is trivial on the isotropic factor of $P \cap \theta(P)$ over k.*
(iv) *A_- is a maximal (θ, k)-split torus of G and $Z_G(A_-) = P \cap \theta(P)$.*

Proof. (i) $\Leftrightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii). Let C, M_1 and M_2 be the central, anisotropic and isotropic factors of $P \cap \theta(P)$ over k respectively. Suppose that $\theta|M_2$ is not trivial. By Proposition 4.3, M_2 has a nontrivial (θ, k)-split torus S. Choose a θ-stable maximal k-split torus of A_1 of M_2 containing S. Let η be a nontrivial element of $X_*(S)$ and $Q(\eta)$ the parabolic k-subgroup of M_2 containing A_1 given by $\Phi(Q(\eta), A_1) = \{\alpha \in \Phi(M_2, A_1) | \langle \eta, \alpha \rangle \geq 0\}$. Note that $\theta(Q(\eta)) = Q(-\eta)$ and $C\,M_1\,Q(\eta)$ is a proper θ-split parabolic k-subgroup of $P \cap \theta(P)$. Clearly this is a contradiction. Thus $\theta|M_2$ is trivial.
(iii) $\Rightarrow$ (iv). From (iii), we have the condition that
(1) (θ, k)-split tori of $P \cap \theta(P)$ are central in $P \cap \theta(P)$.

Thus $P \cap \theta(P) \subset Z_G(A_-)$. By Lemma 4.6, there exists $\lambda \in X_*(A_-)$ such that $Z_G(\lambda) = P \cap \theta(P)$. Clearly $Z_G(A_-) \subset Z_G(\lambda)$. Hence $P \cap \theta(P) = Z_G(A_-)$. Condition (1) yields readily that A_- is a maximal (θ, k)-split torus of G.

(iv) $\Rightarrow$ (ii). By (ii) of Proposition 4.3, θ is trivial on M_2. It follows that any parabolic k-subgroup of $P \cap \theta(P)$ is θ-stable. Thus (ii) follows.

4.8 Lemma. *let P be a minimal θ-split parabolic k-subgroup of G and P_0 a minimal parabolic k-subgroup of G contained in P. Then we have the following conditions:*

(i) $H^0 P = H^0 P_0$.

(ii) $H^0 P_0$ *is open in G.*

Proof. Let A be a θ-stable maximal k-split torus of P_0. By (iv) of Proposition 4.7, A_- is a maximal (θ, k)-split torus of G and $P = Z_G(A_-) \ltimes R_u(P)$. From (ii) and (iii) of Lemma 4.5, $H^0 Z_G(A_-) = H^0 Z_G(A)$. Note that $P = Z_G(A_-)P_0$. Then assertion (i) is immediate. Condition (ii) follows from [29, 1.3. Theorem].

4.9 Proposition. *Let P_1 and P_2 be minimal θ-split parabolic k-subgroups of G and P_0', P_0'' minimal parabolic k-subgroups of G contained in P_1 and P_2 respectively. If $g \in G_k$ satisfies $gP_0'g^{-1} = P_0''$, then $gP_1g^{-1} = P_2$.*

Proof. By (ii) of Lemma 4.8, $H^0 P_0'$ and $H^0 g P_0' g^{-1}$ are both open in G. This yields that $H^0 P_0'$ and $H^0 g P_0'$ are the same open orbit of P_0' in $H^0 \backslash G$. Hence $g \in G_k \cap H^0 P_0'$. It follows that gP_1g^{-1} is a θ-split parabolic k-subgroup of G containing P_0''. Now let A be a θ-stable maximal k-split torus of P_0'' and $\lambda_1, \lambda_2 \in X_*(A_-)$ such that $P(\lambda_1) = gP_1g^{-1}$, $P(\lambda_2) = P_2$. Since $P(\lambda_1) \cap P(\lambda_2) \supset P_0''$, by the same argument as [29, 1.2. Prop. 4] we have that $gP_1g^{-1} \cap P_2 = P(\lambda_1 + \lambda_2)$. Clearly $P(\lambda_1 + \lambda_2)$ is a θ-split parabolic k-subgroup of G. By the minimality condition of P_2, $P_2 = P(\lambda_1 + \lambda_2) \subset gP_1g^{-1}$. By symmetry, we also have that $P_1 \subset g^{-1}P_2 g$. Thus $gP_1g^{-1} = P_2$.

4.10 Lemma. *Let S be a maximal (θ, k)-split torus of G, A a maximal k-split torus of G containing S and $N(S, A, \theta)$ denote the subset of G defined by*

$$N(S, A, \theta) = \{g \in N_G(S) \cap N_G(A) | g^{-1}\theta(g) \in Z_G(S)\}.$$

Then $N(S, A, \theta)_k \not\subset Z_G(S)$.

Proof. Choose $\lambda \in X_*(S)$ such that $P(\lambda)$ is a minimal θ-split parabolic k-subgroup of G containing A. By the preceding proposition, there exists $g \in G_k$ such that

$$gP(\lambda)g^{-1} = P(-\lambda) \quad \text{and} \quad gAg^{-1} = A.$$

Since $\Phi(P(-\lambda), A) = -\Phi(P(\lambda), A)$, g also satisfies

$$gP(-\lambda)g^{-1} = P(\lambda).$$

Applying θ to the above relation, we obtain $g^{-1}\theta(g) \in P(\lambda)$. Observe that $P(\lambda) = Z_G(S) \ltimes R_u(P(\lambda))$. By Lemma 3.7,

$$g^{-1}\theta(g) \in N_{P(\lambda)}(A) = N_{Z_G(S)}(A).$$

This implies that gSg^{-1} is a (θ, k)-split torus of A. By the maximality condition of S, we have that $gSg^{-1} = S$. Hence $g \in N(S, A, \theta)_k$. Note that $Z_G(S) \subset P(\lambda)$. Since $gP(\lambda)g^{-1} = P(-\lambda) \neq P(\lambda)$, $g \notin Z_G(S)$.

4.11 Proposition. *Let P be a minimal θ-split parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Then the following conditions are equivalent:*

(i) $g \in G_k \cap HP$.
(ii) $g \in \{x \in G_k | x^{-1}\theta(x) \in N_{Z_G(A_-)}(A)\} P_k$.
(iii) $g \in G_k$ *and* gPg^{-1} *is a* θ*-split parabolic* k*-subgroup of* G.

Proof. (i) $\Rightarrow$ (ii). By lemma 2.4, there exists a θ-stable maximal k-split torus A_1 of gPg^{-1}. Choose $p \in P_k$ such that $gpA(gp)^{-1} = A_1$. Set $g_1 = gp$. It follows that

$$g_1^{-1}\theta(g_1) \in N_G(A)_k \cap P\theta(P).$$

Note that $P = Z_G(A_-) \ltimes U$ and $\theta(P) = Z_G(A_-) \ltimes U^-$. From the Bruhat decomposition, we have that $N_G(A)_k \cap P\theta(P) = N_G(A)_k \cap Z_G(A_-)$. Hence (ii) follows.

(ii) $\Rightarrow$ (iii). Write $g = \nu p$ such that $p \in P_k$, $\nu \in G_k$ and $\nu^{-1}\theta(\nu) \in N_{Z_G(A_-)}(A)$. Then $gPg^{-1} = \nu P \nu^{-1}$. Observe that $\theta(\nu P \nu^{-1}) = \nu(\nu^{-1}\theta(\nu)\,\theta(P)\theta(\nu)^{-1}\nu)\nu^{-1} = \nu\theta(P)\nu^{-1}$. Hence gPg^{-1} is a θ-split parabolic subgroup of G. Since $g \in G_k$, gPg^{-1} is defined over k.

(iii) $\Rightarrow$ (i). By (ii) of Lemma 4.8, $HgPg^{-1}$ is open in G. Hence $HgP = HP$ and as a consequence $g \in G_k \cap HP$.

4.12 An example. In general, minimal θ-split parabolic k-subgroups are not conjugate to one another by elements of H_k. In this following, we present a simple example. Let $G = SL(2)/\mathbb{Q}$, $\theta(x) = {}^t x^{-1}$, $B =$ the Borel subgroup of upper triangular matrices and A the group of diagonal matrices. Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$, then $g^{-1}\theta(g) \in A_- = A = N_{Z_G(A_-)}(A)$ if and only if $ab + cd = 0$.

Now choose $u, v, t \in \mathbb{Q}$ satisfying the conditions:

$$(1) \qquad \begin{array}{ll} (i) & uv(1+t^2) = 1. \\ (ii) & 1 + t^2 \text{ is not a square of a rational number.} \end{array}$$

Set $g = \begin{pmatrix} ut & -v \\ u & vt \end{pmatrix}$. Then

$$g^{-1}\theta(g) = \begin{pmatrix} v^2(1+t^2) & 0 \\ 0 & u^2(1+t^2) \end{pmatrix}.$$

By Proposition 4.11, gBg^{-1} is a θ-split parabolic $\mathbb{Q}$-subgroup of G. However if $y \in H_{\mathbb{Q}}B_{\mathbb{Q}}$, then $y^{-1}\theta(y) \in A$ implies that $y^{-1}\theta(y) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a \in (\mathbb{Q}^\times)^2$. By condition (ii) of (1), gBg^{-1} is not a conjugate of B by an element of $H_{\mathbb{Q}}$.

5. The root systems associated to θ.

Here we study two root systems associated to θ. Our basic reference for root systems is Bourbaki [4].

5.1. Let V be a finite dimensional real vector space, $\Phi \subset V$ a root system and $W(\Phi)$ the Weyl group of Φ generated by the reflections s_α, $\alpha \in \Phi$. Let $\langle\ ,\ \rangle$ denote a fixed $W(\Phi)$-invariant inner product of V.

5.2. Let V_1 be a subspace of V and $\pi : V \to V_1$ the orthogonal projection. Set

$$\Phi_1 = \pi(\Phi) - \{0\}.$$

Given $\beta \in \Phi_1$, let (β) denote the subset of Φ consisting of $\alpha \in \Phi$ such that $\pi(\alpha)$ is an integral multiple of β. Let $V_1(\beta)$ denote the orthogonal complement of β in V_1, i.e., $V_1(\beta) = \{x \in V_1 | \langle x, \beta\rangle = 0\}$. Φ_1 is *admissible* if for any $\beta \in \Phi_1$, there exists $w_\beta \in W(\Phi)$ satisfying the following conditions:

$$(1) \qquad \begin{array}{ll} (i) & w_\beta \text{ leaves } V_1 \text{ invariant, } w_\beta|V_1(\beta) = 1, \text{ and } w_\beta|V_1 \neq 1. \\ (ii) & \text{Given } \alpha \in \Phi,\ w_\beta(\alpha) = \alpha + \text{ integral combination of } (\beta). \end{array}$$

5.3 Lemma. *Let $W_1 = \{w \in W(\Phi) | w(V_1) = V_1\}$, $W_2 = \{w \in W_1 | w|V_1 = 1\}$ and $W_0 = W_1/W_2$. If Φ_1 is admissible, then W_0 is generated by the reflections s_β, $\beta \in \Phi_1$.*

Proof. Given any $\beta \in \Phi_1$, let $w_\beta \in W(\Phi)$ satisfy the condition (5.2.1). Since w_β is an isometry and $V_1(\beta)$ is of codimension 1 in V_1, (i) of (5.2.1) implies that

$$(1) \qquad w_\beta | V_1 = s_\beta.$$

Now let W_0' be the group generated by s_β, $\beta \in \Phi_1$. Consider the chambers and facets in V_1 (resp. V) with respect to the hyperplanes given by $\beta \in \Phi_1$ (resp. $\alpha \in \Phi$). By [4, Chapter 4, §3.1, Lemma 2], W_0' acts transitively on the set of the chambers of V_1. Note that $W_0' \subset W_0$. It suffices to verify that if C_1 is a chamber of V_1 and $w \in W_1$ with $w(C_1) = C_1$, then $w \in W_2$. Every hyperplane in V_1 is the intersection of V_1 with a hyperplane in V. It follows that there is a unique facet D in V containing C_1. Then $w(D) = D$ and by [4, Chapter 5, §3.3, Prop. 1]

$$w(x) = x, \quad x \in D.$$

Since C_1 is open in V_1, $w|V_1 = 1$. Hence $w \in W_2$.

5.4. Lemma. *If Φ_1 is admissible, then $2\langle\beta, \gamma\rangle\langle\beta, \beta\rangle^{-1} \in \mathbb{Z}$ for $\beta, \gamma \in \Phi_1$.*

Proof. Let $w_\beta \in W(\Phi)$ satisfy (5.2.1). Choose $\alpha \in \pi^{-1}(\gamma) \cap \Phi$. By (ii) of (5.2.1), $w_\beta(\alpha) = \alpha +$ integral combination of (β). Observe that $w_\beta|V_1 = s_\beta$ (5.3.1). Applying the projection π, we have that $s_\beta(\gamma) = \gamma +$ integral multiple of β. Hence the assertion follows.

5.5. Proposition. *Let V_1 and Φ_1 be as in 5.2. If Φ_1 is admissible, then Φ_1 is a root system in V_1.*

Proof. The assertion is immediate from Lemmas 5.3 and 5.4.

Let G be a connected reductive algebraic k-group and θ an involution of G defined over k. Now we apply Proposition 5.5 to investigate root systems associated to θ.

5.6. Let A be a maximal k-split torus of G and $W = N_G(A)/Z_G(A)$. Then W acts on $X_*(A)$ and $X^*(A)$. For $\lambda \in X_*(A)$ and $\alpha \in X^*(A)$, $(\alpha, \lambda) \mapsto \alpha \circ \lambda$ defines a

W-invariant pairing of $X^*(A)$ and $X_*(A)$. Now let $\langle\ ,\ \rangle$ be a fixed W-invariant inner product of $X^*(A)\otimes\mathbb{Q}$. We identify $X_*(A)\otimes\mathbb{Q}$ with $X^*(A)\otimes\mathbb{Q}$ by the map

$$\iota : X_*(A)\otimes\mathbb{Q}\to X^*(A)\otimes\mathbb{Q}$$

such that $\langle\iota(\lambda),\alpha\rangle=\alpha\circ\lambda$, $\lambda\in X_*(A)$, $\alpha\in X^*(A)$. Clearly ι is W-equivariant.

Let B be a subtorus of A. Clearly $X_*(B)\subset X_*(A)$. We identify $X_*(B)\otimes\mathbb{Q}$ with $X^*(B)\otimes\mathbb{Q}$ by the map

$$\tau : X_*(B)\otimes\mathbb{Q}\to X^*(B)\otimes\mathbb{Q}$$

such that $\langle\iota(\beta),\iota(\gamma)\rangle=\tau(\beta)\circ\gamma$, $\beta,\gamma\in X_*(B)$. Consider the map $\pi : X^*(A)\otimes\mathbb{Q}\to X^*(B)\otimes\mathbb{Q}$ defined by $\pi(\alpha)=\alpha|B$, $\alpha\in X^*(A)$. Identify $X^*(B)\otimes\mathbb{Q}$ with $\iota(X_*(B)\otimes\mathbb{Q})$ via $\iota\circ\tau^{-1}$. One checks easily that π is the orthogonal projection.

5.7. Proposition. *Assume that $H\cap[G,G]$ is isotropic over k. Let A_1 be a maximal k-split torus of H and $A_0=(A_1\cap[G,G])^0$. Let $\Phi(G,A_1)$ denote the set of roots of A_1 in G. Then $\Phi(G,A_1)$ is a root system in $(X^*(A_1),X_0)$ in the sense of* [3, §2.1] *where X_0 is the set of characters of A_1 which are trivial on A_0; moreover the Weyl group $W(G,A_1)$ of $\Phi(G,A_1)$ coincides with $N_G(A_1)_k/Z_G(A_1)_k$.*

Proof. Without losing any generality, we may assume that G is semi-simple. Let A be a θ-stable maximal k-split torus of G containing A_1. Then $A_+=A_1$. Given $\beta\in\Phi(G,A_1)$, let $\psi(\beta)$ denote the subset of $\Phi(G,A)$ consisting of $\alpha\in\Phi(G,A)$ such that $\alpha|A_1$ is an integral multiple of β. Clearly $\psi(\beta)$ is a closed and symmetric subset of $\Phi(G,A)$. Let $G_{\psi(\beta)}$ be the group ([3, §3.8]) determined by $\psi(\beta)$. It is a θ-stable reductive k-subgroup of G. The group $G_{\psi(\beta)}$ satisfies the following conditions:

$$(1)\qquad \begin{aligned}(i)&\qquad G_{\psi(\beta)}\subset Z_G(\ker(\beta)).\\ (ii)&\qquad G_{\psi(\beta)}\not\subset Z_G(A_1).\end{aligned}$$

By Proposition 3.4, there exists a proper θ-stable parabolic k-subgroup of $G_{\psi(\beta)}$. Now according to Proposition 3.10, there exists $n\in N_G(A)_k\cap N_{G_{\psi(\beta)}}(A_1)$ with $n\notin Z_G(A_1)$. Let w_β denote the image of n in $W(G,A)$. Since $W(G_{\psi(\beta)},A)$ is generated by the reflections s_α, $\alpha\in\psi(\beta)$,

$$(2)\qquad w_\beta(\alpha)=\alpha+\text{ integral combination of }\psi(\beta).$$

Now set $V=X^*(A)\otimes\mathbb{R}$ and $V_1=X^*(A_1)\otimes\mathbb{R}$. Identify V_1 with a subspace of V as in 5.6. Note that $n\in Z_G(\ker(\beta))$ and $n\notin Z_G(A_1)$. Let $V_1(\beta)$ denote the orthogonal complement of β in V_1. It follows that w_β leaves invariant V_1, $w_\beta|V_1(\beta)=1$ and $w_\beta|V_1\neq 1$. Hence $\Phi(G,A_1)$ is admissible. By Proposition 5.5 and Lemma 5.3, our assertions follow immediately.

5.8. Corollary. *Let P_1 and P_2 be minimal θ-stable parabolic k-subgroups of G. Let A_1 and A_2 be θ-stable maximal k-split tori of P_1 and P_2 respectively. Then there exists $g\in(N_G(A_1)\cap N_G((A_1)_+))_kH_k^0$ such that $g^{-1}P_1g=P_2$.*

Proof. Let $S_1=(A_1)_+$ and $S_2=(A_2)_+$. By Proposition 3.5, $P_1=Z_G(S_1)\ltimes R_u(P_1)$, $P_2=Z_G(S_2)\ltimes R_u(P_2)$. By a conjugation of element in H_k^0, we may assume that $S_1=S_2$. Then $\Phi(P_1,S_1)$ and $\Phi(P_2,S_1)$ are two positive root systems of $\Phi(G,S_1)$. It follows that there exists $n\in(N_G(A_1)\cap N_G(S_1))_k$ such that its image w in the Weyl group takes $\Phi(P_1,S_1)$ to $\Phi(P_2,S_1)$. Then we have that $nP_1n^{-1}=P_2$.

5.9. Proposition. *Assume that θ is nontrivial on the isotropic factor of G over k. Let A_1 be a maximal (θ, k)-split torus of G. Then $\Phi(G, A_1)$ is a root system in $(X^*(A_1), X_0)$ where X_0 is the set of characters of A_1 which are trivial on $(A_1 \cap [G, G])^0$; moreover the Weyl group of $\Phi(G, A_1)$ is $N_G(A_1)_k/Z_G(A_1)_k$.*

Proof. The assertion follows from Lemma 4.10 and an argument similar to that of Proposition 5.7.

Remark. When k is algebraically closed, Proposition 5.9 is due to Richardson [21, §4]. Our Proposition 5.5 is an abstraction of his presentation of the argument of Borel-Tits.

6. Double coset decomposition.

Let G be a connected reductive algebraic k-group, θ an involution of G defined over k and P a minimal parabolic k-subgroup of G. Let H be an open k-subgroup of the fixed point group G^θ of θ. In this section, we discuss the double coset decomposition $H_k\backslash G_k/P_k$.

6.1. Lemma. *Given $y \in G$, set $\theta_y(x) = y\theta(x)y^{-1}$, $x \in G$. Then θ_y is an involution of G if and only if $y\theta(y) \in Z(G)$.*

Proof. Note $\theta_y^2(x) = zxz^{-1}$ where $z = y\theta(y)$. The assertion is obvious.

6.2. Let Q (resp. Q') denote the subset of G defined by

$$Q = \{g^{-1}\theta(g) | g \in G\},$$
$$Q' = \{g \in G | \theta(g) = g^{-1}\}.$$

The set Q is contained in Q'. Given $y \in Q'$ and $z \in G$, the element zy lies in Q' if and only if $\theta_y(z) = z^{-1}$.

6.3. Twisted action. Given $g, x \in G$, the twisted action associated to θ is given by $(g, x) \mapsto g * x = gx\theta(g)^{-1}$. It is known [20, §9] that there are only finite number of twisted G-orbits in Q' and each such orbit is closed. In particular, Q is a connected closed k-subvariety of G.

6.4. Lemma. *Let L be a θ-stable k-subgroup of G, $n \in Q' \cap N_G(L)_k$ and $Q'(n, L) = \{x \in L_k | \theta_n(x) = x^{-1}\}$. Then we have the following conditions*

(i) *$L_k \cdot n \cap Q' = Q'(n, L)n$ is L_k-stable.*
(ii) *There is a bijection between the twisted L_k-orbits (associated to θ) in $L_k \cdot n \cap Q'$ and the twisted L_k-orbits (associated to θ_n) in $Q'(n, L)$.*

Proof. (i) is immediate from 6.2.

(ii) Let $x, y \in L_k$ with $xn \in Q'$. We have that $y(xn)\theta(y)^{-1} = yx\theta_n(y)^{-1}n$. By (i), the assertion is now obvious.

6.5. Let P be a minimal parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Let P^- denote the parabolic k-subgroup of G such that $P \cap P^- = Z_G(A)$. Set $N = N_G(A)$, $U = R_u(P)$ and $U^- = R_u(P)$. By [3, 5.15 Theorem], we have the following conditions:

(i) $G_k = U_k\, N_k\, \theta(U_k)$.
(ii) The map $N \to U\backslash G/\theta(U)$ (resp. $N \to \theta(U)\backslash G/U$) is an injection.

6.6. Proposition. [23] *If $g \in G_k$ satisfies $\theta(g) = g^{-1}$, then there exists $x \in U_k$ such that $xg\,\theta(x)^{-1} \in N_G(A)$.*

Proof. By (i) of 6.5, we write

$$g = u_1 n u_2,$$

with $u_1 \in U_k$, $u_2 \in \theta(U_k)$ and $n \in N_k$. We may assume that

$$n^{-1}u_1 n \in \theta(U^-). \tag{1}$$

We have the condition $\theta(g) = \theta(u_1)\theta(n)\theta(u_2) = u_2^{-1}n^{-1}u_1^{-1}$. Note that $\theta(u_1)$, $u_2^{-1} \in \theta(U_k)$, $\theta(u_2)$, $u_1^{-1} \in U_k$ and $\theta(n)$, $n^{-1} \in N_k$. By (ii) of 6.5, the N_k-component is unique. Hence we have that $\theta(n) = n^{-1}$. Now set $\theta(u_2)^{-1} = v_1 v_2$ such that

$$\begin{aligned} &v_1 \in U_k \cap n\theta(U^-)n^{-1}, \\ &v_2 \in U_k \cap n\theta(U)n^{-1}. \end{aligned} \tag{2}$$

From $\theta(g) = g^{-1}$, we have that $\theta(u_1)n^{-1}v_2^{-1}v_1^{-1} = \theta(v_1)\theta(v_2)n^{-1}u_1^{-1}$. It follows that $(\theta(u_1)n^{-1}v_2^{-1}n)n^{-1}v_1^{-1} = (\theta(v_1)\theta(v_2))n^{-1}u_1^{-1}$. Observe that $\theta(u_1)n^{-1}v_2^{-1}n, \theta(v_1)\theta(v_2) \in \theta(U)$ and $n^{-1}v_1^{-1}n$, $n^{-1}u_1^{-1}n \in \theta(U^-)$ by (1) and (2). Thus $v_1 = u_1$ and $\theta_n(v_2) = v_2^{-1}$. This implies that

$$v_1^{-1}g\theta(v_1) = v_2 n.$$

The unipotent group $U \cap n\theta(U)n^{-1}$ is defined over k and θ_n-stable. By Lemma 0.6, there exists $y \in U_k \cap n\theta(U)n^{-1}$ with $v_2 = y\theta_n(y)^{-1}$. Then

$$\begin{aligned} (v_1 y)^{-1} g\,\theta(v_1 y) &= y^{-1}v_2 n\,\theta(y) \\ &= y^{-1}v_2\theta_n(y)n \\ &= n. \end{aligned}$$

Clearly the element $x = (v_1 y)^{-1} \in U_k$ has the desired property.

Remark. When k is algebraically closed and P is θ-stable, the result is [23, Lemma 4.1 (i)]. In general, G does not have θ-stable minimal parabolic k-subgroups of G. Here we use the pair P, $\theta(P)$ for the special role of a θ-stable Borel subgroup in [23]. Our argument is only a slight refinement of that of Springer.

6.7. Let $\tau : G \to G$ be the map given by

$$\tau(x) = x^{-1}\theta(x), \quad x \in G.$$

By abuse of notation, set $(\tau^{-1}N)_k = \{g \in G_k | g^{-1}\theta(g) \in N_k\}$. Then the group $H_k \times Z_G(A)_k$ acts on $(\tau^{-1}N)_k$ by $(x, y)z = xzy^{-1}$, $(x, y) \in H_k \times Z_G(A)_k$, $z \in (\tau^{-1}N)_k$. Let V denote the set of orbits of $H_k \times Z_G(A)_k$ in $(\tau^{-1}N)_k$. We identify V with a fixed set of the representatives of the orbits in $(\tau^{-1}N)_k$.

6.8. Proposition. *G_k is the disjoint union of the double cosets $H_k v P_k$, $v \in V$.*

Proof. By Proposition 6.6, we have that $G_k = (\tau^{-1}N)_k U_k$. Hence it remains to show that the cosets $H_k v P_k$, $v \in V$ are disjoint. Suppose that $g_1, g_2 \in (\tau^{-1}N)_k$ such that $g_2 \in H_k g_1 P_k$. Write $g_2 = xg_1 y$ with $x \in H_k$ and $y \in P_k$. Since $P_k = Z_G(A)_k \ltimes U_k$, $y = zu$ with $z \in Z_G(A)_k$ and $u \in U_k$. Then

$$g_2^{-1}\theta(g_2) = u^{-1}(g_1 z)^{-1}\theta(g_1 z)\theta(u) \in U_k N_k \theta(U_k).$$

By (ii) of 6.5, $g_2^{-1}\theta(g_2) = (g_1 z)^{-1}\theta(g_1 z)$. It follows that $g_2 = x_1 g_1 z$ with $x_1 \in G_k^\theta$. By our choice, $g_2 = xg_1 zu$. Thus $x^{-1}x_1 = g_1 zu(g_1 z)^{-1} \in g_1 z U_k (g_1 z)^{-1}$. From Lemma 10.1, $x^{-1}x_1 \in H_k^0$ and as a consequence $x_1 \in H_k$. This yields readily that $g_2 \in H_k g_1 Z_G(A)_k$.

6.9. The above description of the double cosets is a slight refinement of Springer [23]. We can also formulate this characterization in a slightly different way, generalizing the characterizations of Rossmann [22] and Matsuki [15] for $k = \mathbb{R}$. Namely, let $\mathcal{A}$ be the set of θ-stable maximal k-split tori of G. The group H_k acts on $\mathcal{A}$ on the left:

$$h \cdot A = hAh^{-1}, \quad h \in H_k,\ A \in \mathcal{A}.$$

Consider the map

$$\pi : H_k \setminus G_k/P_k \to H_k \setminus \mathcal{A}$$

sending $H_k g P_k$ to the H_k-conjugate class [A] of a θ-stable maximal k-split torus A in gPg^{-1}. For a θ-stable maximal k-split torus A of G, choose $g \in G_k$ with $A \subset gPg^{-1}$. Since minimal parabolic k-subgroups of G containing A are conjugate under $N_{G_k}(A)$, the fiber $\pi^{-1}([A])$ is given

$$\pi^{-1}([A]) = H_k \setminus H_k N_{G_k}(A) g P_k/P_k.$$

Let A_1 be a fixed θ-stable maximal k-split torus of P. We may assume that $A = gA_1 g^{-1}$. For $n, n_1 \in N_{G_k}(A)$ with $n_1 g \in H_k ng P_k$, write

$$n_1 g = hngzu$$

with $h \in H_k$, $z \in Z_G(A_1)_k$ and $u \in R_u(P)_k$. Then $ngA_1 g^{-1}n^{-1}$ and $ngzuA_1(ngzu)^{-1}$ are θ-stable maximal k-split tori of $ngPg^{-1}n^{-1}$. By Lemma 2.4, there exists $v \in R_u(P)_k$ such that

(i) $ngvg^{-1}n^{-1} \in G_k^\theta$

(ii) $(ngv)A_1(ngv)^{-1} = (ngzu)A_1(ngzu)^{-1}$.

From (ii) $zuz^{-1} = v$ and as a consequence

$$ngzu = (ngvg^{-1}n^{-1}) \cdot ngz.$$

By (i) and Lemma 10.1, $ngvg^{-1}n^{-1} \in H_k^0$ and so

$$n_1 \in H_k n Z_{G_k}(A).$$

This shows readily that

$$\pi^{-1}([A]) \simeq W_{H_k}(A) \setminus W_{G_k}(A),$$

where $W_{G_k}(A) = N_{G_k}(A)/Z_{G_k}(A)$ and $W_{H_k}(A) = N_{H_k}(A)/Z_{H_k}(A)$. This gives the following characterization of the double coset decomposition of G_k.

6.10. Proposition. Let $\{A_i | i \in I\}$ be representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori in G. Then

$$H_k \setminus G_k/P_k \cong \cup_{i\in I} W_{H_k}(A_i) \setminus W_{G_k}(A_i)$$

6.11. Remark. The characterization of the double cosets in Proposition 6.8 follows from this result as follows. Fix a θ-stable maximal k-split torus A in P. Any other θ-stable maximal k-split torus of G is of the form $g^{-1}Ag$ where $g \in G_k$ satisfies $\theta(g^{-1})A\theta(g) = g^{-1}Ag$, i.e. $g^{-1}\theta(g) \in N_{G_k}(A)$. In other words, every $H_k - P_k$ double coset has a representative g satisfying $g^{-1}\theta(g) \in N_{G_k}(A)$, and g is unique up to right translations from $Z_{G_k}(A)$ and left translations from H_k. So if we put $W_{G_k/H_k}(A) = \{H_k g Z_{G_k}(A) | g\theta(g)^{-1} \in N_{G_k}(A)\}$, then $H_k \setminus G_k/P_k \cong W_{G_k/H_k}(A)$ in such a way that $H_k g P_k \longleftrightarrow H_k g Z_{G_k}(A)$ if $g\theta(g)^{-1} \in N_{G_k}(A)$. This yields the description in (6.8).

6.12. An example. In general $H_k\backslash G_k/P_k$ is infinite. Let $G = SL(2)/\mathbb{Q}$, $\theta(x) = {}^t x^{-1}$, $B =$ the Borel subgroup of upper triangular matrices and A the group of diagonal matrices. Then $\tau((\tau^{-1}N)_{\mathbb{Q}})$ coincides with the set consisting of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a = x^2 + y^2$, $(x, y) \in \mathbb{Q}^2 - \{(0,0)\}$. One checks readily that $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ are in the same twisted $B_{\mathbb{Q}}$ orbit if and only if $a^{-1}b \in (\mathbb{Q}^\times)^2$. It follows that $V \cong \bigoplus_{\substack{p\equiv 1(4)\\ \text{prime}}} \mathbb{Z}/_{2\mathbb{Z}}$. Obviously $H_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/B_{\mathbb{Q}}$ is infinite, where $H = G^\theta$.

6.13. Lemma. *Let $\tau : G \to G$ be the map given by $\tau(x) = x^{-1}\theta(x)$, $x \in G$. Then the differential $d_\tau : T_e(G) \to T_e(\tau(G))$ is surjective.*

Proof. Let $L(G)$ denote the Lie algebra of G and h, q the 1 and -1 eigen subspaces of $d\theta$ respectively. Then $L(H) = h$ and $\dim(\tau(G)) = \dim G - \dim H = \dim q$. Note that $T_e(\tau(G)) \subset q$. Hence $T_e(\tau(G)) = q$. Since $d\tau = d\theta - 1$, $Im(d\tau) = q = T_e(\tau(G))$.

6.14. Let k be a nondiscrete locally compact field. Given an algebraic k-variety V, V_k is endowed with the topology induced by that of k. We call this topology of V_k the t-topology. By Lemma 6.13, $\tau(G_k)$ is t-open in $\tau(G)_k$.

6.15. Proposition. *Let k be a local field. Then $G_k \cap Q'$ has only finitely many twisted P_k-orbits.*

Proof. We establish the assertion in several steps.

Step 1. From Proposition 6.6, there exist $n_1, \ldots, n_\ell \in N_G(A)_k \cap Q'$ such that every U_k orbit in Q'_k meets $\bigcup_{i=1}^{\ell} Z_G(A)_k n_i \cap Q'$. Since $Z_G(A)_k \subset P_k$, it suffices to show that $Z_G(A)_k n_i \cap Q'$ consists of only finitely many twisted $Z_G(A)_k$ orbits.

Step 2. By Lemma 6.4, we may assume that $G = M \cdot A$ where M is anisotropic over k and A is a k-split central torus of G. We verify in this case that $G_k \cap Q'$ has a finite number of twisted G_k-orbits.

Step 3. Consider the group $\overline{G} = G/A$. For $x \in Q' \cap \overline{G}_k$, by Lemma 6.13, $\overline{G}_k * x$ is t-open in $(\overline{G} * x)_k$. However Q' has only finitely many orbits and every orbit is closed in $\overline{G}$. This yields that $\overline{G}_k * x$ is t-open in $Q' \cap \overline{G}_k$. Since $\overline{G}$ is anisotropic over k, by a theorem of Bruhat and Tits [19, Theorem (BTR)], $\overline{G}_k$ is t-compact. As a consequence, $Q' \cap \overline{G}_k$ is also t-compact. Since twisted $\overline{G}_k$ orbits are t-open in $Q' \cap \overline{G}_k$, there are only a finite number of such orbits. Now let $\pi : G \to \overline{G}$ be the projection map. Then G_k acts on $Q' \cap \overline{G}_k$ via π. Note that $\pi(G_k)$ is t-open in $\overline{G}_k$. It follows that $Q' \cap \overline{G}_k$ has only finitely many twisted G_k-orbits. Again by Lemma 6.4, we may assume that $G = A$.

Step 4. $G = A$. In this case, $Q'_k/\tau(Q'_k)$ is finite and the assertion is obvious.

6.16. Corollary. *If k is a local field, then $H_k \backslash G_k / P_k$ is finite.*

Remark. For $k = \mathbb{R}$, the finiteness condition was discussed by J. Wolf [30] and also by T. Matsuki [15].

7. Twisted involutions in Weyl groups.

Here we recall and establish some results on twisted involutions in a Weyl group.

7.1. Let Φ be a root system in a finite dimensional real vector space V, Φ^+ a fixed system of positive roots with basis Δ and $W = W(\Phi)$ the Weyl group of Φ. Let θ be an involution of V such that Φ is θ-stable. Then θ induces an automorphism of W, also denoted by θ, given by

$$\theta(w) = \theta \circ w \circ \theta, \quad w \in W.$$

If s_α is the reflection defined by α, then $\theta(s_\alpha) = s_{\theta(\alpha)}$, $\alpha \in \Phi$.

7.2. An element $w \in W$ is called a twisted involution (relative to θ) if $\theta(w) = w^{-1}$. We write $\mathcal{T}_\theta$ for the set of twisted involutions. The Weyl group W is generated by $\Sigma = \{s_\alpha | \alpha \in \Delta\}$. In the sequel, the length function ℓ and Bruhat order $\le$ on W are defined relative to Σ.

For a subset Π of Δ, Φ_Π is the subset of Φ consisting of integral combinations of Π. Then Φ_Π is a subsystem of Φ with Weyl group W_Π. Let w^0_Π denote the longest element of W_Π.

7.3. Let $w \in GL(V)$ with $w(\Phi) = \Phi$. Let $\psi(w)$ (resp. $\psi^-(w)$) denote the set given by $\psi(w) = \{\alpha \in \Phi^+ | w^{-1}\alpha > 0\}$ (resp. $\psi^-(w) = \Phi^+ - \psi(w)$).

7.4. Given $w \in \mathcal{T}_\theta$, an element $\alpha \in \Phi$ is *complex* (resp. *real*, *imaginary*) relative to w if $w\theta\alpha \ne \pm\alpha$ (resp. $w\theta\alpha = -\alpha$, $w\theta\alpha = \alpha$). We introduce

$$C'(w) = \{\alpha \in \Phi^+ | -\alpha \ne w\theta\alpha < 0\}, \qquad C''(w) = \{\alpha \in \Phi^+ | \alpha \ne w\theta\alpha > 0\},$$
$$R(w) = \{\alpha \in \Phi^+ | -\alpha = w\theta\alpha\}, \qquad I(w) = \{\alpha \in \Phi^+ | \alpha = w\theta\alpha\}.$$

7.5. Lemma. *Let $w \in \mathcal{T}_\theta$. Then we have the following conditions:*

(i) $\psi^-(w\theta) = C'(w) \cup R(w)$ *and* $\psi(w\theta) = C''(w) \cup I(w)$ *are disjoint unions.*
(ii) $\psi^-(w\theta)$, $C'(w)$ *are* $-w\theta$ *stable and* $-w\theta|R(w) = 1$.
(iii) $\psi(w\theta)$, $C''(w)$ *are* $w\theta$ *stable and* $w\theta|I(w) = 1$.

Proof. Since $\theta(w) = w^{-1}$, $(w\theta)^2 = 1$. Then

$$\psi(w\theta) = \{\alpha \in \Phi^+ | w\theta\alpha > 0\},$$
$$\psi^-(w\theta) = \{\alpha \in \Phi^+ | w\theta\alpha < 0\}.$$

It is easy to see that $\psi(w\theta)$ (resp. $\psi^-(w\theta)$) is $w\theta$ stable (resp. $-w\theta$ stable). The assertions are obvious.

In the following discussion till 7.12, we assume that Φ^+ is θ-stable. It follows that Δ and Σ are also θ-stable.

7.6. Lemma. *Let $\alpha \in \Delta$ and $w \in W$ with $w^{-1}\alpha > 0$. If $w_1 = s_\alpha w$, then $\psi^-(w^{-1}) \subset \psi^-(w_1^{-1})$ and $\psi(w_1^{-1}) \subset \psi(w^{-1})$.*

Proof. If $\beta \in \psi^-(w^{-1})$, by definition $w\beta < 0$. It yields that $w_1\beta = s_\alpha(w\beta)$ is either < 0 or $w\beta$ is a negative multiple of α. The later condition implies that $w^{-1}\alpha < 0$ which contradicts our assumption. Hence we have that $\psi^-(w^{-1}) \subset \psi^-(w_1^{-1})$ and equivalently $\psi(w_1^{-1}) \subset \psi(w^{-1})$.

7.7. Lemma. *Let $\alpha \in \Delta$ and $w \in W$ with $w\alpha > 0$. If $w_2 = ws_\alpha$, then $s_\alpha(\psi(w_2^{-1})) \subset \psi(w^{-1})$.*

Proof. If $\beta \in \psi(w_2^{-1})$, then $w(s_\alpha\beta) > 0$. Note that $s_\alpha\beta > 0$ or is a negative multiple of α. Since $w\alpha > 0$, the later condition is impossible. Hence we have that $s_\alpha(\psi(w_2^{-1})) \subset \psi(w^{-1})$.

7.8. Lemma. *Let $w \in \mathcal{T}_\theta$ and $\tau = s_\alpha w\theta(s_\alpha)$ with $\alpha \in \Delta$. Assume that $\ell(\tau) = 2 + \ell(w)$. Then we have the following conditions:*

(i) α, $w\theta\alpha \in C''(w)$, $I(\tau) = s_\alpha(I(w))$ *and* $C''(\tau) = s_\alpha(C''(w) - X)$ *where* $X = \Phi^+ \cap (\mathbb{Z}\alpha \cup \mathbb{Z}w\theta\alpha)$.
(ii) α, $s_\alpha w\theta\alpha \in C'(\tau)$, $R(\tau) = s_\alpha(R(w))$ *and* $C'(\tau) = Y \cup s_\alpha(C'(w))$ *where* $Y = \Phi^+ \cap (\mathbb{Z}\alpha \cup \mathbb{Z}s_\alpha w\theta\alpha)$.

Proof. (i) Step 1. α, $w\theta\alpha \in C''(w)$. If $w\theta\alpha < 0$, then $\ell(w\theta(s_\alpha)) < \ell(w)$. If $w\theta\alpha = \alpha$, then $w = \tau$. Hence $\alpha \neq w\theta\alpha > 0$ and the assertion is obvious.

Step 2. From the condition $\ell(\tau) = 2 + \ell(w)$, we have the following conditions:

$$(1) \qquad \begin{aligned} &(a) \quad s_\alpha w\theta(\alpha) > 0. \\ &(b) \quad w^{-1}\alpha > 0. \end{aligned}$$

Now consider the element $\tau^{-1} = \theta(s_\alpha)(w^{-1}s_\alpha)$. By (a) of (1) and Lemma 7.6, $\psi(\tau) \subset \psi(s_\alpha w)$. Next consider the element $w^{-1}s_\alpha$. By (b) of (1) and Lemma 7.7, $\psi(s_\alpha w) \subset s_\alpha\psi(w)$. Thus we obtain the relation

$$(2) \qquad \psi(\tau) \subset s_\alpha\psi(w).$$

Step 3. Note that $s_\alpha(\alpha) = -\alpha < 0$ and $\tau^{-1}(s_\alpha w\theta\alpha) = -\theta\alpha < 0$. Condition (2) can be improved to

$$\psi(\tau) \subset s_\alpha(\psi(w) - X), \tag{3}$$

where $X = \Phi^+ \cap (\mathbb{Z}\alpha \cup \mathbb{Z}w\theta\alpha)$. Conversely if $\beta \in \psi(w) - X$, then $s_\alpha\beta > 0$ and $\tau^{-1}(s_\alpha\beta) = \theta(s_\alpha)w^{-1}\beta > 0$. This yields that

$$s_\alpha(\psi(w) - X) \subset \psi(\tau). \tag{4}$$

From (3) and (4), we have that

$$\psi(\tau) = s_\alpha(\psi(w) - X). \tag{5}$$

Step 4. Since $\theta(\Phi^+) = \Phi^+$, $\psi(\tau) = \psi(\tau\theta)$ and $\psi(w) = \psi(w\theta)$. Observe that $\tau\theta(s_\alpha\beta) = s_\alpha(w\theta\beta)$, $\beta \in \Phi$. Since $\alpha \neq w\theta\alpha$, from (5) one concludes readily that $I(\tau) = s_\alpha(I(w))$. By Lemma 7.5, $C''(\tau) = \psi(\tau) - I(\tau) = s_\alpha(\psi(w) - X - I(w)) = s_\alpha(C''(w) - X)$.

(ii). From (5), we have that $\psi^-(\tau) = Y \cup s_\alpha(\psi^-(w))$ where $Y = \Phi^+ \cap (\mathbb{Z}\alpha \cup \mathbb{Z}s_\alpha w\theta\alpha)$. If $\tau\theta\alpha = -\alpha$, then $w\theta\alpha = -\alpha$ and as a consequence $w = s_\alpha w\theta(s_\alpha)$. Certainly this is a contradiction. Hence $\tau\theta\alpha \neq -\alpha$ and $\tau\theta\alpha < 0$. We have that $\alpha, s_\alpha w\theta\alpha = -\tau\theta\alpha \in C'(\tau)$. Observe that $\tau\theta(s_\alpha\beta) = s_\alpha(w\theta\beta)$, $\beta \in \Phi$. This yields readily that $R(\tau) = s_\alpha R(w)$. By Lemma 7.5, $C'(\tau) = \psi^-(\tau) - R(\tau) = Y \cup s_\alpha(\psi^-(w) - R(w)) = Y \cup s_\alpha(C'(w))$.

Remark. This lemma is a refinement of [23, Lemma 3.9].

7.9. Proposition. [23] *If $w \in \mathcal{T}_\theta$, there exist $s_1, \dots, s_h \in \Sigma$ and a θ-stable subset Π of Δ satisfying the following conditions:*

(i) $w = s_1 \dots s_h w^0_\Pi \theta(s_h) \dots \theta(s_1)$ *and* $\ell(w) = 2h + \ell(w^0_\Pi)$.
(ii) $w^0_\Pi \theta\alpha = -\alpha$, $\alpha \in \Phi_\Pi$.
(iii) *Let $t_1, \dots, t_m \in \Sigma$ and Λ a θ-stable subset of Δ satisfy conditions (i) and (ii) for w. Then $m = h$, $s_1 \dots s_h\Pi = t_1 \dots t_h\Lambda$ and $s_1 \dots s_h\theta(s_h) \dots \theta(s_1) = t_1 \dots t_h\theta(t_h) \dots \theta(t_1)$.*

Proof. (i) and (ii) are (a) and (b) of [23, Prop. 3.3] respectively. It remains to verify (iii). From (ii), we have that $R(w^0_\Pi) = \Phi^+_\Pi$. By (ii) of Lemma 7.8,

$$R(w) = s_1 \dots s_h\Phi^+_\Pi = t_1 \dots t_m\Phi^+_\Lambda.$$

Now consider the set $\Phi(w) = \{\alpha \in \Phi | w\theta\alpha = -\alpha\}$. Clearly it is a subsystem of Φ and $R(w) = \Phi^+ \cap \Phi(w)$ is a system of positive roots of $\Phi(w)$. Then

$$\Delta' = s_1 \dots s_h\Pi = t_1 \dots t_m\Lambda$$

is the unique basis defined by $R(w)$.

It is easy to see that

$$w^0_{\Delta'} = s_1 \dots s_h w^0_\Pi s_h \dots s_1 = t_1 \dots t_m w^0_\Lambda t_m \dots t_1.$$

As a consequence, $\ell(w^0_\Lambda) = \ell(w^0_\Pi)$ and $m = h$. Since $w = s_1 \dots s_h w^0_\Pi \theta(s_h) \dots \theta(s_1) = t_1 \dots t_h w^0_\Lambda \theta(t_h) \dots \theta(t_1)$, the above condition yields immediately that

$$s_1 \dots s_h\theta(s_h) \dots \theta(s_1) = t_1 \dots t_h\theta(t_h) \dots \theta(t_1).$$

7.10. Corollary. *If $\alpha \in \Delta$ satisfies $w\theta\alpha = -\alpha$, then $\alpha \in s_1 \dots s_h \Pi$.*

Proof. $R(w)$ is a system of positive roots of the root system $R(w) \cup -R(w)$ and $s_1 \dots s_h \Pi$ is the basis defined by $R(w)$. Since $\alpha \in \Delta$, it is not the sum of two positive roots. It follows easily that $\alpha \in s_1 \dots s_h \Pi$.

Remark. (iii) of Proposition 7.9 is a kind of uniqueness condition of the decomposition of w. This result seems to be new and the corollary generalizes slightly [23, (c) Prop. 3.3] which asserts α or $-\alpha$ belongs to $s_1 \cdots s_h \Pi$.

7.11. Proposition. [23]. *Let $w \in \mathcal{T}_\theta$ be an element such that $sw\theta(s) = w$ for $s \in \Sigma$ with $sw > w$. Then*

$$w = w_\Pi^0 w_\Delta^0, \tag{1}$$

where $\Pi = \{\alpha \in \Delta | w\theta\alpha = \alpha\}$; moreover (1) implies that $\psi(w) = \Phi_\Pi^+$.

Proof. The first assertion is [23, 3.5]. Note that $w_\Delta^0 \Phi^+ = -\Phi^+$. The condition $w^{-1}\beta > 0$ is equivalent to $w_\Pi^0(\beta) < 0$. Clearly (1) implies that $\psi(w) = \Phi_\Pi^+$.

7.12. Proposition. *Let $w \in \mathcal{T}_\theta$. There exists $w_1 \in W$ with $w = w_1\theta(w_1)^{-1}$ if and only if $R(w) = \phi$.*

Proof. $\Rightarrow$). If $\alpha \in \Phi$ satisfies $w\theta\alpha = -\alpha$, then $\theta(w_1^{-1}\alpha) = -w_1^{-1}\alpha$. However $\theta(\Phi^+) = \Phi^+$. Clearly we have a contradiction.

$\Leftarrow$). By Proposition 7.9, we write $w = s_1 \dots s_h w_\Pi^0 \theta(s_h) \dots \theta(s_1)$ such that Π is θ-stable and $w_\Pi^0 \theta\alpha = -\alpha$ for $\alpha \in \Phi_\Pi$. It is easy to check that $\{\alpha \in \Phi | w\theta\alpha = -\alpha\} = s_1 \dots s_h \Phi_\Pi$. By assumption, the set is empty. Hence $\Pi = \phi$ and $w_1 = s_1 \dots s_h$ has the desired property.

7.13. From now on, θ is an involution of V which leaves Φ invariant but does not necessarily leave Φ^+ invariant. Then there exists $w_0 \in W$ such that

$$\theta(\Phi^+) = w_0(\Phi^+). \tag{1}$$

From (1), we have the following conditions:

$$\begin{aligned} &(i) && w_0 \in \mathcal{T}_\theta. \\ &(ii) && \theta' = \theta w_0 \quad \text{is an involution of } V. \\ &(iii) && \theta'(\Phi^+) = \Phi^+. \end{aligned} \tag{2}$$

7.14. Given a subset Π of Δ, $w \in W$ and $\Pi' = w\Pi$, the subset $\Phi_{\Pi'}$ of Φ of the integral combinations of Π' is a subsystem of Φ with Weyl group $W(\Phi_{\Pi'})$. We write $w_{\Pi'}^0$ for the longest element of $W(\Phi_{\Pi'})$ with respect to the set s_α, $\alpha \in \Pi'$.

7.15. Lemma. *Let w_0 be as in 7.13(1), $w \in W$, $\Pi \subset \Delta$ and $\Pi' = w\Pi$. The following two conditions are equivalent:*

(i) $w_0 = w w_\Pi^0 \theta'(w^{-1})$.

(ii) $w_0 = \theta(w^{-1}) w_{\Pi'}^0 w$.

Proof. (i) $\Rightarrow$ (ii). Since $\theta' = \theta w_0$, (i) yields that $e = w w_\Pi^0 w_0^{-1} \theta(w^{-1})$. Hence $w_0^{-1} = w_\Pi^0 w^{-1} \theta(w) = w^{-1} w_{\Pi'}^0 \theta(w)$ and (ii) follows.

(ii) $\Rightarrow$ (i) follows by reversing the above argument.

7.16. Lemma. *Let w_0, w, Π and Π' be as above. Suppose that $w_0 = w w^0_{\Pi} \theta'(w^{-1})$. The following two conditions are equivalent:*

(i) $w^0_{\Pi}\theta'|\Pi = -1$.

(ii) $\theta|\Pi' = -1$.

Proof. We have that

$$(1)\qquad \begin{aligned} w^0_{\Pi}\theta' &= w^0_{\Pi}\theta w_0 \\ &= (w^{-1}w^0_{\Pi'}w)\cdot\theta\cdot(\theta(w^{-1})w^0_{\Pi'}w)(\text{by Lemma 7.15}) \\ &= w^{-1}w^0_{\Pi'}\theta w^0_{\Pi'}w. \end{aligned}$$

Note that $\Pi = w^{-1}\Pi'$. From (1),

$$\begin{aligned} w^0_{\Pi}\theta'|\Pi = -1 &\Leftrightarrow w^0_{\Pi'}\theta w^0_{\Pi'}|\Pi' = -1 \\ &\Leftrightarrow \theta|w^0_{\Pi'}(\Pi') = -1 \Leftrightarrow \theta|\Pi' = -1 \end{aligned}$$

7.17. A subset ψ of Φ is parabolic if ψ is closed and $\psi \cup -\psi = \Phi$. Given any subset ψ of Φ, let ψ_s denote the set $\psi \cap -\psi$ and ψ_u the complement of ψ_s in ψ.

7.18. Lemma. *Let ψ be a θ-stable parabolic subset of Φ. Then ψ is a minimal θ-stable parabolic set of Φ if and only if $\psi_s = \{\alpha \in \Phi|\theta\alpha = -\alpha\}$.*

Proof. There exists $v \in V^{\theta}$ such that

$$\psi = \{\beta \in \Phi| < v, \beta > \geq 0\}.$$

Clearly ψ_s contains the set $\{\alpha \in \Phi|\theta\alpha = -\alpha\}$. Now the assertion is immediate.

7.19. A parabolic subset ψ of Φ is called

$$(\theta, \Phi^+)\text{-special if } \psi \supset \Phi^+ \cap \theta(\Phi^+).$$

7.20. Let ψ be a (θ, Φ^+)-special minimal θ-stable parabolic subset of Φ. Set

$$(1)\qquad \psi^+ = (\psi_s \cap \Phi^+) \cup \psi_u.$$

Then ψ^+ is a system of positive roots of Φ. Consider the sets

$$(2)\qquad \begin{aligned} \Omega_0 &= \Phi^+ \cap \psi_s = \{\alpha \in \Phi^+|\theta\alpha = -\alpha\}, \\ \Omega_+ &= \Phi^+ \cap \psi_u, \\ \Omega_- &= \Phi^+ \cap -\psi_u. \end{aligned}$$

It follows readily that we have the decompositions

$$(3)\qquad \begin{aligned} \Phi^+ &= \Omega_0 \cup \Omega_+ \cup \Omega_-, \\ \psi^+ &= \Omega_0 \cup \Omega_+ \cup -\Omega_-. \end{aligned}$$

Since ψ is (θ, Φ^+)-special, $\psi_u \supset \Phi^+ \cap \theta(\Phi^+)$ and so

$$(4)\qquad \Omega_+ \supset \Phi^+ \cap \theta(\Phi^+).$$

Now let Ω'_+ denote the complement of $\Phi^+ \cap \theta(\Phi^+)$ in Ω_+. From (4), we have that

$$(5)\qquad \theta(\Omega'_+), \quad \theta(\Omega_-) \subset -\Phi^+.$$

7.21. Lemma. *Let ψ be a (θ, Φ^+)-special minimal θ-stable parabolic subset of Φ, ψ^+ be as in 7.20(1) and $w \in W$ such that $w(\Phi^+) = \psi^+$. Let Π' be the set of simple roots of Ω_0 and Π the subset of Δ with $\Pi' = w\Pi$. Then we have the following conditions:*

(i) $w_0 = ww^0_\Pi \theta'(w^{-1})$.

(ii) $l(w_0) = 2l(w) + l(w^0_\Pi)$.

(iii) $w^0_\Pi \theta' | \Pi = -1$.

(iv) $\theta | \Pi' = -1$.

Proof. Note that $w_0(\Phi^+) \cap \Phi^+ = \theta(\Phi^+) \cap \Phi^+ \subset \Omega_+$ and $\Phi^+ = \Omega_0 \cup \Omega'_+ \cup \Omega_- \cup (\theta(\Phi^+) \cap \Phi^+)$ is a disjoint union. It follows easily that

$$l(w_0) = Card(\Omega_0) + Card(\Omega'_+) + Card(\Omega_-). \tag{1}$$

Since ψ is θ-stable, so are the sets ψ_u and $\Omega'_+ \cup -\Omega_-$. From 7.20(5), we have that

$$\theta(\Omega'_+) = -\Omega_-. \tag{2}$$

From (1) and (2), $l(w_0) = Card(\Omega_0) + 2\,Card(\Omega_-)$.

Clearly $Card(\Omega_0) = l(w^0_{\Pi'}) = l(w^0_\Pi)$ and by 7.20(3) $l(w) = Card(\Omega_-)$. Hence we obtain

$$l(w_0) = 2l(w) + l(w^0_\Pi).$$

Thus (ii) is established. Now consider the set $\theta(\psi^+)$. Observe that $\theta | \psi_s = -1$ and ψ_u is θ-stable. From the decomposition $\psi^+ = (\psi_s \cap \Phi^+) \cup \psi_u$, it follows readily

$$\theta(\psi^+) = w^0_{\Pi'} \psi^+. \tag{3}$$

On the other hand, we have that

$$\begin{aligned} \theta(\psi^+) &= \theta(w\Phi^+) \\ &= (\theta(w) w_0 w^{-1})\psi^+. \end{aligned} \tag{4}$$

Then (3) and (4) yield that $w^0_{\Pi'} = \theta(w) w_0 w^{-1}$. Clearly $w_0 = \theta(w^{-1}) w^0_{\Pi'} w$ and by Lemma 7.15, assertion (i) follows.

Since ψ is a minimal θ-stable parabolic subset of Φ, by Lemma 7.18, $\theta | \psi_s = -1$. Thus (iv) is obvious and by Lemma 7.16, (iii) follows.

7.22. Lemma. *Let $w \in W$ and $\Pi \subset \Delta$. Suppose that*

$$w_0 = ww^0_\Pi \theta'(w')$$

such that $l(w_0) = 2l(w) + l(w^0_\Pi)$ and $w^0_\Pi \theta' | \Pi = -1$. Then there is a unique (θ, Φ^+)-special minimal θ-stable parabolic subset ψ of Φ such that

$$w(\Phi^+) = (\psi_s \cap \Phi^+) \cup \psi_u.$$

Proof. Set

$$\Pi' = w\Pi.$$

By Lemma 7.15, $w_0 = \theta(w^{-1})w^0_{\Pi'}w$ and so

(1) $$w^0_{\Pi'} = \theta(w)w_0w^{-1}.$$

Note that $w_0\theta' = w_0\theta w_0 = \theta$. By (ii) of Lemma 7.8 and a simple induction, we have that

(2) $$w\Phi^+_\Pi = \{\alpha \in \Phi^+ | \theta\alpha = -\alpha\}.$$

Consider the set

(3) $$\psi^+ = w(\Phi^+).$$

Then $\theta(\psi^+) = \theta(w)w_0w^{-1}(\psi^+)$ and by (1)

(4) $$\theta(\psi^+) = w^0_{\Pi'}(\psi^+).$$

Now set

(5) $$\Omega_0 = w\Phi^+_\Pi = \Phi^+_{\Pi'}.$$

From (2), $\Omega_0 \subset \Phi^+$. Thus there exists $\Omega_+, \Omega_- \subset \Phi^+$ such that

(6) $$\begin{aligned} &(i)\ \Phi^+ = \Omega_0 \cup \Omega_+ \cup \Omega_-, \\ &(ii)\ \psi^+ = \Omega_0 \cup \Omega_+ \cup -\Omega_-. \end{aligned}$$

are disjoint unions. Note that $w\Delta$ is the set of simple roots of ψ^+, $\Pi' \subset w\Delta$. From (4) and (ii) of (6), we have the following conditions:

(7) $$\begin{aligned} &(i) \text{The set } \psi = -\Omega_0 \cup \psi^+ \text{ is a minimal } \theta\text{-stable parabolic subset of } \Phi. \\ &(ii) \psi_u = \Omega^+ \cup -\Omega_- \text{ is } \theta\text{-stable.} \end{aligned}$$

Let Ω'_+ (resp. Ω'_-) denote the intersection of Ω_+ (resp. Ω_-) with $\Phi^+ \cap \theta(\Phi^+)$. Let

(8) $$\begin{aligned} \Omega_+ &= \Omega'_+ \cup \Omega''_+, \\ \Omega_- &= \Omega'_- \cup \Omega''_- \end{aligned}$$

be disjoint unions. From (ii) of (7), it yields that Ω'_+ and Ω'_- are θ-stable. As a consequence,

(9) $$\theta(\Omega''_+) = -\Omega''_-.$$

Clearly we have that $l(w_0) = Card(\Omega_0) + Card(\Omega''_+) + Card(\Omega'_-)$ and by (9),

(10) $$l(w_0) = Card(\Omega_0) + 2\,Card(\Omega''_-).$$

Observe that $l(w^0_\Pi) = l(w_{\Pi'}) = Card(\Omega_0)$ and $l(w) = Card(\Omega_-)$. Thus by assumption on $l(w_0)$,

(11) $$l(w_0) = Card(\Omega_0) + 2\,Card(\Omega_-).$$

Then (10) and (11) imply that $\Omega'_- = \phi$ and

(12) $$\Phi^+ \cap \theta(\Phi^+) \subset \Omega_+ \subset \psi.$$

From (4), (i) of (7) and (12) it follows that ψ has the desired property. The uniqueness is obvious.

7.23. Given $w \in \mathcal{T}_\theta$, a decomposition

$$ww_0 = \tau w^0_\Pi \theta'(\tau^{-1})$$

with $\tau \in W$ and $\Pi \subset \Delta$ is called a *Springer decomposition* of w if

(i) $l(ww_0) = 2l(\tau) + l(w^0_\Pi)$,
(ii) $w^0_\Pi\theta'|\Pi = -1$.

7.24. Proposition. *let $w \in \mathcal{T}_\theta$ and $\xi = w\theta$. Then we have the following conditions:*

(i) ξ is an involution leaving Φ invariant.

(ii) Given any (ξ, Φ^+)-special minimal ξ-stable parabolic subset ψ of Φ, let $\tau \in W$ be the element such that

$$\tau(\Phi^+) = (\psi_s \cap \Phi^+) \cup \psi_u.$$

Then there exists $\Pi \subset \Delta$ such that

$$(a)\ \tau(\Phi^+_\Pi) = \psi_s \cap \Phi^+,$$

$$(b)\ ww_0 = \tau w^0_\Pi \theta'(\tau^{-1}) \text{ is a Springer decomposition}.$$

(iii) There is a one to one correspondence, given in (ii), between the set of (ξ, Φ^+)-special minimal ξ-stable parabolic subsets of Φ and the set of Springer decompositions of w.

Proof. (i) is obvious. Note that $\xi(\Phi^+) = ww_0(\Phi^+)$ and

$$\xi' = \xi w w_0 = \theta w_0 = \theta'.$$

Then (ii) and (iii) follow immediately from Lemmas 7.21 and 7.22.

7.25. Remark. Besides yielding a new proof of Springer's result [21, Prop. 3.3] of a more constructive nature, the above result also classifies such decompositions. The approach here is inspired by the work of Matsuki [16].

8. Some dimension formulas.

Let L be any minimal parabolic k-subgroup of G. In this section, we study the group $L^\theta = L \cap G^\theta$ and establish some dimension formulas which are needed for our discussion on orbit closures.

8.1. Let P be a fixed minimal parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Let $\Phi(G, A)$ denote the set of roots of A in G and $\Phi^+ = \Phi(P, A)$ with basis Δ. For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root subspace of the Lie algebra $L(G)$ of G corresponding to α. We have the decomposition $L(G) = L(Z_G(A)) \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Given $\alpha \in \Delta$, let P_α denote the standard parabolic k-subgroup of G containing P such that $\Phi(P_\alpha, A) = (\mathbb{Z}\alpha \cap \Phi) \cup \Phi^+$. It is easy to see that

$$\dim(P_\alpha) = \dim(P) + \dim(\oplus_{\gamma \in \mathbb{Z}\alpha \cap \Phi^+} \mathfrak{g}_\gamma). \tag{1}$$

8.2. Let M be a subgroup of G and $M^\theta = \{x \in M | \theta(x) = x\}$. Given $y \in G$, let $a = y\theta(y)^{-1}$ and ζ denote the involution $\zeta(x) = a\theta(x)a^{-1}$, $x \in G$. Then $\theta(y^{-1}xy) = y^{-1}\zeta(x)y$, $x \in G$. Hence we have the condition that

$$(y^{-1}My)^\theta = y^{-1}M^\zeta y.$$

8.3. Lemma. *Let $y_1, y_2 \in Q'$ (6.2) and ψ and ϕ be the involutions of G defined by $\psi(x) = y_1\theta(x)y_1^{-1}$, $\phi(x) = y_2\theta(x)y_2^{-1}$, $x \in G$. Let M be any subgroup of G. If $y_2 = zy_1\theta(z)^{-1}$, then $(zMz^{-1})^{\phi} = zM^{\psi}z^{-1}$.*

Proof. Note that $\phi(zxz^{-1}) = z\psi(x)z^{-1}$, $x \in G$. The assertion is obvious.

8.4. Let L be any minimal parabolic k-subgroup of G. There exists $g \in G_k$ with $L = g^{-1}Pg$. By Proposition 6.6, $G_k = U_k(\tau^{-1}N)_k$. Here

$$(\tau^{-1}N)_k = \{v \in G_k | v\theta(v)^{-1} \in N_G(A)\}.$$

Hence to study L^{θ}, we may assume that $L = v^{-1}Pv$ with $v \in (\tau^{-1}N)_k$. Let $n = v\theta(v)^{-1}$ and w be its image in the Weyl group W of $\Phi(G, A)$. Observe that $L \cap \theta(L) = v^{-1}(P \cap n\theta(P)n^{-1})v$. It follows that

$$L \cap \theta(L) = v^{-1}(Z_G(A) \ltimes U_{\psi(w\theta)})v, \tag{1}$$

where $\psi(w\theta)$, defined in 7.3, is $\Phi^+ \cap w\theta\Phi^+$.

8.5. Proposition. *Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$, w the image of n in W, $L = v^{-1}Pv$ and ζ the involution of G given by $\zeta(x) = n\theta(x)n^{-1}$, $x \in G$. We have the following conditions:*

(i) $L^{\theta} \simeq Z_G(A)^{\zeta} \ltimes U^{\zeta}_{\psi(w\theta)}$.

(ii) *Let $I(w) = \{\alpha \in \Phi^+ | w\theta\alpha = \alpha\}$, $C''(w) = \{\alpha \in \Phi^+ | \alpha \neq w\theta\alpha > 0\}$ and $\mathfrak{g}_w = \bigoplus_{\alpha \in C''(w)} \mathfrak{g}_{\alpha}$. Then*

$$\dim(L^{\theta}) = \dim(Z_G(A)^{\zeta}) + \dim(U^{\zeta}_{I(w)}) + \frac{1}{2}\dim(\mathfrak{g}_w).$$

Proof. (i) By (8.4.1), $L \cap \theta(L) = v^{-1}(Z_G(A) \ltimes U_{\psi(w\theta)})v$. The groups $v^{-1}Z_G(A)v$ and $v^{-1}U_{\psi(w\theta)}v$ are θ-stable. Hence by 8.2, $L^{\theta} = v^{-1}(Z_G(A)^{\zeta} \ltimes U^{\zeta}_{\psi(w\theta)})v$. Then (i) is obvious.

(ii) The set $I(w)$ is a closed subset of Φ^+ and $L(U_{I(w)}) = \bigoplus_{\alpha \in I(w)} \mathfrak{g}_{\alpha}$. By (i) of Lemma 7.5, $\psi(w\theta) = I(w) \cup C''(w)$ is a disjoint union. It follows that $L(U_{\psi(w\theta)}) = L(U_{I(w)}) \oplus \mathfrak{g}_w$. Note that $\zeta\alpha = w\theta\alpha$, $\alpha \in \Phi$. Since $I(w)$ and $C''(w)$ are $w\theta$-stable by Lemma 7.5, $L(U_{\psi(w\theta)})^{\zeta} = L(U_{I(w)})^{\zeta} \oplus \mathfrak{g}_w^{\zeta}$. By Lemma 0.6, we have that $\dim(U^{\zeta}_{\psi(w\theta)}) = \dim(U^{\zeta}_{I(w)}) + \dim(\mathfrak{g}^{\zeta}_w)$. It remains to show that $\dim(\mathfrak{g}^{\zeta}_w) = \frac{1}{2}\dim(\mathfrak{g}_w)$. From the definition of $C''(w)$ and the fact that $\zeta = w\theta$ on Φ, $\zeta\alpha \neq \alpha$ for $\alpha \in C''(w)$. Hence there exists a subset J of $C''(w)$ such that $C''(w) = J \cup \zeta(J)$ is a disjoint union. Now set $V = \bigoplus_{\alpha \in J} \mathfrak{g}_{\alpha}$. We have that $\mathfrak{g}_w = V \oplus \zeta(V)$. It is easy to see that $\mathfrak{g}^{\zeta}_w \simeq V$ and $\dim(\mathfrak{g}^{\zeta}_w) = \frac{1}{2}\dim(\mathfrak{g}_w)$.

8.6. Corollary. *Let $v \in (\tau^{-1}N)_k$ and $v_1 = mv$ with $m \in N_G(A)_k$. Let $n = v\theta(v)^{-1}$, $n_1 = v_1\theta(v_1)^{-1} = mn\theta(m)^{-1}$ and w, w_1, t the images of n, n_1, m in W respectively. Then $\dim\left((v_1^{-1}Pv_1)^{\theta}\right) - \dim\left((v^{-1}Pv)^{\theta}\right) = \frac{1}{2}(\dim(\mathfrak{g}_{w_1}) - \dim(\mathfrak{g}_w))$, if $t(I(w)) = I(w_1)$.*

Proof. Let ζ (resp. η) denote the involution of G given by $\zeta(x) = n\theta(x)n^{-1}$, $x \in G$ (resp. $\eta(x) = n_1\theta(x)n_1^{-1}$, $x \in G$). Since $t(I(w)) = I(w_1)$, $mU_{I(w)}m^{-1} = U_{I(w_1)}$. By

Lemma 8.3, we have that $Z_G(A)^\eta = m(Z_G(A)^\zeta)m^{-1}$ and $U^\eta_{I(w_1)} = m(U^\zeta_{I(w)})m^{-1}$. From Proposition 8.5, we have the identities that

$$\dim((v_1^{-1}Pv_1)^\theta) = \dim(Z_G(A)^\eta) + \dim(U^\eta_{I(w_1)}) + \frac{1}{2}\dim(\mathfrak{g}_{w_1}),$$
$$\dim((v^{-1}Pv)^\theta) = \dim(Z_G(A)^\zeta) + \dim(U^\zeta_{I(w)}) + \frac{1}{2}\dim(\mathfrak{g}_{w}).$$

Now the assertion follows.

8.7. The group $\theta(P)$ is also a minimal parabolic k-subgroup of G containing A. By the conjugacy theorem, there exists $n_0 \in N_G(A)_k$ with $n_0Pn_0^{-1} = \theta(P)$. Let w_0 denote the image of n_0 in W. By our choice, $\theta(\Phi^+) = w_0(\Phi^+)$. It yields that $\theta(w_0) = w_0^{-1}$. Set

$$\theta' = \theta w_0. \tag{1}$$

It is easy to see that θ' is an involution of W satisfying the conditions:

$$(2) \qquad \begin{array}{ll} (i) & \theta'(\Phi^+) = \Phi^+. \\ (ii) & \mathcal{T}_{\theta'} = \mathcal{T}_\theta \cdot w_0. \end{array}$$

8.8. Lemma. *Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$ and $n_\alpha \in N_G(A)_k$ with image s_α in W. Let w be the image of n in W and $w' = ww_0$. Suppose that $\alpha \in \Delta$ and $\ell(s_\alpha w'\theta'(s_\alpha)) = 2 + \ell(w')$. Then* $\dim((v_1^{-1}Pv_1)^\theta) = \dim((v^{-1}Pv)^\theta) + \dim P - \dim P_\alpha$, *where $v_1 = n_\alpha v$.*

Proof. Note that $w'\theta' = w\theta$ and $s_\alpha w'\theta'(s_\alpha)\theta' = s_\alpha w\theta(s_\alpha)\theta$. It follows that

$$I(w,\theta) = I(w',\theta'), \quad I(s_\alpha w\theta(s_\alpha),\theta) = I(s_\alpha w'\theta'(s_\alpha),\theta'),$$

$$C''(w,\theta) = C''(w',\theta'), \quad C''(s_\alpha w\theta(s_\alpha),\theta) = C''(s_\alpha w'\theta'(s_\alpha),\theta').$$

Applying Lemma 7.8 to w', s_α and θ', we obtain that

$$(1) \qquad \begin{array}{ll} (i) & I(s_\alpha w\theta(s_\alpha)) = s_\alpha I(w), \\ (ii) & C''(s_\alpha w\theta(s_\alpha)) = s_\alpha(C''(w) - X), \end{array}$$

where $X = \Phi^+ \cap (\mathbb{Z}\alpha \cup \mathbb{Z}w\theta\alpha)$. By (i) of (1) and Lemma 8.6, $\dim\left((v_1^{-1}Pv_1)^\theta\right) - \dim\left((v^{-1}Pv)^\theta\right) = \frac{1}{2}\left(\dim(\mathfrak{g}_{s_\alpha w\theta(s_\alpha)}) - \dim(\mathfrak{g}_w)\right)$. From (ii) of (1), this number coincides with $(-1)\cdot\dim\left(\bigoplus_{\gamma\in\Phi^+\cap\mathbb{Z}\alpha}\mathfrak{g}_\alpha\right)$. Our assertion now follows from (8.1.1).

9. The big cell and orbit closures.

Let G be a connected reductive algebraic k–group and θ an involution of G defined over k. Let Q denote the set given by

$$Q = \{g\theta(g)^{-1} | g \in G\}.$$

Then the group G has a twisted action on Q defined by $g * x = gx\theta(g)^{-1}$, $g \in G$, $x \in Q$. Let P be a fixed minimal parabolic k–subgroup of G and A a θ–stable maximal k–split torus of P. Set

$$(\tau^{-1}N)_k = \{g \in G_k | g\theta(g)^{-1} \in N_G(A)\}.$$

From (ii) of Lemma 4.8 and Proposition 6.8, there exists $v \in (\tau^{-1}N)_k$ such that for $n = v\theta(v)^{-1}$ the orbit $P * n$ is open in Q. Then $P * n$ is the unique open orbit of P in Q, called the big cell. Here we study properties of such element v. Given $g \in G_k$, consider the double coset PgH. We study the closure $c\ell(PgH)$ of PgH in G. When k is a local field, G_k is endowed with the topology, called t–topology, induced from that of k. We also discuss the t–closure of $P_k g H_k$ in G_k.

Let $\Phi = \Phi(G, A)$, $\Phi^+ = \Phi(P, A)$ and w_0 the element in the Weyl group $W = N_G(A)/Z_G(A)$ with $w_0(\Phi^+) = \theta(\Phi^+)$. In the following, $\theta' = \theta w_0$.

9.1. Lemma. *Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$, w the image of n in W and $w' = ww_0$. Suppose that $\alpha \in \Delta$ is a simple root satisfies $\ell(s_\alpha w'\theta'(s_\alpha)) = \ell(w') + 2$. If $n_\alpha \in N_G(A)_k$ has image s_α in W, then for $n_1 = n_\alpha n\theta(n_\alpha)^{-1}$*

$$\dim(P * n_1) = \dim(P * n) + \dim P_\alpha - \dim P,$$

where P_α is the standard parabolic k–subgroup of G containing P with $\Phi(P_\alpha, A) = (\mathbb{Z}\alpha \cap \Phi) \cup \Phi^+$.

Proof. Set $v_1 = n_\alpha v$. By Lemma 8.8, we have that

$$\dim((v_1^{-1}Pv_1)^\theta) = \dim((v^{-1}Pv)^\theta) + \dim P - \dim P_\alpha.$$

Note that $\dim(P * n) = \dim(P) - \dim((v^{-1}Pv)^\theta)$ and $\dim(P * n_1) = \dim(P) - \dim((v_1^{-1}Pv_1)^\theta)$. Now the assertion is obvious.

9.2. Proposition. *Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$, w the image of n in W and $w' = ww_0$. The following conditions are equivalent:*

(i) *$P * n$ is open in $Q = \{x\theta(x)^{-1} | x \in G\}$.*

(ii) *Let ζ be the involution of G given by $\zeta(x) = n\theta(x)n^{-1}$, $x \in G$ and $\Pi = I(w) \cap \Delta$. Then $C''(w) \cap \Delta = \phi$ and ζ is trivial on the isotropic factor of G_{Φ_Π} where Φ_Π is the subsystem of $\Phi(G, A)$ consisting of integral combinations of Π.*

(iii) *$w' = w^0_\Pi w^0_\Delta$ and ζ is trivial on the isotropic factor of G_{Φ_Π}.*

(iv) *$v^{-1}P_\Pi v$ is a minimal θ-split parabolic k-subgroup of G, where P_Π is the standard parabolic k–subgroup of G containing P with $\Phi(P_\Pi, A) = \Phi_\Pi \cup \Phi^+$.*

(v) *There exists a minimal θ-split parabolic k-subgroup of G containing $v^{-1}Pv$.*

Proof. (ii) $\Rightarrow$ (iii). Note that $w'\theta' = w\theta$. Since $C''(w) \cap \Delta = \phi$, by [23, (ii) of Lemma 3.2] $s_\alpha w'\theta'(s_\alpha) = w'$ for all $\alpha \in \Delta$ with $s_\alpha w' > w'$. Hence by Proposition 7.11, $w' = w^0_\Pi w^0_\Delta$.

(iii) $\Rightarrow$ (iv). Observe that $v^{-1}P_\Pi v \cap \theta(v^{-1}P_\Pi v) = v^{-1}(P_\Pi \cap n\theta(P_\Pi)n^{-1})v$. From Proposition 7.11, $(\Phi_\Pi \cup \Phi^+) \cap w\theta(\Phi_\Pi \cup \Phi^+) = \Phi_\Pi \cup \Phi^+_\Pi = \Phi_\Pi$. It follows that $v^{-1}P_\Pi v \cap \theta(v^{-1}P_\Pi v) = v^{-1}G_{\Phi_\Pi}v$. Consequently $v^{-1}P_\Pi v$ is a θ-split parabolic k-subgroup of G. Since ζ is trivial on the isotropic factor of G_{Φ_Π}, θ is trivial on the isotropic factor of $v^{-1}G_{\Phi_\Pi}v$. By (iii) of Proposition 4.7, $v^{-1}P_\Pi v$ is a minimal θ-split parabolic k-subgroup of G.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i). By Lemma 4.8, $v^{-1}PvH$ is open in G. Hence PvH is open in G and (i) is immediate.

(i) $\Rightarrow$ (ii). By Lemma 9.1, $C''(w) \cap \Delta = \phi$. As in (ii) $\Rightarrow$ (iii), this condition yields that $w' = w^0_\Pi w^0_\Delta$ and as in (iii) $\Rightarrow$ (iv) $v^{-1}P_\Pi v \cap \theta(v^{-1}P_\Pi v) = v^{-1}G_{\Phi_\Pi} v$. Now let $P_0 = P \cap G_{\Phi_\Pi}$. A simple dimension argument shows that $P_0 * n$ is a big cell in $G_{\Phi_\Pi} * n$. It follows that $P_0 G^\zeta_{\Phi_\Pi}$ is open in G_{Φ_Π}. However $\zeta|\Phi_\Pi = 1$ and P_0 is ζ–stable. Now by Lemma 1.7, $G_{\Phi_\Pi} = P_0 G^\zeta_{\Phi_\Pi}$ and by Lemma 1.8, ζ is trivial on the isotropic factor of G_{Φ_Π}.

9.3. Lemma. *Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$, w the image of n in W and $w' = ww_0$. Let $\Pi = \Delta \cap R(w) = \{\alpha \in \Delta | w\theta\alpha = -\alpha\}$ and P_Π the standard parabolic k-subgroup of G containing P defined by Π. We have the following conditions:*

(i) *If $w' = w^0_\Pi$, then the group $P' = v^{-1}P_\Pi v$ is θ-stable and the derived group $D(P'/R_u(P'))$ of $P'/R_u(p')$ has a θ-split maximal k-split torus.*

(ii) *Conversely if P'' is any θ-stable parabolic k-subgroup of G satisfying the condition that $D(P''/R_u(P''))$ has a θ-split maximal k-split torus, then there exist $v \in (\tau^{-1}N)_k$ and a θ'-stable subset Π of Δ such that $P'' = v^{-1}P_\Pi v$, $w^0_\Pi\theta'|\Pi = -1$, and $v\theta(v)^{-1}$ has image $w^0_\Pi w_0^{-1}$ in W.*

Proof. (i) First note that $w\theta = w'\theta'$. From the condition $w' = w^0_\Pi$, $w'\theta'(\Phi_\Pi \cup \Phi^+) = \Phi_\Pi \cup \Phi^+$. By [3, 3.22], we have that $P' \cap \theta(P') = v^{-1}P_\psi v$ with $\psi = (\Phi_\Pi \cup \Phi^+) \cap w\theta(\Phi_\Pi \cup \Phi^+) = \Phi_\Pi \cup \Phi^+$. It follows that P' is θ-stable. Let A_Π denote the identity component of $A \cap D(G_{\Phi_\Pi})$. Since $w\theta|\Phi_\Pi = -1$, $v^{-1}A_\Pi v$ is a θ-split maximal k-split torus of $D(v^{-1}G_{\Phi_\Pi} v)$.

(ii) Let L be a θ-stable Levi k-subgroup of P'' and A_1 a θ-stable maximal k-split torus of L such that $(A_1 \cap D(L))^0$ is a θ-split maximal k-split torus of $D(L)$. There is a unique $\Pi \subset \Delta$ such that P'' is conjugate to P_Π by an element of G_k. Now choose $v \in G_k$ with $P'' = v^{-1}P_\Pi v$, $L = v^{-1}G_{\Phi_\Pi} v$ and $A_1 = v^{-1}Av$. Since A and A_1 are θ-stable, $n = v\theta(v)^{-1} \in N_G(A)$. Let w be the image of n in W. By the assumption on A_1, $\theta|\Phi(L, A_1) = -1$. It follows that $w\theta\alpha = -\alpha$ for $\alpha \in \Phi_\Pi$. Note that $R_u(P_\Pi) = U_\eta$ where $\eta = \Phi^+ - \Phi^+_\Pi$. Since $R_u(P'')$ is θ-stable, it follows that $\Phi^+ - \Phi^+_\Pi$ is $w\theta$-stable. As $w'\theta' = w\theta$, w' satisfies the condition that $w'\theta'\alpha = -\alpha$, $\alpha \in \Phi_\Pi$ and $\Phi^+ - \Phi^+_\Pi$ is $w'\theta'$-stable. Then by [23, Cor. 3.4], Π is θ'-stable and $w^0_\Pi = w'$.

9.4. Lemma. *Let $\tau(g) = g\theta(g)^{-1}, g \in G, v \in (\tau^{-1}N)_k$ and $P' = v^{-1}P_\Pi v$ be as in Lemma 9.3. Then we have the conditions:*

(i) *$\tau(P')$ is a closed irreducible and smooth k-subvariety of $Q = \{g\theta(g)^{-1} | g \in G\}$.*

(ii) *$\tau(v^{-1}Pv)$ is dense in $\tau(P')$.*

Proof. Since P' is θ-stable, by Lemma 1.7, $P'H$ is a closed subvariety of G. It follows that $\tau(P')$ is a closed irreducible k-subvariety of Q. As $\tau(P')$ is homogeneous, it is smooth. Note that $v^{-1}G_{\Phi_\Pi} v$ is θ-stable. By the assumption on Π, $v^{-1}G_{\Phi^+_\Pi} v$ is a θ-split parabolic k-subgroup of $v^{-1}G_{\Phi_\Pi} v$. It follows from Lemma 4.8 that $v^{-1}G_{\Phi^+_\Pi} vH$ is dense in $v^{-1}G_{\Phi_\Pi} vH$. Hence $v^{-1}PvH$ is dense in $P'H$. Now the assertion is obvious.

Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$, w the image of n in W and $w' = ww_0$. We write w' as in Proposition 7.9

$$w' = s_1 \dots s_h w^0_\Pi \theta'(s_h) \dots \theta'(s_1),$$

with $\ell(w') = 2h + \ell(w^0_\Pi)$. Choose $n_1, \ldots, n_h \in N_G(A)_k$ with images $s_1, \ldots, s_h$ in W respectively. Set $u = n_h^{-1} \ldots n_1^{-1} v$, $m = u\theta(u)^{-1}$, $s_i = s_{\alpha_i}$ and $P_i = P_{\alpha_i}$ (8.1), $1 \leq i \leq h$.

9.5. Proposition. *(i)* $c\ell(P * n) = P_1 * \cdots * P_h * P_\Pi * m$.

(ii) $\dim(P * n) = \Sigma_{i=1}^h (\dim(P_i) - \dim(P)) + \dim(P_\Pi * m)$.

Proof. We prove the assertion by induction on h, starting $h = 0$. When $h = 0$, the result is immediate from Lemma 9.4. Set $v_1 = n_1^{-1} v = n_2 \ldots n_h u$. We may assume that

$$c\ell(P * v_1\theta(v_1)^{-1}) = P_2 * \cdots * P_h * P_\Pi * m.$$

By Lemma 9.1, we have that

$$\dim(P * n) = \dim(P * v_1\theta(v_1)^{-1}) + \dim P_1 - \dim P. \tag{1}$$

Clearly $\dim(P_1 * P * v_1\theta(v_1)^{-1}) \leq \dim(P * v_1\theta(v_1)^{-1}) + \dim P_1 - \dim P$. Hence we have the relation that $\dim(P_1 * P * v_1\theta(v_1)^{-1}) \leq \dim(P * n)$. However $P_1 * P * v_1\theta(v_1)^{-1}$ contains $P * n$. This yields that

$$\dim(P_1 * P * v_1\theta(v_1)^{-1}) = \dim(P * n). \tag{2}$$

Since $P_1 * c\ell(P * v_1\theta(v_1)^{-1})$ is closed and irreducible, we conclude that $c\ell(P * n) = c\ell(P_1 * P * v_1\theta(v_1)^{-1}) = P_1 * c\ell(P * v_1\theta(v_1)^{-1}) = P_1 * \cdots * P_h * P_\Pi * m$. (ii) follows from (1) by an easy induction.

Remark. When k is algebraically closed, the above result is due to Springer [23, Theorem 6.5]. Our argument is a refinement of his. The dimension formula in Lemma 8.8 yields additional information on the dimension of the orbit.

9.6. Lemma. *Let ψ be a unipotent quasi-closed subset of $\Phi(G, A)$. Then the projection map $U_\psi(k) \to (U_\psi/U_\psi \cap H)(k)$ is surjective.*

Proof. By [3, Prop. 3.22], $U_\psi \cap \theta(U_\psi) = U_\psi \cap U_{\theta(\psi)} = U_{\psi\cap\theta(\psi)}$. It follows that the unipotent group $U_{\psi\cap\theta(\psi)}$ is k-split from [3, Cor. 3.18]. Hence by [3, (ii) of 2.7] the projection map $U_\psi(k) \to (U_\psi/U_{\psi\cap\theta(\psi)})(k)$ is surjective. By Lemma 0.6, the projection map $U_{\psi\cap\theta(\psi)}(k) \to (U_{\psi\cap\theta(\psi)}/U_\psi \cap H)(k)$ is also surjective. These two surjective conditions imply readily our desired assertion.

9.7. Proposition. *Let $v \in (\tau^{-1}N)_k$, $n = v\theta(v)^{-1}$, w the image of n in W and $w' = ww_0$. Let $\alpha \in \Delta$ be a simple root such that $\ell(s_\alpha w'\theta'(s_\alpha)) = 2 + \ell(w')$. If $n_\alpha \in N_G(A)_k$ has image s_α in W, then $P_\alpha(k) * n$ has only two $P(k)$-orbits $P(k) * n$ and $P(k) * (n_\alpha)n\theta(n_\alpha)^{-1}$.*

Proof. We show the assertion in several steps. Let ψ be the set given by $\psi = \{\beta \in \Phi^+ | w\theta\beta > 0\}$ and ξ the involution of G defined by $\xi(x) = n\theta(x)n^{-1}$, $x \in G$.

Step 1. $(v^{-1}P_\alpha v)^\theta = v^{-1}(Z_G(A)^\xi \ltimes U_\psi^\xi)v$. Let $\Phi_\alpha = \Phi^+ \cup (\mathbb{Z}\alpha \cap \Phi)$. Clearly $P_\alpha = G_{\Phi_\alpha}$ and by [3, 3.22] $v^{-1}P_\alpha v \cap \theta(v^{-1}P_\alpha v) = v^{-1}G_{\Phi_\alpha \cap w\theta(\Phi_\alpha)}v$. The length condition on $s_\alpha w'\theta'(s_\alpha)$ implies $\alpha \neq w\theta\alpha > 0$ and $\Phi_\alpha \cap w\theta(\Phi_\alpha) = \psi$. It follows that $v^{-1}P_\alpha v \cap \theta(v^{-1}P_\alpha v) = v^{-1}G_\psi v$. By 8.2, $(v^{-1}P_\alpha v)^\theta = (v^{-1}G_\psi v)^\theta = v^{-1}G_\psi^\xi v$. Note that $G_\psi = Z_G(A) \ltimes U_\psi$. Clearly $G_\psi^\xi = Z_G(A)^\xi \ltimes U_\psi^\xi$.

Step 2. $R_u(P) = U_\psi^\xi R_u(P_\alpha)$. Given $X \in \mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{2\alpha}$), we have that

$$X + \xi(X) \in L(U_\psi^\xi) \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{w\theta\alpha}).$$

Since $\alpha \neq w\theta\alpha > 0$, $\mathfrak{g}_{w\theta\alpha} \subset L(R_u(P_\alpha))$. Hence $X \in L(U_\psi^\xi) + L(R_u(P_\alpha))$. It follows that $L(R_u(P)) = L(U_\psi^\xi) + L(R_u(P_\alpha))$ and as a consequence $R_u(P) = U_\psi^\xi R_u(P_\alpha)$.

Step 3. $R_u(P)(k) = R_u(P_\alpha)(k)U_\psi^\xi(k)$. By Lemma 9.6, the projection $R_u(P_\alpha)(k) \to (R_u(P_\alpha)/R_u(P_\alpha) \cap U_\psi^\xi)(k)$ is surjective. Now the assertion follows from the fact that $R_u(P)/U_\psi^\xi$ and $R_u(P_\alpha)/R_u(P_\alpha) \cap U_\psi^\xi$ are k-isomorphic as k-varieties.

Step 4. Let M denote the group $U_\psi^\xi(k)$. Since $R_u(P_\alpha)(k)$ is normal in $P_\alpha(k)$, we have that

$$P(k)\backslash P_\alpha(k)/M \simeq P(k)\backslash P_\alpha(k)/R_u(P_\alpha)(k)M.$$

By Step 3, $R_u(P)(k) = R_u(P_\alpha)(k)M$. It implies that

$$P(k)\backslash P_\alpha(k)/M \simeq P(k)\backslash P_\alpha(k)/R_u(P)(k).$$

Now the desired assertion follows readily.

9.8. Corollary. *Assume that $\ell(s_\alpha w'\theta'(s_\alpha)) = \ell(w') - 2$. Then $P_\alpha(k) * n$ consists of only two $P(k)$ orbits $P(k) * n$ and $P(k) * (n_\alpha n\theta(n_\alpha)^{-1})$.*

Proof. Consider the element $n_\alpha v$. Applying Proposition 9.7, we know that $P_\alpha(k) * (n_\alpha n\theta(n_\alpha)^{-1}) = P(k)*(n_\alpha n\theta(n_\alpha)^{-1}) \cup P(k)*n$. Since $P_\alpha(k)*n = P_\alpha(k)*(n_\alpha n\theta(n_\alpha)^{-1})$, the assertion is obvious.

Now we are ready to discuss the t-closure of P_k orbit in Q_k when k is a local field. The result bears a striking resemblance to Proposition 9.5.

9.9. Proposition. *Let k be a local field, $v \in \tau^{-1}N_k$, $n = v\theta(v)^{-1}$, w the image of n in W and $w' = ww_0$. let us write w' according to Proposition 7.9, $w' = s_1, \ldots s_h w_\Pi^0 \theta'(s_h) \ldots \theta'(s_1)$ with $\ell(w') = 2h + \ell(w_\Pi^0)$. Let $n_1, \ldots, n_h \in N_G(A)_k$ with images $s_1, \ldots, s_h$ respectively in W, $s_i = s_{\alpha_i}$, $\alpha_i \in \Delta$, $P_i = P_{\alpha_i}$, $1 \le i \le h$ and $m = u\theta(u)^{-1}$ with $u = n_h^{-1} \ldots n_1^{-1} v$. Then we have the condition that*

$$t\text{-}c\ell(P(k) * n) = P_1(k) * \cdots * P_h(k) * t\text{-}c\ell(P(k) * m).$$

Proof. Set $v_1 = n_2 \ldots n_h u$. By Lemma 9.7, $P_1(k) * (v_1\theta(v_1)^{-1})$ consists of two $P(k)$ orbits $P(k) * v_1\theta(v_1)^{-1}$ and $P(k) * n$. By Lemma 9.1, $\dim(P(k) * v_1\theta(v_1)^{-1}) < \dim(P(k) * n)$. Since $P_1(k) * v_1\theta(v_1)^{-1}$ is an analytic k-variety, $P(k) * n$ is t-dense in $P_1(k) * v_1\theta(v_1)^{-1}$. It follows that

$$\text{t-c}\ell(P(k) * n) = \text{t-c}\ell(P_1(k) * P(k) * (v_1\theta(v_1)^{-1})).$$

Since $P_1(k)/P(k)$ is compact, the right member coincides with $P_1(k) * \text{t-c}\ell(P(k) * v_1\theta(v_1)^{-1})$. Now the assertion follows easily by induction on h.

Remark. Note that in Step 4 of the proof of Proposition 9.7 $M \subset (G^\xi)^0 = v(G^\theta)^0 v^{-1}$. We actually prove a slightly stronger result. Let H be an open k-subgroup of G^θ, $v \in (\tau^{-1}N)_k$ and $n_\alpha \in N_G(A)_k$ with image s_α. If $l(s_\alpha w'\theta'(s_\alpha)) = l(w') \pm 2$, then $P_\alpha(k)vH_k = P_k vH_k \cup P_k n_\alpha vH_k$. Moreover as a consequence if $v = n_1 \ldots n_h u$ satisfies the conditions in Proposition 9.9, we have $\text{t-cl}(P_k vH_k) = P_1(k) \ldots P_h(k) \cdot \text{t-cl}(P_k uH_k)$.

10. $(PH)_k$**.**

Let P be a minimal parabolic k-subgroup of G and H an open k-subgroup of G^θ. In this section, we study the set $(PH)_k$. As an application, we discuss the Iwasawa decomposition.

10.1. Lemma. *Let ψ be a unipotent quasi-closed subset of $\Phi(G, A)$. Then the group $U_\psi^\theta = \{x \in U_\psi | \theta(x) = x\}$ is connected and defined over k; in particular for any parabolic k-subgroup P_1 of G, $R_u(P_1)^\theta$ is contained in H^0.*

Proof. By [3, 3.22], $U_\psi \cap \theta(U_\psi) = U_{\psi\cap\theta(\psi)}$. The assertion is immediate from Lemma 0.6.

10.2. Lemma. *Let P be a minimal parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Then we have the following conditions:*

(i) $PH \cap (\tau^{-1}N)_k = (Z_G(A)H)_k$.
(ii) $(PH)_k = U_k(Z_G(A)H)_k$, *where* $U = R_u(P)$.
(iii) *For* $v \in (\tau^{-1}N)_k$, $(PvH)_k = U_k(Z_G(A)vH)_k$.

Proof. (i) Given $x \in PH \cap (\tau^{-1}N)_k$, we write $x = uzh$ with $u \in U$, $z \in Z_G(A)$ and $h \in H$. By the definition of $(\tau^{-1}N)_k$, $x\theta(x)^{-1} = uz\theta(z)^{-1}\theta(u)^{-1} = n \in N_G(A)$. It yields that $Uz\theta(z)^{-1}\theta(U) = Un\theta(U)$. By [3, 5.15], $z\theta(z)^{-1} = n$ and as a consequence $z^{-1}uz \in U^\theta$. By Lemma 10.1, $U^\theta \subset H$. Hence $x = z(z^{-1}uz)h \in Z_G(A)H$. (ii) The assertion is immediate from (i) and Proposition 6.6. (iii) Consider the group $v^{-1}Pv$. The assertion is obvious by (ii).

10.3. Proposition. *Let A_1 and A_2 be maximal (θ, k)-split tori of G and A a maximal k-split torus of G containing A_1. Then there exists $g \in (H^0 Z_G(A))_k$ such that $gA_1g^{-1} = A_2$.*

Proof. Choose $\lambda \in X_*(A_1)$ such that $\alpha \in \Phi(Z_G(A_1), A)$ for $\alpha \in \Phi(G, A)$ with $\langle\lambda, \alpha\rangle = 0$. By Proposition 4.7, the parabolic k-subgroup $P(\lambda)$ of G containing A defined by λ (3.1) is a minimal θ-split parabolic k-subgroup of G. This shows that there exists a minimal θ-split parabolic k-subgroup P_1 (resp. P_2) of G containing A_1 (resp. A_2). By Proposition 4.9, there exists $x \in G_k$ such that $xP_1x^{-1} = P_2$. By Lemma 4.8, $H^0P_1 = H^0xP_1$ and as a consequence $x \in (H^0P_1)_k$. Now let P be a minimal parabolic k-subgroup of P_1 containing A. By Lemma 4.8, $H^0P_1 = H^0P$. Note that A is θ-stable. By Lemma 10.2, there exists $u \in R_u(P)_k$ such that $xu \in (H^0Z_G(A))_k$. Set $g = xu$. Clearly $gP_1g^{-1} = P_2$ and gA_1g^{-1} is (θ, k)-split. Since $gA_1g^{-1} \subset P_2\cap\theta(P_2) = Z_G(A_2)$, $gA_1g^{-1}A_2$ is a (θ, k)-split torus of G. The maximality condition implies easily that $A_2 = gA_1g^{-1}$.

10.4. Lemma. *Let G be a connected reductive algebraic k-group, A a maximal k-split torus of G and ψ a unipotent quasi-closed subset of $\Phi(G, A)$. If S is a subtorus of A such that $\alpha|S \neq 1$ for $\alpha \in \psi$, then $S \cdot U_\psi$ is generated by S and $(U_\psi)_k$.*

Proof. By [3, Remark to 3.18], the group U_ψ has an S-invariant normal series $U_1 \subset U_2 \cdots \subset U_\ell = U_\psi$ of k-subgroups such that for each i, U_i/U_{i-1} has a k-vector space structure and S acts on it by a rational character. By an easy induction, S and $(U_i)_k$ generate $S \cdot U_i$ for every i.

10.5 Proposition. *Let G be a connected reductive k-group and θ an involution of G defined over k. The following conditions (i), (ii) and (iii) are equivalent:*

(i) *the group $[G, G] \cap H$ is anistropic over k.*
(ii) *Any parabolic k-subgroup of G is θ-split.*
(iii) *Any minimal parabolic k-subgroup of G is θ-split.*

These conditions imply that

(iv) $G_k = (PH^0)_k$ *for any minimal parabolic k-subgroup P of G.*

Conversely if (iv) holds, then there exists an almost direct product $G = G_1 \cdot G_2$ of k-groups such that $\theta|G_2$ is trivial and $\theta|G_1$ satisfies the above equivalent conditions.

Proof. To establish the equivalent conditions, we may assume that G is semi-simple.

(i) $\Rightarrow$ (ii). Let P_1 be any parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P_1. Then there exists $\lambda \in X_*(A)$ such that P_1 is the parabolic k-subgroup $P(\lambda)$ of G containing A defined by λ. Clearly $A = A_-$ and $\theta(P_1) = P(-\lambda)$. It follows that P_1 is θ-split.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Let A be any θ-stable maximal k-split torus of G. It suffices to show that $A = A_-$. Let P denote a minimal parabolic k-subgroup of G containing A, $\Phi^+ = \Phi(P, A)$ and Δ the set of simple roots of Φ^+. For $\alpha \in \Delta$, let $n_\alpha \in N_G(A)_k$ with image s_α in the Weyl group. We have that $P = G_{\Phi^+}$ and $n_\alpha P n_\alpha^{-1} = G_{s_\alpha(\Phi^+)}$. From (iii), $\theta(P) = G_{\theta(\Phi^+)} = G_{-\Phi^+}$ and $\theta(G_{s_\alpha(\Phi^+)}) = G_{\theta s_\alpha(\Phi^+)} = G_{-s_\alpha(\Phi^+)}$. By [3, 3.23], $\theta(\Phi^+) = -\Phi^+$ and $\theta s_\alpha(\Phi^+) = -s_\alpha(\Phi^+)$. It follows that $s_\alpha(\Phi^+) = s_{\theta(\alpha)}(\Phi^+)$ and as a consequence $s_\alpha = s_{\theta(\alpha)}$. We must have that $\theta(\alpha) = \alpha$ or $\theta(\alpha) = -\alpha$. Since P is θ-split, $\theta(\alpha) = -\alpha$. From $\theta(\alpha) = -\alpha$, $\alpha \in \Delta$, one concludes easily that $\theta|A = -1$.

(iii) $\Rightarrow$ (iv). Given any $g \in G_k$, by Lemma 4.8 $PgH^0 = PH^0$ is the unique open double coset. This implies that $g \in (PH^0)_k$.

Now assume that G satisfies condition (iv). Write G as an almost direct product $G = G_1 \cdot G_2$ of k-groups such that G_2 is maximal with the property $\theta|G_2 = 1$. We show that $\theta|G_1$ satisfies condition (i). Suppose the assertion to be false. Let S be a nontrivial k-split torus of the group $[G_1, G_1] \cap H$ and A a θ-stable maximal k-split torus of G containing S. Let $\lambda \in X_*(S)$ be a nonzero element and P_1 (resp. P_1^-) denote the parabolic k-subgroup of G containing A defined by λ (resp. $-\lambda$). Then P_1 and P_1^- are θ-stable opposite parabolic k-subgroups of G.

Condition (iv) implies that $G_k = (P_1 H^0)_k = (P_1^- H^0)_k$. From this, the groups $g^{-1} P_1 g$ and $g^{-1} P_1^- g$ are θ-stable for $g \in G_k$. Hence we have that $g\theta(g)^{-1} \in P_1 \cap P_1^-$, $g \in G_k$. Now set $M = P_1 \cap P_1^-$, $P_1 = M \ltimes U$, and $P_1^- = M \ltimes U^-$. Clearly U and U^- are θ-stable. It follows that for $g \in U_k$ (resp. U_k^-), $g\theta(g)^{-1} \in M \cap U = \{e\}$ (resp. $M \cap U^- = \{e\}$). Due to our construction of $P(\lambda)$, $\langle \lambda, \alpha \rangle > 0$ (resp. < 0) for $\alpha \in \Phi(U, A)$ (resp. $\Phi(U^-, A)$). By Lemma 10.4, U and U^- are contained in $G_1 \cap H$. It follows that U and U^- generate a nontrivial connected normal k–subgroup of G_1 contained in H. Clearly this contradicts the maximality property of G_2.

10.6. Corollary. *Let A be a θ-stable maximal k-split torus of G. If H is anisotropic over k, then $N_G(A) = N_{H^0}(A) Z_G(A)$.*

Proof. Let P denote a minimal parabolic k-subgroup of G containing A and $U = R_u(P)$. Given $n \in N_G(A)_k$, by the preceding proposition we can write $n = uzh$ with

$u \in U, z \in Z_G(A)$ and $h \in H^0$. Consider the element $n\theta(n)^{-1} = uz\theta(z)^{-1}\theta(u)^{-1}$. By **[3**, 5.15 **]**, $n\theta(n)^{-1} = z\theta(z)^{-1}$. Since P is θ-splitting, $U \cap \theta(U) = \{e\}$. This implies that $u = \{e\}$ and $n = zh \in Z_G(A)N_{H^0}(A)$. As $N_G(A) = N_G(A)_k Z_G(A)$, the desired assertion follows.

10.7. Lemma. *Let k be a field with $ch(k) = 0$ and M a reductive k-subgroup of $GL(n)$. If $B(X, Y)$ is the bilinear form of $L(M)$ given by $B(X, Y) = Tr(XY)$, $X, Y \in L(M)$, then it is nondegenerate.*

Proof. Case 1: M is a torus.

By taking a conjugation over the algebraic closure $\bar{k}$, we may assume that M is contained in the group of diagonal matrices. In this case, M is defined over $\mathbb{Q}$ and $L(M) = L(M)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{k}$. Since $B(X, Y)$ is positive over $L(M)_{\mathbb{Q}}$, our assertion is obvious.

Case 2: M is a reductive group.

Let ϑ denote the set given by $\vartheta = \{X \in L(M) | B(X, Y) = 0 \text{ for all } Y \in L(M)\}$. Then ϑ is an ideal of $L(M)$. If $\vartheta \neq 0$, it contains a nonzero semi-simple element X_0. By Case 1, $B(X, Y)$ is nondegenerate on the smallest algebraic subalgebra of $L(M)$ containing X_0. Certainly this is a contradiction and we must have that $\vartheta = 0$.

10.8. Proposition. *Let G be a connected reductive algebraic k-group with $ch(k) = 0$, $Q = \{g\theta(g)^{-1} | g \in G\}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the decomposition of $\mathfrak{g} = L(G)$ into eigen subspaces of θ where $\mathfrak{h} = L(H)$. Suppose that $H \cap [G, G]$ is anisotropic over k. Then we have the following conditions:*

(i) *Q_k consists of semi-simple elements.*
(ii) *$\mathfrak{q}_k$ consists of semi-simple elements.*
(iii) *If M is a connected k-subgroup of G containing H^0, then there exists a connected normal k subgroup G' of G such that $M = H^0 G'$.*

Proof. (i) Given $g \in Q_k$, let $g = g_s g_u$ denote the Jordan decomposition of g. By [20, Lemma 6.2], $g_s, g_u \in Q_k$. Suppose that $g_u \neq e$. Let V be the closure of the group generated by g_u in G. Since $\theta(g_u) = g_u^{-1}$, V is a θ-stable unipotent k-subgroup of G. By **[3**, 8.3**]**, there exists a minimal parabolic k-subgroup P of G with $V \subset R_u(P)$. However by Proposition 10.5, P is θ-split and consequently $V \subset R_u(P) \cap R_u(\theta(P)) = \{e\}$. Certainly this is a contradiction.

(ii) Given $X \in \mathfrak{q}_k$, let $A(X)$ denote the smallest algebraic subgroup of G with $X \in L(A(X))$. Then $A(X)$ is a θ-stable abelian k-subgroup of G. Since $X \in \mathfrak{q}_k$, $\theta(X) = -X$ and so $\theta | A(X) = -1$ and by (i), $A(X)$ is a torus. Now the assertion is obvious.

(iii) Let G' be the maximal connected normal k-subgroup of G contained in M. Write $G = G' \cdot G''$ as an almost direct product of k-groups. Because $L(M)$ is θ-stable, so are M and G'. We may replace G by G'' and assure that $G' = \{e\}$.

The group $R_u(M)$ is a θ-stable k-subgroup. Since $H \cap [G, G]$ is anisotropic over k, $R_u(M)^\theta = \{e\}$. Thus by Lemma 0.6, $R_u(M) \subset Q$ and by (i) $R_u(M) = \{e\}$.

Now let $B(X, Y)$, $X, Y \in L(G)$ denote the nondegenerate invariant form $Tr(XY)$. Set $\mathfrak{m} = L(M)$ and let $\mathfrak{q}_1 = \mathfrak{m}^\perp$ (in $\mathfrak{g}$) denote the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{g}$. By Lemma 10.7, $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}_1$. Let $\mathfrak{h}_1 = [\mathfrak{q}_1, \mathfrak{q}_1]$ and $\vartheta = \mathfrak{h}_1 + \mathfrak{q}_1$. Clearly $\mathfrak{q} = \mathfrak{h}^\perp$ (in $\mathfrak{g}$) and we have that $\mathfrak{q}_1 \subset \mathfrak{q}$ and $\mathfrak{h}_1 \subset \mathfrak{h}$. Since $B(X, Y)$ is invariant, $[\mathfrak{m}, \mathfrak{q}_1] \subset \mathfrak{q}_1$. It follows readily that ϑ is an ideal of $\mathfrak{g}$. Thus the ideal $\vartheta^\perp$ (in $\mathfrak{g}$) is contained in $\mathfrak{q}_1^\perp = L(M)$. By the above additional assumption on G, we have that $\vartheta^\perp = 0$. Hence $\mathfrak{m} = [\mathfrak{q}_1, \mathfrak{q}_1] \subset \mathfrak{h}$ and as a consequence $M = H^0$.

Remark. If G has no anisotropic factors over k, (iii) yields that H^0 is a maximal connected anisotropic k-subgroup of G. The results in 10.5 and 10.8 resemble strikingly the usual properties of Cartan involutions for real groups. In the following, we derive the Iwasawa decomposition for real groups from these results.

10.9. Proposition. *Let G be an algebraic $\mathbb{R}$-group and K a compact Lie subgroup of $G_{\mathbb{R}}$. Assume that G acts on an affine $\mathbb{R}$-variety V by an algebraic action defined over $\mathbb{R}$. If $x \in V_{\mathbb{R}}$, then $K \cdot x = c\ell(K \cdot x)_{\mathbb{R}}$.*

Proof. Suppose that $z \in V_{\mathbb{R}} - K \cdot x$. Since K is compact, so is $K \cdot z \cup K \cdot x$. By the Stone-Weierstrass theorem, there exists $f \in \mathbb{R}[V]$ such that f is positive on $K \cdot x$ and negative on $K \cdot z$. As the action is algebraic and defined over $\mathbb{R}$, there exist $f_i \in \mathbb{R}[V]$ and $h_i \in \mathbb{R}[G]$, $i = 1, \cdots, \ell$ such that $f(g \cdot y) = \sum_{i=1}^{\ell} h_i(g) f_i(y)$, $g \in G$, $y \in V$. It follows that the function f^K, given by

$$f^K(y) = \int_K f(k \cdot y) dk,$$

lies in $\mathbb{R}[V]$. Here dk is a Haar measure of K. Now consider the function $\xi(y) = f^K(y) - c$, $y \in V$ with $c = f^K(x)$. Clearly ξ vanishes on $K \cdot x$ and $\xi(z) < 0$. This implies easily that $z \notin c\ell(K \cdot x)$.

10.10. Corollary. *Let G be an anisotropic reductive $\mathbb{R}$-group such that $G_{\mathbb{R}}$ meets every component of G. For any algebraic $\mathbb{R}$-subgroup L of G, the map $G_{\mathbb{R}} \to (G/L)_{\mathbb{R}}$ defined by the inclusion map is surjective.*

Proof. Since G is reductive and anisotropic over $\mathbb{R}$, so is L and as a consequence G/L is an affine $\mathbb{R}$-variety. Consider the natural action of G on G/L. Clearly $G_{\mathbb{R}}$ is compact and Zariski dense in G. Now the assertion is obvious from the preceding proposition.

10.11. Proposition. *Let G be a connected reductive algebraic $\mathbb{R}$-group and θ an involution of G defined over $\mathbb{R}$. Let A be a θ-stable maximal $\mathbb{R}$-split torus of G, P a minimal parabolic $\mathbb{R}$-subgroup of G containing A and $U = R_u(P)$. Assume that G has no anisotropic factors over $\mathbb{R}$ and H is anisotropic over $\mathbb{R}$. Then we have the following conditions:*

(i) *Let $(A_{\mathbb{R}})^0$ denote the topological identity component of $A_{\mathbb{R}}$. Then $G_{\mathbb{R}} = U_{\mathbb{R}}(A_{\mathbb{R}})^0 H_{\mathbb{R}}$.*
(ii) *$H_{\mathbb{R}} = (A_{\mathbb{R}} \cap H) H^0_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$.*

Proof. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ denote the decomposition of $L(G)$ into eigen subspaces of θ with $\mathfrak{h} = L(H)$. By (iii) of Proposition 10.8, H^0 is a maximal connected anisotropic $\mathbb{R}$-subgroup of G. It follows that $\theta|\mathfrak{g}_{\mathbb{R}}$ is a Cartan involution. Let $B(X, Y)$ stand for the form $Tr(XY)$, $X, Y \in L(G)$. Then $B(X, Y)$ is positive definite on $\mathfrak{q}_{\mathbb{R}}$ and negative definite on $\mathfrak{h}_{\mathbb{R}}$. As a consequence if M is a connected θ-stable anisotropic $\mathbb{R}$-subgroup of G, then $M \subset H$ for $B(X, Y)$ is negative definite on $L(M)_{\mathbb{R}}$. In particular we have that $Z_G(A)H^0 = AH^0$.

By Proposition 10.5, $G_{\mathbb{R}} = (PH^0)_{\mathbb{R}}$ and by Lemma 10.2, $G_{\mathbb{R}} = U_{\mathbb{R}}(Z_G(A)H^0)_{\mathbb{R}}$. Consider the quotient $A\backslash AH^0$. According to Corollary 10.10 the map $H^0_{\mathbb{R}} \to (A \cap H^0 \backslash H^0)_{\mathbb{R}} = (A\backslash AH^0)_{\mathbb{R}}$ is surjective. It follows that $(AH^0)_{\mathbb{R}} = A_{\mathbb{R}} H^0_{\mathbb{R}}$. Note that $A_{\mathbb{R}} \cap H = \{a \in A_{\mathbb{R}} | a^2 = e\}$. We have that $A_{\mathbb{R}} = (A_{\mathbb{R}})^0 (A_{\mathbb{R}} \cap H)$. Now we arrive to the condition $G_{\mathbb{R}} = U_{\mathbb{R}}(A_{\mathbb{R}})^0 (A_{\mathbb{R}} \cap H) H^0_{\mathbb{R}}$. Since $U_{\mathbb{R}}(A_{\mathbb{R}})^0$ has no nontrivial compact subgroups, now both (i) and (ii) follow immediately.

Remark. In [17], Matsumoto proved that $G_{\mathbb{R}} = (G_{\mathbb{R}})^0 A_{\mathbb{R}}$. Assertion (ii) can be viewed as a refinement of his result.

11. Cartan involutions.

In this section, we generalize the concept of Cartan involutions to algebraic groups and discuss properties of such involutions. Let G be a connected reductive algebraic k-group, θ an involution of G defined over k and H a k-open subgroup of G^θ. We denote by τ_θ, or simply τ when there is no confusion, the map of G defined by $\tau(g) = g\theta(g)^{-1}$, $g \in G$.

11.1. Lemma. *If S is a maximal θ-split k-torus of G, then S is a maximal θ-split torus of G.*

Proof. (1) First we show that if $\theta \neq 1$, then there exists a nontrivial θ-split k-torus of G. We may assume that G is semi-simple and by Proposition 4.3, we may assume that G is anisotropic over k. In this case, k is infinite and G_k is Zariski-dense in G. It follows from [21, Lemma 3.3] that there is $x \in G_k$ such that the element $g = x\theta(x)^{-1}$ is semi-simple and noncentral in G. Then consider the group $Z_G(g)^0$. By [21, Prop. 6.3], g belongs to a θ-split torus of G, $\theta|Z_G(g)^0 \neq 1$. Clearly $Z_G(g)^0$ is defined over k and $\dim(Z_G(g)^0) < \dim(G)$. The assertion follows by an easy induction on $\dim(G)$.

(2) Consider the group $Z_G(S)/S$. By (1) and maximality condition of S, θ is trivial on this group. It follows immediately that S is a maximal θ-split torus of G.

11.2. Lemma. *Let $g \in \tau(G)_k$ be a semi-simple element. If S is a maximal θ-split k-torus of $Z_G(g)^0$, then $g \in S$.*

Proof. By [21, Prop. 6.3], there exists a maximal θ-split torus S_1 of G containing g. Clearly $S_1 \subset Z_G(g)^0$. By Lemma 11.1, S is also a maximal θ-split torus of $Z_G(g)^0$. Then S and S_1 are conjugate by an element of $Z_G(g)^0 \cap H$. Since g is central in $Z_G(g)^0$, $g \in S$.

From now on, k is an infinite field.

11.3. Proposition. *Let P be a minimal parabolic k-subgroup of G, $U = R_u(P)$ and A a θ-stable maximal k-split torus of P. Assume that H is anisotropic over k. Then the following conditions are equivalent:*

(i) $(Z_G(A)H)_k = A_k H_k$.
(ii) *For any open subgroup H_1 of H, $(Z_G(A)H_1)_k = A_k H_1(k)$.*
(iii) $G_k = U_k A_k H_k^0$.
(iv) $G_k = U_k A_k H_k$.

Proof. (i) $\Rightarrow$ (ii). Write $Z_G(A) = M \cdot A$ as an almost direct product of k-groups. Consider the map $\tau(x) = x\theta(x)^{-1}, x \in M$. Condition (i) yields that $\tau(M_k) \subset A_k$. Since A centralizes M, $\tau|M_k$ is a homomorphism of M_k into A_k. The set M_k is Zariski-dense in M and as a consequence τ is a k-homomorphism of M into A. Then the group $\tau(M)$, being k-split and anisotropic over k, is trivial. It follows that $M \subset H^0$ and $Z_G(A)H_1 = AH_1$. Now given $x \in (Z_G(A)H_1)_k$, write $x = ah = a_1h_1$ with $a \in A, h \in H_1$ and $a_1 \in A_k, h_1 \in H_k$. Then $a_1^{-1}a \in A \cap H$. Since $A \cap H$ consists of elements of A of order 2, $A \cap H \subset A_k$ and so $a \in A_k$.

(ii) $\Rightarrow$ (iii). From (iv) of Proposition 10.5 and (ii) of Lemma 10.2, we know that $G_k = U_k(Z_G(A)H^0)_k$. By (ii), $G_k = U_k A_k H_k^0$.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). Given $x \in (Z_G(A)H)_k$, write $x = zh = uah'$ with $z \in Z_G(A)$, $h \in H$ and $u \in U_k$, $a \in A_k$ and $h' \in H_k$. Then we have that $z\theta(z)^{-1} = ua^2\theta(u)^{-1}$. According to [3, 5.15], $z\theta(z)^{-1} = a^2$. Since $U \cap \theta(U) = \{e\}$, $u = e$. It follows that $x = ah' \in A_k H_k$.

11.4. Corollary. *Assume that H is anisotropic over k and $G_k = U_k A_k H_k$. Then we have the following conditions:*

(i) *A is maximal θ-split.*
(ii) *Any two maximal (θ, k)-split tori are conjugate by an element of H_k^0.*
(iii) *Let P' be any minimal parabolic k-subgroup of G, $U' = R_u(P')$ and A' a maximal k-split torus of P'. Then $G_k = U'_k A'_k H_k$.*
(iv) *$N_{G_k}(A_k) = A_k \cdot N_{H_k^0}(A_k)$.*

Proof. (i) Let S be any maximal θ-split k-torus of G containing A. Then $S_k^2 \subset U_k A_k^2 \theta(U_k)$. By [3, 5.15], $S_k^2 = A_k^2$ and so $S = A$ for S_k is Zariski-dense in S. By Lemma 11.1, A is maximal θ-split.

(ii) Let A' be any maximal (θ, k)-split torus of G. By Propositon 10.3, there exists $x \in (Z_G(A)H^0)_k = A_k H_k^0$ such that $A' = x^{-1}Ax$. It follows that A' and A are conjugate by an element of H_k^0. Now the assertion is immediate.

(iii) We may assume that A' is θ-stable. By (ii), there is $y \in H_k^0$ with $A' = y^{-1}Ay$. Then $(Z_G(A')H)_k = y^{-1}(Z_G(A)H)_k y = y^{-1}A_k H_k y = A'_k H_k$. By Proposition 11.3, the assertion follows.

(iv) Given $n \in N_{G_k}(A_k)$, write $n = uah$ with $u \in U_k$, $a \in A_k$ and $h \in H_k^0$. Then $n\theta(n)^{-1} = ua^2\theta(u)^{-1}$ and by [3, 5.15] $n\theta(n)^{-1} = a^2$. Note that $U \cap \theta(U) = \{e\}$. This implies that $u = e$ and $h \in N_{H_k^0}(A_k)$.

11.5. Lemma. *Assume that $\tau(G_k)$ consists of k-split semi-simple elements. Then we have the following conditions:*

(i) *If L is a connected θ-stable reductive anistropic k-subgroup of G, then $L \subset H$.*
(ii) *Any θ-split k torus of G is k-split.*
(iii) *Any maximal θ-split k-torus of G is maximal (θ, k)-split.*
(iv) *Given any $x \in \tau(G_k)$, there is a maximal (θ, k)-split torus of G containing x.*

Proof. (i) Suppose the assertion to be false. By Lemma 11.1, there is a nontrivial θ-split k-torus S of L. It follows that S_k^2, consisting of k-split semi-simple elements, is k-diagonalizable. Since S_k^2 is Zariski-dense in S, S is k-split. From the assumption on L, S is anisotropic over k. Certainly this is a contradiction.

(ii) and (iii) are immediate from (i).

(iv) is immediate from Lemma 11.2 and (iii).

11.6. Proposition. *Assume that H is anisotropic over k and $G_k = H_k A_k H_k$. Then we have the following conditions:*

(i) *The set $\tau(G_k)$ consists of k-split semi-simple elements.*
(ii) *$G_k = U_k A_k H_k^0$.*
(iii) *$G_k = H_k^0 A_k H_k^0$.*
(iv) *If $(k^\times)^2 = (k^\times)^4$, then $G_k = \tau(G_k) G_k^\theta$.*

Proof. (i) The condition $G_k = H_k A_k H_k$ yields easily that $\tau(G_k)$ is the union of $xA_k^2x^{-1}$, $x \in H_k$. The assertion now is obvious.

(ii) We write $Z_G(A) = M \cdot A$ as an almost direct product of k-groups. The group M in θ-stable, connected, reductive and anisotropic over k. By (i) of Lemma 11.5, $M \subset H$ and $Z_G(A)H = AH$.

Give $x \in (Z_G(A)H)_k$, write $x = ah = h_1a_1h_2$ with $a \in A$, $h \in H$ and $h_1, h_2 \in H_k, a_1 \in A_k$. Then we have the condition that $a^2 = h_1a_1^2h_1^{-1}$. If λ is any eigen value of a, then either λ or $-\lambda$ is an eigen value of a_1. Since $a_1 \in A_k$, this implies that $\lambda \in k$. By assumption, A is k-split. Now it is easy to see that $a \in A_k$. It follows that $(Z_G(A)H)_k = A_kH_k$ and by Proposition 11.3, the assertion now follows.

(iii) From (ii), $H_k = (U_kA_k \cap H_k)H_k^0$. Given $x \in U_kA_k \cap H_k$, write $x = ua$ with $u \in U_k$ and $a \in U_k$. Then $ua = \theta(u)a^{-1}$. By [3, 5.15], $a = a^{-1}$. Since $U \cap \theta(U) = \{e\}, u = e$. This shows that $U_kA_k \cap H_k = A_k \cap H$ and $H_k \subset A_kH_k^0$. Now the condition $G_k = H_k^0A_kH_k^0$ is immediate from $G_k = H_kA_kH_k$.

(iv) The condition on k implies that $A_k^2 = A_k^4$. Since $\tau(G_k) = \cup xA_k^2x^{-1}$, $x \in H_k$, we have that $\tau(G_k)^2 = \tau(G_k)$. The assertion $G_k = \tau(G_k)G_k^\theta$ is obvious.

11.7. Proposition. *Assume that H is anisotropic over k and* $(k^\times)^2 = (k^\times)^4$. *The following conditions are equivalent:*

(i) $G_k = H_k^0A_kH_k^0$.
(ii) $G_k = H_kA_kH_k$.
(iii) $G_k = U_kA_kH_k$ *and* $\tau(G_k)$ *consists of k-split semi-simple elements.*
(iv) $G_k = U_kA_kH_k^0$ *and* $\tau(G_k)$ *consists of k-split semi-simple elements.*
(v) $G_k = \tau(G_k)G_k^\theta$ *and* $\tau(G_k)$ *consists of k-split semi-simple elements.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). It follows from (i) and (ii) of Proposition 11.6.

(iii) $\Rightarrow$ (iv). The assertion is immediate from Proposition 11.3.

(iv) $\Rightarrow$ (v). Given $x \in \tau(G_k)$, by (iv) of Lemma 11.5, there is a θ-stable maximal k-split torus A' of G containing x. Now let P' be a minimal parabolic k-subgroup of G containing A' and $U' = R_u(P')$. By (iii) of Corollary 11.4, $G_k = U_k'A_k'H_k$. Let $y \in G_k$ with $y\theta(y)^{-1} = x$. Write $y = uah$ with $u \in U_k'$, $a \in A_k'$ and $h \in H_k$. It follows that $x = ua^2\theta(u)^{-1}$. By [3, 5.15], $x = a^2 \in (A_k')^2$. Since $(k^\times)^2 = (k^\times)^4$, we have that $(A_k')^2 = (A_k')^4$. As a consequence there is $b \in (A_k')^2 \subset \tau(G_k)$ with $b^2 = x$. This establishes that $\tau(G_k)^2 = \tau(G_k)$ and so $G_k = \tau(G_k)G_k^\theta$.

(v) $\Rightarrow$ (i). Write $Z_G(A) = M \cdot A$ as an almost direct product of k-groups and $H = G^\theta$. By (i) of Lemma 11.5, $M \subset H$ and $Z_G(A)H = AH$. Given $x \in AH \cap \tau(G_k)$, write $x = ah$ with $a \in A$ and $h \in H$. Observe that $x^2 = a^2$. For any eigen value λ of a, either λ or $-\lambda$ is an eigen value of x. By assumption on $\tau(G_k)$, x is k-split and $\lambda \in k$. This yields easily that $a \in A_k$ and $h \in H_k$. Hence $AH \cap \tau(G_k) \subset A_kH_k$ and as a consequence $(Z_G(A)H)_k = (AH)_k = (AH \cap \tau(G_k))H_k = A_kH_k$. By Proposition 11.3 and (ii) of Corollary 11.4, any two maximal θ-stable maximal k-split tori of G are conjugate by an element of H_k^0.

Now let $y \in \tau(G_k)$. By (iv) of Lemma 11.5, there is a θ-stable maximal k-split torus A' containing y. The above discussion shows that there is $z \in H_k^0$ such that $A' = zAz^{-1}$. Then $y \in zA_kz^{-1} \subset H_k^0A_kH_k^0$. This yields that $G_k = H_kA_kH_k$. By Proposition 11.6, (i) follows.

11.8. *Let G be a connected reductive algebraic k-group and θ an involution of G defined over k. We call θ a Cartan involution of G (over k) if H is anisotropic over k, $(k^\times)^2 = (k^\times)^4$ and G_k satisfies the equivalent conditions in Proposition 11.7. We call θ a quasi-Cartan involution of G (over k) if H is anisotropic over k, $(k^\times)^2 = (k^\times)^4$ and $\tau(G_k)$ consists of k-split semi-simple elements.*

11.9. Corollary. *Let θ be a Cartan involution of G over k and M a θ-stable connected reductive k-subgroup of G. Then $\theta|M$ is a Cartan involution of M over k.*

Proof. Clearly $\tau(M_k)$ consists of k-split semi-simple elements. Given $x \in \tau(M_k)$, by (iv) of Lemma 11.5, there is a θ-stable k-split torus S of M containing x. Let $y \in M_k$ with $y\theta(y)^{-1} = x$. According to condition (i) of Proposition 11.7, write $y = hah'$ with $h, h' \in H_k$ and $a \in A_k$. Then $x = ha^2h^{-1}$ and as a consequence any eigen value λ of x belongs to $(k^\times)^2$. This yields that $x \in S_k^2$. Since $S_k^2 = S_k^4$ and S is θ-split, $x \in \tau(M_k)^2$. It follows that $M_k = \tau(M_k)M_k^\theta$. Thus by definition $\theta|M$ is a Cartan involution of M over k.

11.10. An example. *Let k be a subfield of $\mathbb{R}$ and k_+ the set of positive elements of k. Assume that $k_+ = (k_+)^2$. Let $G = SL(2)$ and $\theta(g) = {}^tg^{-1}$, $g \in G$. Then θ is a Cartan involution of G over k.*

11.11. Lemma. *Let θ_1 and θ_2 be quasi-Cartan involutions of G over k. If $\theta_1\theta_2 = \theta_2\theta_1$, then $\theta_1 = \theta_2$.*

Proof. Let H_1 and H_2 be the fixed point groups of θ_1 and θ_2 respectively. Clearly H_1 (resp. H_2) is θ_2-stable (resp. θ_1-stable). By (i) of Lemma 11.5, $H_1^0 \subset H_2^0$ and $H_2^0 \subset H_1^0$. So we have that $H_1^0 = H_2^0$ and by Proposition 1.2, $\theta_1 = \theta_2$.

11.12. Lemma. *Let θ be a quasi-Cartan involution of G over k and Z a θ-stable central k-subgroup of G. Then the induced involution of G/Z, denoted again by θ, is a quasi-Cartan involution.*

Proof. Let G' denote the quotient group G/Z and $\pi : G \to G'$ the projection map.

(1) Case: Z is a k-split torus.

In this case, $G'_k = \pi(G_k)$. Given any $y \in G'_k$, there is $x \in G_k$ with $\pi(x) = y$. By assumption, $x\theta(x)^{-1}$ is k-split semi-simple. Hence $y\theta(y)^{-1}$, being the image of $x\theta(x)^{-1}$ under π, is k-split semi-simple.

(2) Case: Z is anisotropic over k.

Let m be the cardinality of Z/Z^0. Consider the map $f : G \to G$ defined by $f(x) = (x\theta(x^{-1}))^m$, $x \in G$. By (i) of Lemma 11.5, $Z^0 \subset H$ and it follows that f factors through π. There is a map $f' : G' \to G$ with $f = f' \circ \pi$. Clearly f is defined over k and so is f'. It yields that for $y \in G'_k$, $(y\theta(y)^{-1})^{2m} \in \pi(\tau(G_k))$. Since 2 and m are prime to the characteristic of k, $y\theta(y)^{-1}$ is semi-simple for $(y\theta(y)^{-1})^{2m}$ is semi-simple. By Lemma 11.2, there exists a θ-split k-torus S of G' containing $y\theta(y)^{-1}$. Let $\widetilde{S} = (\pi^{-1}(S))^0$. Clearly $\pi(\widetilde{S}_-) = S$. By (ii) of Lemma 11.5, $\widetilde{S}_-$ is k-split and so is S. This implies that $y\theta(y)^{-1}$, being an element of S_k, is k-split semi-simple.

Now the desired assertion is immediate from (1) and (2) by a simple reduction argument.

11.13. Lemma. *Let σ and θ be involutions of G defined over k. If $\sigma\theta = \theta\sigma$ and θ is a quasi-Cartan involution of G, then there exists (σ, θ)-stable maximal k-split torus of G.*

Proof. We may assume that G is semi-simple.

(1) Case: G^σ is anisotropic over k.

Clearly G^σ is θ-stable and by (i) of Lemma 11.5, $(G^\sigma)^0 \subset G^\theta$. By Proposition 1.10, G has an almost direct product $G = G_1 \cdot G_2$ of algebraic groups such that $\sigma|G_1 = \theta|G_1$ and $\theta|G_2 = 1$. Since σ and θ are defined over k, we may assume that G_1 and G_2 are k-groups. Then any σ-stable maximal k-split torus of G is θ-stable.

(2) Case: $(G^\sigma)^0$ is isotropic over k.

Now let S be a θ-stable maximal k-split torus of G^σ. Consider the group $M = Z_G(S)$. Since G is semi-simple and $S \neq \{e\}$, $\dim(M) < \dim(G)$. Our assertion is true for M by induction on $\dim(G)$.

From now on, k is an infinite field satisfying the condition $-1 \notin (k^\times)^2 = (k^\times)^4$. The algebraic closure of k is denoted by $\overline{k}$.

11.14. Lemma. *Let $x \in GL(n, k)$ be a k-split semi-simple element with eigenvalues in $(k^\times)^2$. Then we have the following conditions:*

(i) For every positive integer m, there is a unique semi-simple element $y \in GL(n, \overline{k})$ such that $y^{2^m} = x$ and the eigenvalues of y are contained in $(k^\times)^2$; moreover the element y belongs to $GL(n, k)$.

(ii) For $g \in GL(n, \overline{k})$ with $gxg^{-1} = x$, then $gyg^{-1} = y$.

Proof. (i) Since x is k-split semi-simple with eigenvalues in $(k^\times)^2 = ((k^\times)^2)^{2^m}$, the existence of such y in $GL(n, k)$ is immediate. Now let y be any semi-simple element of $GL(n, \overline{k})$ with the described condition. Let $\lambda_1, \cdots, \lambda_\ell$ be the distinct eigenvalues of x and $V_1, \cdots, V_\ell$ the corresponding eigen subspaces of x. Clearly $xy = yx$ and y leaves each V_i, $i = 1, \cdots, \ell$, invariant. Since $-1 \notin (k^\times)^2$, it follows that there is a unique $\nu_i \in (k^\times)^2$ with $\nu_i^{2^m} = \lambda_i$. Since $y^{2^m} = x$ and eigenvalues of y are contained in $(k^\times)^2$, $y|V_i$ coincides with the scalar multiplication by ν_i. Hence y is uniquely determined.

(ii) is immediate from the uniqueness condition of y.

11.15. Proposition. *Let θ_1 and θ_2 be quasi-Cartan involutions of G over k. Let P be any minimal parabolic k-subgroup of G, $U = R_u(P)$, A a θ_1-stable maximal k-split torus of P and $Int(A)$ the image of A in the inner automorphism group $Int(G)$ of G. Then there exist $u \in U_k$ and $t \in Int(A)^2_k$ such that $\theta_2 = x \circ \theta_1 \circ x^{-1}$ where $x = t \circ Int(u)$.*

Proof. There exists a θ_2-stable maximal k-split torus A' of P. Choose $u \in U_k$ with $uA'u^{-1} = A$. Consider the involution $\theta_2' = Int(u) \circ \theta_2 \circ Int(u)^{-1}$. Clearly θ_2' is a quasi-Cartan involution of G over k. We may replace θ_2 by θ_2' and assume that A is (θ_1, θ_2)-stable. Write $Z_G(A) = M \cdot A$ as an almost direct product of k-groups. Let H_1 and H_2 denote the fixed point groups of θ_1 and θ_2 respectively. By (i) of Lemma 11.5, $M \subset H_1$ and $M \subset H_2$. Since H_1 and H_2 are anisotropic over k by assumption, $\theta_1|A = \theta_2|A = -1$. It follows that $\theta_1\theta_2|Z_G(A) = 1$. Hence $\theta_1\theta_2$ is inner. Let S be a maximal k-torus of G containing A. Clearly $S = S_+S_-$ with $A = S_-$. Since $\theta_1\theta_2$ is inner, $\theta_1\theta_2 = \alpha \in Int(S)_k$. Write $\alpha = \alpha_1\alpha_2$ wih $\alpha_1 \in Int(S_+)$ and $\alpha_2 \in Int(A)$. From $\theta_2^2 = (\theta_1\alpha)^2 = 1$, $\theta_1\alpha\theta_1 = \alpha^{-1}$. This yields that $\alpha_1^2 = 1$ and $\beta = \alpha^2 \in Int(A)_k$. Now set $\theta_1' = \theta_2\theta_1\theta_2$. Then we have the condition $(\theta_1\theta_1')^2 = \beta^2 \in Int(A)^2_k$. By Lemma 11.14, there is $s \in Int(A)^2_k$ such that $s^4 = (\theta_1\theta_1')^2$. Note that $\theta_1\theta_1'$ commutes with $(\theta_1\theta_1')^2$ and $\theta_1(\theta_1\theta_1')^2\theta_1 = \theta_1'(\theta_1\theta_1')^2\theta_1 = (\theta_1\theta_1')^{-2}$. By the uniqueness condition in Lemma 11.14, s commutes with $\theta_1\theta_1'$ and $\theta_1 s\theta_1 = \theta_1' s\theta_1' = s^{-1}$. This implies that $(s\theta_1' s^{-1})\theta_1 = s^2\theta_1'\theta_1 = s^{-2}\theta_1\theta_1' = \theta_1(s\theta_1' s^{-1})$. By Lemma 11.12, θ_1 and $s\theta_1' s^{-1}$ are quasi-Cartan involutions of $Int(G)$ over k. Hence by Lemma 11.11, $\theta_1 = s\theta_1' s^{-1}$. As a consequence, $(\theta_1\theta_2)^2 = s^2 \in Int(A)_k^2$. Then we can repeat the argument and conclude that there is $t \in Int(A)_k^2$ with $\theta_1 = t\theta_2 t^{-1}$. Set $x = t \circ Int(u)$. One checks readily that x has the desired property.

11.16. Corollary. *Let θ be a quasi-Cartan involution of G over k. Then θ is a Cartan involution of $Int(G)$ over k.*

Proof. Let P' be a minimal parabolic k-subgroup of $Int(G)$, $U' = R_u(P')$ and A' a θ-stable maximal k-split torus of P'. Given $g \in Int(G)_k$, consider the involutions θ and $g\circ\theta\circ g^{-1}$. By the preceding proposition, there is $x \in U_k'A_k'$ with $x\circ\theta\circ x^{-1} = g\circ\theta\circ g^{-1}$. Then $x^{-1}g \in Int(G)_k^\theta$ and so $g \in U_k'A_k'Int(G)_k^\theta$. By Lemma 11.12, θ is a quasi-Cartan involution of $Int(G)$ over k and by definition, θ is a Cartan involution of $Int(G)$ over k.

11.17. Proposition. *Let G be a connected reductive algebraic k-group with a quasi-Cartan involution θ over k. If σ is any involution of G defined over k, then there exists $t \in \tau(Int(G)_k)$ such that $t\circ\theta\circ t^{-1}$ commutes with σ.*

Proof. Consider the involutions $\sigma\theta\sigma$ and θ. By Proposition 11.15, there is $x \in Int(G)_k$ such that $\sigma\theta\sigma = x\theta x^{-1}$. It follows that $(\sigma\theta)^2 = x\theta x^{-1}\theta \in \tau(Int(G)_k)$. By (iv) of Lemma 11.5, there is a (θ, k)-split torus S of $Int(G)$ containing $x\theta x^{-1}\theta$. By Corollary 11.16, θ is a Cartan involution of $Int(G)$ over k. This implies that $(\sigma\theta)^2 = x\theta x^{-1}\theta \in S_k^2$. By Lemma 11.14, there is a unique $t \in S_k^2$ with $(\sigma\theta)^2 = t^4$. Then the element $t\circ\theta\circ t^{-1}$ has the desired property.

Remark. For $k = \mathbb{R}$ this result is due to Berger [1]. Moreover, in this case every involution of G is a Cartan involution of G for some real form $G(\mathbb{R})$ of G (see [10]). So for $k = \mathbb{R}$ one can study the structure of semisimple algebraic groups with a pair of commuting involutions instead of the structure of real semisimple algebraic groups with an involution.

11.18. Proposition. *Let θ be a Cartan involution of G over k and σ an involution of G defined over k with $\sigma\theta = \theta\sigma$. Let H be an open k-subgroup of G^σ. Then we have the following conditions:*

(i) *Given any σ-stable maximal k-split torus A of G, there is $h \in H(k)$ such that hAh^{-1} is θ-stable.*
(ii) *If two (σ, θ)-stable maximal k-split tori A_1 and A_2 are $H(k)$-conjugate, then they are $H^\theta(k)$-conjugate.*

Proof. (i) Consider the semi-direct product $G' = F \ltimes G$ of k-groups where $F = \{1, \sigma, \theta, \sigma\theta\}$. Clearly $G_k' = F \ltimes G_k$ and θ is an involution of G' in the obvious manner.

(1) Set $\tau = \tau_\theta$. Then $\tau(G_k') = \tau(G_k) = \tau(G_k)^2$ consists of k-split semi-simple elements with eigen values in $(k^\times)^2$.

Since $F \subset (G')^\theta$, the condition $\tau(G_k') = \tau(G_k)$ is obvious. The other assertion follows from the condition that θ is a Cartan involution of G over k.

(2) There is $x \in G_k$ such that $\theta_1 = x\theta x^{-1}$ normalizes A and commutes with σ.

Let P be a minimal parabolic k-subgroup of G containing A and $U = R_u(P)$. Since P has a θ-stable maximal k-split torus, there is $u \in U_k$ such that $u^{-1}Au$ is θ-stable. Now consider the element $\theta' = u\theta u^{-1}$ in G'_k and its induced involution $Int(u) \circ \theta \circ Int(u)^{-1}$, also denoted by θ', on G'. Then θ' normalizes A. Let $\tau'(x) = x\theta'(x)^{-1}$, $x \in G'$. Then $\tau'(G'_k) = u\tau(G'_k)u^{-1}$ and by (1) $\tau'(G'_k) = u\tau(G_k)^2u^{-1} = \tau'(G_k)$. It follows that $(\sigma\theta')^2$, an element of $\tau'(G'_k)$, is k-split semi-simple with eigen values in $(k^\times)^2$. Observe that $\theta'|A = -1$ and $(\sigma\theta')^2|A = 1$. Thus by Lemma 11.2, $(\sigma\theta')^2 \in A$ and from the condition on its eigen values, it lies in A^2_k. By Lemma 11.14, there is a unique $t \in A^2_k$ with $t^4 = (\sigma\theta')^2$. As in 11.16, one shows that for $x = tu$, $x\theta x^{-1}$ has the desired property.

(3) The element $(\theta_1\theta)^2 \in \tau_\theta(G'_k)$ and by (1) and (iv) of Lemma 11.5, there is a (θ, k)-split torus S of G with $(\theta_1\theta)^2 \in S^2_k$. By Lemma 11.14, there is a unique $h \in S^2_k$ with $(\theta_1\theta)^2 = h^{-4}$. Note that $S, A \subset Z_G((\theta_1\theta)^2)$. This yields that $(\theta_1\theta)^2 \in A^2_k$ and there exist $h_n \in A^2_k$ with $h_n^{2^n} = h$, $(n = 1, 2, \dots)$. Since σ commutes with $(\theta_1\theta)^2$, by (ii) of Lemma 11.4, $h, h_n \in A^\sigma_k$. Observe that $A^\sigma/(A^\sigma)^0$ has no nontrivial 2-divisible elements. It follows that $h \in (A^\sigma)^0 \subset H$. Now as in 11.16, $h^{-1}\theta h$ commutes with θ_1. By Lemma 11.11 $Int(x) \circ \theta \circ Int(x)^{-1} = Int(h)^{-1} \circ \theta \circ Int(h)$ and by (2), h has the desired property.

(ii). Suppose that $y \in H(k)$ such that $y^{-1}A_1y = A_2$. Since $\theta|A_1 = -1$ and $\theta|A_2 = -1$, it yields that $y\theta(y)^{-1} \in Z_G(A_1) \cap \tau_\theta(G_k)$. Set $A = A_1$. Since $y\theta(y)^{-1}$ is k-split semi-simple and has eigen values in $(k^\times)^2$, by Lemma 11.2 $y\theta(y)^{-1} \in A^2_k$. As in (3) of (i), there exists $a \in A^2_k \cap H^0$ with $y\theta(y)^{-1} = a^{-2}$. Then $z = ay \in H^\theta(k)$ satisfies the desired condition $z^{-1}A_1z = A_2$.

Remark. A_1 and A_2 are conjugate by an element of $(H^0 \cap G^\theta)_k$ provided they are conjugate by an element of H^0_k. When $k = \mathbb{R}$, such result was obtained by Matsuki [15].

11.19. Corollary. *All (σ, θ)-stable maximal k-split tori A of G with maximal A_+ (resp. A_-) parts (with respect to the involution σ) are $(G^\sigma \cap G^\theta)^0_k$-conjugate.*

Proof. Let A and A' be two such k-split tori of G. It follows from Lemma 11.13 and the maximality condition that A_+ and A'_+ (with respect to σ) are θ-stable maximal k-split tori of G^σ by considering the groups $Z_G(A_+)$ and $Z_G(A'_+)$. By Corollary 11.9, $\theta|(G^\sigma)^0$ is a Cartan involution over k and by (ii) of Corollary 11.4, A_+ and A'_+ are conjugate by an element of $(G^\sigma \cap G^\theta)^0_k$. Thus we may assume that $A_+ = A'_+$. Consider the groups $M = Z_G(A_+)$, $D(M) = [M, M]$, $A_1 = (A \cap D(M))^0$ and $A'_1 = (A' \cap D(M))^0$. Then A_1 and A'_1 are θ-stable maximal k-split tori of $D(M)^{\sigma\theta}$. The same reasoning yields that A_1 and A'_1 are conjugate by an element of $(M^{\sigma\theta} \cap M^\theta)^0_k = (M^\sigma \cap M^\theta)^0_k$. Hence it follows readily that A and A' are $(G^\sigma \cap G^\theta)^0_k$- conjugate.

For maximal A_-, the assertion follows from the above discussion by considering the pair $(\sigma\theta, \theta)$.

11.20. Corollary. *Let σ be an involution of G defined over k and H the identity component of G^σ. If G has a Cartan involution over k, then all maximal (σ, k)-split tori are H_k-conjugate.*

Proof. By Proposition 11.17, there is a Cartan involution θ of G over k with $\sigma\theta = \theta\sigma$. The assertion is now immediate from (i) of Proposition 11.18 and Corollary 11.19.

In the following, we assume that G has a Cartan involution over k. Given any involution σ of G defined over k, by Proposition 11.17, there is a Cartan involution θ of G

over k commuting with σ. By Lemma 11.13, there exists a (σ,θ)-stable maximal k-split torus A of G. Let P be a minimal parabolic k-subgroup of G containing A, $U = R_u(P)$ and $\tau_{\sigma,\theta}^{-1}N$ the set defined by

$$\tau_{\sigma,\theta}^{-1}N = \{x \in G | x\sigma(x)^{-1}, x\theta(x)^{-1} \in N_G(A)\}.$$

For simplicity of notations, set $H = G^\sigma and K = G^\theta$.

11.21. Proposition. *Let P, A, $\tau_{\sigma,\theta}^{-1}N$ be as above and H_1 a k-open subgroup of H. Then the natural map $Z_G(A)_k\backslash(\tau_{\sigma,\theta}^{-1}N)_k/H_1^+(k) \to P_k\backslash G_k/H_1(k)$, induced by the inclusion map, is a bijection, where $H_1^+ = H_1 \cap K$.*

Proof. Let $\tau_\sigma^{-1}N = \{x \in G | \tau_\sigma(x) \in N_G(A)\}$. By Proposition 6.6, $G_k = P_k(\tau_\sigma^{-1}N)_k$. By (i) of Proposition 11.18, $(\tau_\sigma^{-1}N)_k \subset (\tau_{\sigma,\theta}^{-1}N)_k H_1(k)$ and so the map is surjective. Given $v_1, v_2 \in (\tau_{\sigma,\theta}^{-1}N)_k$ with $v_2 \in P_k v_1 H_1(k)$, write

$$v_2 = uzv_1h,$$

with $u \in U_k$, $z \in Z_G(A)_k$ and $h \in H_1(k)$. Then $v_2\sigma(v_2)^{-1} = uzv_1\sigma(v_1)^{-1}\sigma(z)^{-1}\sigma(u)^{-1}$ and by [3, 5.15], $v_2\sigma(v_2)^{-1} = zv_1\sigma(v_1)^{-1}\sigma(z)^{-1}$. It follows that $v_2 = zv_1h_1$ where $h_1 \in H$ and $h_1 = (zv_1)^{-1}u(zv_1)\cdot h$. By Lemma 10.1, $(zv_1)^{-1}u(zv_1) \in H^0$ and so $h_1 \in H_1(k)$. Consider the (σ,θ)-stable maximal k-split tori $v_1^{-1}Av_1$ and $v_2^{-1}Av_2$. Since $v_2 = zv_1h_1$, they are conjugate by an element of $H_1(k)$. By (ii) of Proposition 11.18, we may assume that $v_1^{-1}Av_1 = v_2^{-1}Av_2$. Then the element $t = h_1\theta(h_1)^{-1} \in Z_G(v_1^{-1}Av_1)$. Note that θ is a Cartan involution of G over k. As in 11.18(3) by Lemmas 11.2 and 11.5, $t \in v_1^{-1}A_k^2v_1$ and by Lemma 11.14, there is $a \in (v_1^{-1}Av_1) \cap H_1(k)$ such that $t = a^{-2}$. It follows that $ah_1 \in H_1^+(k)$, $zv_1a^{-1} \in Z_G(A)_kv_1$ and so $v_2 \in Z_G(A)_kv_1H_1^+(k)$. This shows that the map is injective.

Remark. The result is an interesting refinement of Springer's result [23, (i) of Theorem 4.2]. When $k = \mathbb{R}$, in a slight different form, the result is due to Matsuki [15].

11.22. Corollary. *Let $\{A_i | i \in I\}$ be representatives of the $H_1^+(k)$-conjugacy classes of (σ,θ)-stable maximal k-split tori in G. Then*

$$P_k \setminus G_k/H_1(k) \cong \cup_{i\in I} W_{G_k}(A_i)/W_{H_1(k)}(A_i)$$

Let A be a (σ,θ)-stable maximal k-split torus of G such that the group A_- (with respect to σ) is a maximal (σ,k)-split torus of G. There exists a minimal σ-split parabolic k-subgroup P_1 of G containing A_-. Let P denote a minimal parabolic k-subgroup of P_1 containing A_-.

11.23. Proposition. *Let A, A_-, P_1 and P be as above and $v \in (\tau_{\sigma,\theta}^{-1}N)_k$. The following conditions are equivalent:*

(i) *The orbit PvH is open in G/H.*
(ii) *$v^{-1}P_1v$ is a minimal σ-split parabolic subgroup of G.*
(iii) *$v \in N_G(A, A_-)_k(H\cap K)_k^0$, where $N_G(A, A_-) = N_G(A) \cap N_G(A_-)$.*

Proof. (i) $\Rightarrow$ (ii). By (v) of Proposition 9.2, there exists a minimal σ-split parabolic k-subgroup P_2 containing $v^{-1}Pv$. By Proposition 4.9, we have that $v^{-1}P_1v = P_2$.

(ii) $\Rightarrow$ (iii). By (iv) of Proposition 4.7, $(v^{-1}Av)_-$ is a maximal (σ,k)-split torus of G. Since $v \in \tau_{\sigma,\theta}^{-1}N$, $v^{-1}Av$ is (σ,θ)-stable. According to Corollary 11.19, there is $h \in (H\cap K)_k^0$ with $h^{-1}v^{-1}Avh = A$. Thus we may assume that $v \in N_G(A)_k$. By Lemma 4.6, there exists $\lambda \in X_*(A_-)$ such that $v^{-1}P_1v = P(\lambda)$. Then $\Phi(v^{-1}P_1v, A_-)$ defines a system of positive roots of $\Phi(G, A_-)$. By Proposition 5.7, one finds an element $n \in N_G(A, A_-)_k$ with $\Phi(v^{-1}P_1v, A_-) = \Phi(n^{-1}P_1n, A_-)$. As a consequence $n^{-1}P_1n = v^{-1}P_1v$ and $vn^{-1} \in N_G(A) \cap P_1 \subset Z_G(A_-)$. This shows that $v \in N_G(A, A_-)_k$.

(iii) $\Rightarrow$ (i). We may assume that $v \in N_G(A, A_-)_k$. By Lemma 4.6, $P_1 = P(\lambda)$ with $\lambda \in X_*(A_-)$. Then $v^{-1}P_1v = P(w^{-1}(\lambda))$, where w is the image of v in $W = N_G(A)/Z_G(A)$. Since $v \in N_G(A_-)$, $w^{-1}(\lambda) \in X_*(A_-)$ and $P(w^{-1}(\lambda))$ is σ-split. The minimality condition of P_1 implies that $v^{-1}P_1v$ is also a minimal σ-split parabolic k-subgroup of G. Then by Proposition 9.2, assertion (i) follows.

Let P be a minimal parabolic k-subgroup of G and H_1 an open k-subgroup of H. Assume that PH_1 is open in G. Now we are ready to compute the cardinality of $P_k\backslash(PH_1)_k/H_1(k)$. By (i) of Proposition 11.18, we may assume that P has a (σ,θ)-stable maximal k-split torus A.

11.24. Proposition [15,27]. *Let P, H_1 and A be as above. Let $H_1^+ = H_1\cap K$, $W(A, A_-) = N_G(A, A_-)/Z_G(A)$ and $W_\sigma^+(A) = N_{H_1^+(k)}(A)/Z_{H_1^+(k)}(A)$. Then $|P_k\backslash(PH_1)_k/H_1(k)| = |W(A, A_-)|/|W_\sigma^+(A)|$.*

Proof. Given $g \in G_k$, $g \in (PH_1)_k$ if and only if PgH is open in G. By Propositions 11.21 and 11.24, the cardinality of $P_k\backslash(PH_1)_k/H_1(k)$ coincides with that of $Z_G(A)_k\backslash N_G(A, A_-)_kH_1^+(k)/H_1^+(k)$. Then the assertion is obvious.

12. (σ,θ)-stable maximal k-split tori.

In this section, we discuss the H_k^+-conjugacy classes of (σ,θ)-stable maximal k-split tori of G. Such conjugacy classes are essential in the decomposition of $P_k\backslash G_k/H_k$ in 11.22. We present a satisfactory description of such conjugacy classes in terms of conjugacy classes in a restricted Weyl group.

12.1.. Here we set the notations. Let G be a connected reductive k-group, σ an involution of G defined over k and θ a Cartan k-involution of G commuting with σ. Let H be a k-open subgroup of G^σ, $K = G^\theta$ and $H^+ = H \cap K$. Given a σ-stable torus T, we reserve the notation T^+ and T^- for T_σ^+ and T_σ^- respectively. For other involutions of T, we shall keep the subscript. Let $\mathcal{A}$ denote the set of (σ,θ)-stable maximal k-split tori of G.

12.2. Definition. *For $A_1, A_2 \in \mathcal{A}$, the pair (A_1, A_2) is called standard if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. In this case, we also say that A_1 is standard with respect to A_2.*

As in the case of a single involution [11], the (σ,θ)-stable maximal k-split tori of G can be put in a standard position.

12.3. Lemma. *Let A_1, $A_2 \in \mathcal{A}$ such that $A_1^+ \supset A_2^+$ (resp. $A_1^- \subset A_2^-$). Then there exists $x \in Z_{H_k^+}(A_2^+)$ (resp. $Z_{H_k^+}(A_1^-)$) such that (A_1, xA_2x^{-1}) is standard. In particular if A_1^+ and A_2^+ (resp. A_1^- and A_2^-) are H_k^+-conjugate, so are A_1 and A_2.*

Proof. Consider the group $M = Z_G(A_2^+)$. Then A_1^- and A_2^- are (σ, θ)-stable k tori of M. By Lemma 11.9, $\theta|M$ is a Cartan k-involution of M. Since A_2^- is (σ, k)-split, by Corollary 11.19 there exists $x \in Z_{H_k^+}(A_2^+)$ with $x^{-1}A_1^-x \subset A_2^-$. Clearly x has the desired property. Consider the standard pair (A_2, A_1) for $(\sigma\theta, \theta)$. The other assertion follows.

A standard pair (A_1, A_2) of (σ, θ)-stable maximal k-split tori of G gives rise to an involution in $W(A_1)$ (resp. $W(A_2)$).

12.4. Lemma. *Let (A_1, A_2) be a standard pair of (σ, θ)-stable maximal k-split tori of G. Then we have the following conditions:*

(i) There exists $g \in Z_{K_k}(A_1^- A_2^+)$ such that $gA_1g^{-1} = A_2$.
(ii) If $n_1 = \sigma(g)^{-1}g$ and $n_2 = \sigma(g)g^{-1}$, then $n_1 \in N_G(A_1)$ and $n_2 \in N_G(A_2)$.
(iii) Let w_1 and w_2 be the images of n_1 and n_2 in $W(A_1)$ and $W(A_2)$ respectively. Then $w_1^2 = e$, $w_2^2 = e$, and $(A_1)_{w_1}^+ = (A_2)_{w_2}^+ = A_1^- A_2^+$ which characterizes w_1 and w_2.

Proof. (i) Consider the group $M = Z_G(A_1^- A_2^+)$. By Lemma 11.9, $\theta|M$ is a Cartan k-involution of M. Clearly A_1 and A_2 are (θ, k)-split. Hence by (ii) of Lemma 11.4, there is $g \in (M \cap K)_k^0$ with $gA_1g^{-1} = A_2$.

(ii) and (iii) Given $x \in A_1^+ \cap g^{-1}A_2^- g$, write $x = g^{-1}ag$ with $a \in A_2^-$. Note that $Int(n_1) = \sigma \cdot Int(g^{-1}) \cdot \sigma \cdot Int(g)$. It follows that $n_1 x n_1^{-1} = \sigma(g^{-1}a^{-1}g) = \sigma(x^{-1}) = x^{-1}$. We have the condition that

$$Int(n_1)|A_1^- A_2^+ = 1, \quad Int(n_1)|A_1^+ \cap g^{-1}A_1^- g = -1. \tag{1}$$

Observe that $A_1 = (A_1^- A_2^+)(A_1^+ \cap g^{-1}A_2^- g)$. Then (1) yields immediately that $n_1 \in N_G(A_1)$, $w_1^2 = e$ and $(A_1)_{w_1}^+ = A_1^- A_2^+$. Consider the standard pair (A_2, A_1) for $(\sigma\theta, \theta)$. The assertion for n_2 and w_2 follows.

Remark. By (iii) of Lemma 12.4, w_1 and w_2 are independent of our choice of $g \in Z_{K_k}(A_1^- A_1^+)$ with $gA_1g^{-1} = A_2$.

12.5. Definition. *Let A_1, A_2, $w_1 \in W(A_1)$ and $w_2 \in W(A_2)$ be as in Lemma 12.4. We call w_1 (resp. w_2) the A_2-standard involution (resp. A_1-standard involution) of $W(A_1)$ (resp. $W(A_2)$).*

For a (σ, θ)-stable k-torus T of G, we write $W(T, H_k^+)$ for $N_{H_k^+}(T)/Z_{H_k^+}(T)$.

In the following, we fix an element $A_0 \in \mathcal{A}$ (resp. $S \in \mathcal{A}$) such that A_0^- (resp. S^+) is a maximal (σ, k)-split torus of G (resp. a maximal k-split torus of H).

12.6. Proposition. *Assume that A_1, $A_2 \in \mathcal{A}$ such that they are standard with respect to A_0 (resp. S). Let w_1 and w_2 be the A_1-standard and A_2-standard involutions in $W(A_0)$ (resp. $W(S)$) respectively. Then A_1 and A_2 are H_k^+-conjugate if and only if w_1 and w_2 are conjugate under $W(A_0, H_k^+)$ (resp. $W(S, H_k^+)$).*

Proof. The assertion for S follows from that for A_0 by using the pair $(\sigma\theta, \theta)$ instead of (σ, θ). Thus it suffices to establish the result for A_0.

$\Rightarrow$) Assume that $x \in H_k^+$ with $xA_1x^{-1} = A_2$. Then $xA_1^+x^{-1} = A_2^+$ and $xA_1^-x^{-1} = A_2^-$. Set $M = Z_G(A_2^-)$. It follows that A_0 and xA_0x^{-1} are (σ, θ)-stable maximal k-split tori of M with maximal A_0^- and $xA_0^-x^{-1}$ parts respectively. From Corollary 11.19, there exists $y \in (M^\sigma \cap M^\theta)_k^0$ such that $yxA_0x^{-1}y^{-1} = A_0$. One checks readily that

$$Int(yx)A_0^+ = A_0^+, \quad Int(yx)A_1^- = A_2^-. \tag{1}$$

Note that $(A_0)_{w_1}^+ = A_1^-A_0^+$ and $(A_0)_{w_2}^+ = A_2^-A_0^+$. Condition (1) yields easily that the image w of yx in $W(A_0, H_k^+)$ satisfies the desired condition $ww_1w^{-1} = w_2$.

$\Leftarrow$) Assume that $w \in W(A_0, H_k^+)$ with $ww_1w^{-1} = w_2$. Clearly $(A_0)_{w_1}^+ = A_1^-A_0^+$ and $(A_0)_{w_2}^+ = A_2^-A_0^+$ are σ-stable. Since $A_1^- = ((A_0)_{w_1}^+)^-$ and $A_2^- = ((A_0)_{w_2}^+)^-$, the condition $ww_1w^{-1} = w_2$ implies immediately that $w(A_1^-) = A_2^-$. Let x be a preimage of w in $N_{H_k^+}(A_0)$. We have that $xA_1^-x^{-1} = A_2^-$. Hence by Lemma 12.2, A_1 and A_2 are H_k^+-conjugate.

The above proposition provides a sound criterion when elements in $\mathcal{A}$ are H_k^+-conjugate. To complete the characterization of H_k^+-conjugacy classes of $\mathcal{A}$, it reduces to single out those $w \in W(A_0)$ (resp. $W(S)$) which are the A-standard involutions for some $A \in \mathcal{A}$.

12.7.. Recall that a k-involution τ of a connected reductive k-group M is called k-split if there exists a τ-split maximal k-split torus of M.

Let $A \in \mathcal{A}$ and $w \in W(A)$ satisfying $w^2 = e$ and $w\sigma = \sigma w$. Set $G_w = Z_G(A_w^+)$. Let n be a preimage of w in $N_G(A)$. Then $n \in Z_G(A_w^+)$ and $A_w^- \cap Z(G_w)$ is finite. As a consequence, A_w^- is a (σ, θ)-stable maximal k-split torus of $[G_w, G_w]$.

12.8. Lemma. *let A, w and G_w be as above. Then we have the following conditions:*

(i) If $\sigma\theta|[G_w, G_w]$ is k-split, then $(A_w^+)^- = A_1^-$ for some $A_1 \in \mathcal{A}$.
(ii) If $\sigma|[G_w, G_w]$ is k-split, then $(A_w^+)^+ = A_1^+$ for some $A_1 \in \mathcal{A}$.
(iii) If A^- is a maximal (σ, k)-split torus of G (resp. A^+ is a maximal $(\sigma\theta, k)$-split torus of G) and G_w satisfies conditions (i) and (ii), then w is the A'-standard involution in $W(A)$ for some $A' \in \mathcal{A}$.

Proof. (i) Since $\sigma\theta|[G_w, G_w]$ is k-split, there is a $(\sigma\theta, \theta)$-split maximal k-split torus S of $[G_w, G_w]$. Clearly $A_1 = S \cdot A_w^+$ has the desired property.

(ii) The condition follows from (i) by using the pair $(\sigma\theta, \theta)$.

(iii) We show the assertion for the case that A^- is maximal. By (ii) and the minimality condition of A^+, $A^+ \subset A_w^+$ and $A_w^+ = A^+ \cdot (A_w^+)^-$. Let S be as in (i) and $A' = S \cdot A_w^+$. Then (A', A) is standard and $(A')^-A^+ = A_w^+$. Hence w is the A'-standard involution in $W(A)$.

12.9. Definition. *Let $A \in \mathcal{A}$ and $w \in W(A)$. We say that w is (σ, θ)-singular if*
(1) $w^2 = e$
(2) $\sigma w = w\sigma$
(3) the involutions $\sigma|[G_w, G_w]$ and $\sigma\theta|[G_w, G_w]$ are k-split.
A root $\alpha \in \Phi(A)$ is called (σ, θ)-singular if the corresponding reflection $s_\alpha \in W(A)$ is (σ, θ)-singular.

Analogous to [11], we have the following characterization of the H_k^+-conjugacy classes of (σ, θ)-stable maximal k-split tori of G.

12.10. Proposition. *Let A be a (σ, θ)-stable maximal k-split torus of G with maximal A^- (resp. A^+). Then there is a one to one correspondence between the H_k^+-conjugacy classes of $\mathcal{A}$ and the $W(A, H_k^+)$-conjugacy classes of (σ, θ)-singular involutions of $W(A)$.*

Proof. By the duality between (σ, θ) and $(\sigma\theta, \theta)$, one needs only to prove the assertion for the case that A^- is maximal. By Proposition 12.6 and (iii) of Lemma 12.8, it suffices to show that the standard involutions are (σ, θ)-singular. Let $A_1 \in \mathcal{A}$ such that (A_1, A) is standard and let $w \in W(A)$ denote the A_1-standard involution in $W(A)$. By (iii) of Lemma 12.4, $A_w^+ = A_1^- A^+$. Clearly A_w^+ is σ-stable and so $w\sigma = \sigma w$. Note that A^-, $A_1^+ \subset G_w$ and their images in $G_w/Z(G_w)$ are maximal k-split tori of $G_w/Z(G_w)$. It follows easily that $\sigma|[G_w, G_w]$ and $\sigma\theta|[G_w, G_w]$ are k-split. Thus w is (σ, θ)-singular.

12.11. Corollary. *Let $A \in \mathcal{A}$ with maximal A^- (resp. A^+) and $A_i \in \mathcal{A}$, $i \in I$ such that (A_i, A) (resp. (A, A_i)) are standard and the A_i-standard involutions in $W(A)$ are representatives of $W(A, H_k^+)$-conjugacy classes of (σ, θ)-singular involutions of $W(A)$. If P is a minimal parabolic k-subgroup of G, then*

$$P_k \setminus G_k/H_k \simeq \bigcup_{i\in I} W(A_i, G_k)/W(A_i, H_k^+);$$

in particular $Card(P_k \setminus G_k/H_k) \leq |W(A)|^2$.

Proof. It is immediate from 11.22 and 12.10.

12.12. Lemma. *Suppose that $A \in \mathcal{A}$ and $\alpha \in \Phi(A)$ with $\sigma\alpha = \alpha$ (resp. $\sigma\alpha = -\alpha$). Then α is (σ, θ)-singular if and only if $\sigma|G_\psi^* \neq 1$ (resp. $\sigma\theta|G_\psi^* \neq 1$) where $\psi = \mathbb{Q}\alpha \cap \Phi(A)$.*

Proof. We show the case for $\sigma\alpha = \alpha$. Let $w = s_\alpha$ be the corresponding reflection. Clearly G_ψ^* is the k-isotropic factor of $[G_w, G_w]$ and as $\sigma\alpha = \alpha$ we have that $A^- \subset A_w^+$. This yields that $\theta\sigma|G_\psi^*$ is k-split. By Proposition 4.3, $\sigma|G_\psi^*$ is k-split if and only if $\sigma|G_\psi^* \neq 1$. By duality, the other assertion follows.

12.13. Lemma. *If σ and $\sigma\theta$ are k-split, then $-1 \in W(A)$*

Proof. Choose A_1, $A_2 \in \mathcal{A}$ such that $A_1 = A_1^+$ and $A_2 = A_2^-$. Clearly (A_1, A_2) is standard and the A_1-standard involution w of $W(A_2)$ is determined by $(A_2)_w^+ = A_1^- \cdot A_2^+ = \{e\}$. Obviously $w = -1$. Since A and A_2 are conjugate, the assertion for A is immediate.

12.14. Lemma. *Suppose that σ and $\sigma\theta$ are k-split. Then for $A \in \mathcal{A}$, $\Phi(A)$ contains a (σ, θ)-singular root.*

Proof. We may assume that G is semi-simple and has no anisotropic factor over k.

Case 1. $A = A^+$ or $A = A^-$.
Note that $\sigma \neq 1$ and $\sigma\theta \neq 1$. The assertion is immediate for Lemma 12.12.
Case 2. A^+ and A^- are nontrivial.
Consider the group $M = Z_G(A^+)$. Clearly the involutions of M/A^+ induced by σ and $\sigma\theta$ are k-split. It follows that $[M, M]$ has the same property as for G. Then by induction on k-rank, the assertion is true for M and so is true for G.

12.15. Lemma. *Let $A \in \mathcal{A}$ (resp. $S \in \mathcal{A}$) such that A^- (resp. S^+) is a maximal (σ, k)-split torus of G (resp. maximal $(\sigma\theta, k)$-split torus of G). If $\alpha \in \Phi(A)$ (resp. $\Phi(S)$) is (σ, θ)-singular, then $\sigma(\alpha) = -\alpha$ (resp. $\sigma(\alpha) = \alpha$).*

Proof. We prove the assertion for A. Let $w = s_\alpha$ be the corresponding reflection of α. Since $\sigma|G_w$ is k-split, by (ii) of Lemma 12.8 and the minimality condition of A^+ we have that

$$A^+ \subset A_w^+.$$

Note that $A_w^+ = (\ker\alpha)^0$. Now the assertion is obvious. By duality, the other assertion also follows.

12.16. Proposition. *Let A be a (σ, θ)-stable maximal k-split torus of G. Assume that G has the condition:*

(a) σ and $\sigma\theta$ are k-split.

Then there exist orthogonal (σ, θ)-singular roots $\alpha_1, \dots, \alpha_\ell$ such that $-1 = s_{\alpha_1} \dots s_{\alpha_\ell}$.

Proof. We prove the proposition by induction in several steps.

(1) If both A^+ and A^- are nontrivial, then the derived subgroups of $Z_G(A^+)$ and $Z_G(A^-)$ respectively satisfy condition (a). Hence the induction argument works. We may assume that $A = A^-$. Note that such tori are H_k^+-conjugate. It suffices for us to establish the result for some $A \in \mathcal{A}$ with $A = A^-$.

(2) Let $S \in \mathcal{A}$ with $S = S^+$. Given a root $\alpha \in \Phi(S)$, set

$$G_{\langle\alpha\rangle} = G^*_{\mathbb{Z}\alpha\cap\Phi(S)}, \quad S_1 = (S \cap G_{\langle\alpha\rangle})^0 \text{ and } S_2 = (\ker\alpha)^0.$$

Suppose that $\sigma|G_{\langle\alpha\rangle} \neq 1$. Then by Propositions 4.3 and 11.18 there exists a nontrivial (σ, θ, k)-split torus A_1 of $G_{\langle\alpha\rangle}$. Consider the group $Z_G(A_1)$. Clearly $A_1 = A_1^-$ and $S_2 \subset Z_G(A_1)$. This yields that the isotropic factor G_1 of $Z_G(A_1)$ over k satisfies the following conditions:

(a) $rk_k(G_1) = rk_k(G) - 1$.

(b) $\sigma|G_1$ and $\sigma\theta|G_1$ are k-split.

Choose a (σ, θ)-stable maximal k-split torus A_2 of G_1 with $A_2 = A_2^-$. Set $A = A_1 \cdot A_2$. There exist $x \in G_{\langle\alpha\rangle}(k)$ and $y \in G_1(k)$ such that $xS_1x^{-1} = A_1$ and $yS_2y^{-1} = A_2$. Then the element $z = yx$ has the property that

$$zSz^{-1} = A \quad \text{and} \quad z(\ker\alpha)^0z^{-1} = A_2.$$

Let $\beta = \alpha \circ Int(z^{-1})$. Observe that $A = A_1A_2$, $\beta|A_2 = 0$ and $\gamma|A_1 = 0$ for $\gamma \in \Phi(G_1, A)$. It follows that

(c) $\beta \perp \Phi(G_1, A)$.

(3) Suppose that $G_{\langle\beta\rangle}$ commutes with G_1. Since $y \in G_1$ and $G_{\langle\beta\rangle} = yG_{\langle\alpha\rangle}y^{-1}$, $G_{\langle\alpha\rangle} = G_{\langle\beta\rangle}$ and the assertions for $G_{\langle\alpha\rangle}$ and G_1 yields the result for A and S.

(4) We may assume that G satisfies the following conditions:

(a) $\Phi(A)$ is irreducible.

(b) Condition (c) of (2) fails to yield the computing condition $[G_{\langle\beta\rangle}, G_1] = \{e\}$; in particular for any long root $\gamma \in \Phi(A)$ (resp. $\alpha \in \Phi(S)$), $G_{\langle\gamma\rangle} \subset G^{\sigma\theta}$ (resp. $G_{\langle\alpha\rangle} \subset G^\sigma$).

Let $\Phi_0(A)$ denote the set of indivisible roots of A. By (b) of (4), the type of $\Phi_0(A)$ is B_ℓ, C_ℓ or F_4. Note that $Z_G(A_1)$ is the Levi k-subgroup containing A of certain parabolic k-subgroup of G. Take compatible positive root systems $\Phi^+(G_1, A)$ and $\Phi^+(G, A)$. Then simple roots of $\Phi^+(G_1, A)$ are also simple roots of $\Phi^+(G, A)$.

(5) Assume that the type of $\Phi_0(A)$ is B_ℓ (resp. C_ℓ). Let $\epsilon_1, \ldots, \epsilon_\ell$ be an orthonormal basis. The roots are

$$\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, (i \neq j), \epsilon_j \text{ (resp. } 2\epsilon_i).$$

Set $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell$ and $\alpha_\ell = \epsilon_\ell$ (resp. $2\epsilon_\ell$). We take $\alpha_1, \ldots, \alpha_\ell$ as the simple roots of $\Phi^+(G, A)$. If $|\alpha| = \sqrt{2}$, then so is β and condition (c) of (2) yields that $\Phi^+(G_1, A)$ has simple roots $\alpha_1, \alpha_3, \ldots, \alpha_\ell$ and $\beta = \pm(\epsilon_1 + \epsilon_2)$. By (b) of (4), this is possible only if $2\epsilon_i \in \Phi(A)$. In this case, $G_{\langle\alpha_1\rangle}$ is normal in G_1. Since G_1 has property (a), so has $G_{\langle\alpha_1\rangle}$. By (b) of (4), $G_{\langle 2\epsilon_2\rangle} \subset G^{\sigma\theta}$ and by (iv) of Corollary 11.4 there is $n \in N_{H_k^+}(A)$ with image s_{ϵ_2} in $W(A)$. Clearly $G_{\langle\beta\rangle} = nG_{\langle\alpha_1\rangle}n^{-1}$ has property (a) and the induction argument works. Thus we may assume that in addition $G_{\langle\alpha\rangle} \subset G^\sigma$ for $\alpha \in \Phi(S)$ with $|\alpha| = \sqrt{2}$ (resp. $G_{\langle\gamma\rangle} \subset G^{\sigma\theta}$ for $\gamma \in \Phi(A)$ with $|\gamma| = \sqrt{2}$). Then a (σ, θ)-singular root α has length 1 and $G_{\langle\beta\rangle}$ always has property (a) for otherwise $G = G^{\sigma\theta}$. Hence the assertion is true by induction.

(6) Assume that $\Phi_0(A)$ is of type F_4. In this case $\Phi(A) = \Phi_0(A)$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ and ϵ_4 be orthonormal. The roots are

$$\epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j)(i \neq j), \pm\epsilon_i, \frac{1}{2}(e_1\epsilon_1 + e_2\epsilon_2 + e_3\epsilon_4 + e_4\epsilon_4)$$

with $e_i = \pm 1$. Let $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3$ and $\alpha_4 = \frac{1}{2}(\epsilon_4 - \epsilon_1 - \epsilon_2 - \epsilon_3)$. We take α_1, α_2, α_3 and α_4 as the simple roots of $\Phi^+(G, A)$. By (b) of (4), $G_{\langle\gamma\rangle} \subset G^{\sigma\theta}$ for $\gamma \in \Phi(A)$ with $|\gamma| = \sqrt{2}$. Then $|\alpha| = 1$ and (c) of (2) yields that $\Phi(G_1, A)$ has simple roots α_1, α_2 and α_3, and $\beta = \pm\epsilon_4$. Since G_1 has property (a), there is a (σ, θ)-singular root δ of $\Phi(G_1, A)$ with $|\delta| = 1$. Note $\delta = \pm\epsilon_i$ $(i = 1, 2, 3)$. Since $G_{\langle\gamma\rangle} \subset G^{\sigma\theta}$ for $|\gamma| = \sqrt{2}$, $W(A, H_k^+) \supset$ the symmetric group S_4 and consequently $G_{\langle\beta\rangle}$ has property (a) for $\beta = \pm\epsilon_4$. Then the induction works.

12.17. Proposition. *For $A \in \mathcal{A}$ and a (σ, θ)-singular involution w of $W(A)$, there exist orthogonal (σ, θ)-singular roots $\alpha_1, \ldots, \alpha_\ell$ of A such that $w = s_{\alpha_1} \ldots s_{\alpha_\ell}$.*

Proof. Consider the groups A_w^- and $[G_w, G_w]$. Now the assertion is immediate from Proposition 12.16.

Remark. These results generalize similar results of Matsuki [15] for $k = \mathbb{R}$.

In order to classify the $W(A, H_k^+)$-conjugacy classes of (σ, θ)-singular involutions, we need to determine $W(A, H_k^+)$. Similarly as in [11] we have the following lemma.

12.18. Lemma. *Let A be a (σ, θ)-stable maximal k-split torus of G such that A^- is maximal. If $w_1, w_2 \in W(A)$ are involutions such that $A_{w_i}^- \subset A^-$, then w_1 and w_2 are conjugate under $W(A)$ if and only if w_1 and w_2 are conjugate under $W_1(A) = \{w \in W(A) | w(A^-) \subset A^-\}$.*

Remarks. (1) A similar result holds again in the case that A contains a maximal k-split torus of H.

(2) For some of the pairs (σ, θ) the Weyl group elements in $W_1(A)$ have representatives in H^+ (see [10, 6.15]). In the other cases one has to include some quadratic elements of A^-. Using the above results and the classification of pairs of commuting involutions in [10] one can determine the conjugacy classes of these (σ, θ)-singular involutions in the case that $k = \mathbb{R}$. This classification will appear in a forthcoming paper.

13. Orbits over local fields.

In this section, we characterize orbits with minimal (resp. maximal) dimension in terms of t-topology when k is a local field.

13.1. A minimal parabolic k-subgroup P of G is called quasi θ-stable (resp. quasi θ-split) if P is contained in a minimal θ-stable parabolic k-subgroup of G (resp. minimal θ-split parabolic k-subgroup of G).

13.2. In the following, k is a local field, G a connected reductive algebraic group defined over k and θ an involution of G defined over k. The topology on G_k is the one induced from that of k.

13.3. Proposition. *Let H be an open k-subgroup of G^θ and P a minimal parabolic k-subgroup of G. The following conditions are equivalent:*

(i) $\dim(PH) = \min\{\dim(PgH) | g \in G_k\}$
(ii) *$P_k H_k$ is closed in G_k.*
(iii) *P is quasi θ-stable.*

Proof. (i) $\Rightarrow$ (ii). Consider the Zariski closure $\mathrm{cl}(PH)$ of PH in G. Given $g \in \mathrm{cl}(PH)_k$, $\dim(PgH) \leq \dim(PH)$. By the minimality condition, we have that $\dim(PgH) = \dim(PH)$ and as a consequence

$$\mathrm{cl}(PH)_k = (PH)_k. \tag{1}$$

Since $(P \backslash G)_k = P_k \backslash G_k$, we view $P_k \backslash (PH)_k$ as $(P \backslash PH)_k$. It follows that the H_k-orbits in $P_k \backslash (PH)_k$ are open, hence closed. Thus the double cosets of P_k and H_k in $(PH)_k$ are closed in $(PH)_k$ and by (1) closed in G_k. In particular $P_k H_k$ is closed in G_k.

(ii) $\Rightarrow$ (iii). Let A be a θ-stable maximal k-split torus of P and $U = R_u(P)$. Since $Z_G(A)^\theta / A_+$ is anisotropic over k, by [19] $Z_G(A)^\theta_k / (A_+)_k$ is compact. This implies that $(P_k \cap H_k)/(A_+)_k U^\theta_k$ is compact. By assumption, $P_k H_k$ is closed in G. Then $H_k / H_k \cap P_k$, being a closed subset of G_k / P_k, is compact. We have the condition that

$$(A_+)_k U^\theta_k \text{ is cocompact in } H_k. \tag{2}$$

Note that $U \cap \theta(U)$ is k-split. By (i) of Lemma 0.6, U^θ is k-split. It follows easily that $A_+ U^\theta$ is k-split. As a consequence, there exists a minimal parabolic k-subgroup Q of H^0 containing $A_+ U^\theta$. Let A_1 be a maximal k-split torus of Q containing A_+ and $U_1 = R_u(A)$. From (2),

$$(A_+)_k U^\theta_k \text{ is cocompact in } (A_1)_k (U_1)_k. \tag{3}$$

Let $l = \dim(A_1/A_+)$ and $m = \dim(U_1/U^\theta)$. Since A_1, A_+, U_1 and U^θ are k-split, $(A_1)_k/(A_+)_k \simeq (k^\times)^l$ and $(U_1)_k/U_k^\theta \simeq k^m$ as topological spaces. Then (3) yields that $l = m = 0$; in particular

(4) $$A_+ \text{ is a maximal } k\text{-split torus of } H.$$

Now consider the root system $\Phi = \Phi(A, G)$. Let $\Phi^+ = \Phi(A, P)$ and ψ a minimal θ-stable parabolic subset of Φ containing $\Phi^+ \cap \theta(\Phi^+)$. By Lemma 7.18

(5) $$\psi_s = \{\alpha \in \Phi | \alpha | A_+ = 0\}.$$

Let $\psi^+ = (\psi_s \cap \Phi^+) \cup \psi_u$ and $w, w_0 \in W(\Phi)$ such that

$$w_0(\Phi^+) = \theta(\Phi^+) \text{ and } w(\Phi^+) = \psi^+.$$

Set

$$\theta' = \theta w_0, w_1 = w^{-1}\theta(w)w_0.$$

Then we have the decomposition

(6) $$w_0 = w w_1 \theta'(w^{-1}).$$

Let Δ be the set of simple roots and the length function l on $W(\Phi)$ given with respect to s_α, $\alpha \in \Delta$. By Proposition 7.24, (6) is a Springer decomposition. In particular,

(7) $$l(w_0) = 2l(w) + l(w_1).$$

Write $w = s_1 \dots s_h$ with $h = l(w)$ and $s_i = s_{\alpha_i}$, $\alpha_i \in \Delta$. Choose $n_i \in N_G(A)_k$ with image s_i in $W(\Phi)$ and $P_i = P_{\alpha_i}$ (8.1), $1 \leq i \leq h$. Now set

$$n = n_1 \dots n_h.$$

By Proposition 9.9, $P_k H_k = \text{t-cl}(P_k n n^{-1} H_k) = P_1(k) \dots P_h(k)\text{t-cl}(P_k n^{-1} H_k)$. It follows that

(8) $$P_k H_k = P_k n^{-1} H_k.$$

From $w(\Phi^+) = \psi^+$, $nPn^{-1} \subset P_\psi$. By (4), (5) and Proposition 3.5 P_ψ is a minimal θ-stable parabolic k-subgroup of G. From (8), we can write $n^{-1} = px$ with $p \in P_k$ and $x \in H_k$. Then $P \subset xP_\psi x^{-1}$. Since $x \in H_k$, $xP_\psi x^{-1}$ is a minimal θ-stable parabolic k-subgroup of G.

(iii) $\Rightarrow$ (i). Let $g \in G_k$ be such that

$$\dim(PgH) = \min\{\dim PgH | g \in G_k\}.$$

By (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii), $g^{-1}Pg$ is quasi θ-stable. Let P_1 be a minimal θ-stable parabolic k-subgroup of G containing $g^{-1}Pg$. Let $U_1 = R_u(P_1)$ and A_1 a θ-stable maximal k-split torus of P_1. Then $P_1 = Z_G((A_1)_+)U_1$ and $(A_1)_+$ is a maximal k-split torus of H. It follows that

$$P_1 \cap H^0 = Z_{H^0}((A_1)_+)U_1^\theta$$

is a minimal parabolic k-subgroup of H^0. Note that $g^{-1}Pg \cap Z_G((A_1)_+)$ is θ-split. By Lemma 4.8, $(g^{-1}Pg \cap Z_G((A_1)_+)) \cdot Z_{H^0}((A_1)_+)$ is open in $Z_G((A_1)^+)$. Hence $\dim(g^{-1}PgH) = \dim(P_1H) = \dim(P_1) + s$ where s is the codimension of a minimal parabolic k-subgroup of H.

Now let P_2 be a minimal θ-stable parabolic k-subgroup of G containing P. By the same argument, we also have that $\dim(PH) = \dim(P_2) + s$. By Corollary 5.8, P_1 and P_2 are conjugate to each other. Thus

$$\dim(PH) = \dim(PgH)$$

has the desired condition (i).

13.4. Proposition. *Let H and P be as in Proposition 13.3. The following conditions are equivalent:*

(i) *P is quasi θ-split.*
(ii) *P_kH_k is open in G_k.*

Proof. By Proposition 9.2, P is quasi θ-split if and only if PH is open in G. The assertion now is obvious.

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