Multiplicity one for representations corresponding to spherical distribution vectors of class ρ

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Abstract. In this paper one considers a unimodular second countable locally compact group G and the homogeneous space $X := H\backslash G$, where H is a closed unimodular subgroup of G. Over X complex vector bundles are considered such that H acts on the fibers by a unitary representation ρ with closed image. The natural action of G on the space of square integrable sections is unitary and possesses an integral decomposition in so-called spherical distributions of class ρ. The uniqueness of this decomposition can be characterized by a number of equivalent properties. Uniqueness is shown to hold for a class of semidirect products. In the case that H is compact, the multiplicity free decomposition is shown to be equivalent with the commutativity of a suitable convolution algebra. As an example, one takes for X a symmetric k-variety $\mathcal{H}_k\backslash\mathcal{G}_k$, with k a locally compact field of characteristic not equal to two, and for ρ a character of $\mathcal{H}_k$, whose square is trivial. Here $\mathcal{G}$ is a reductive algebraic group defined over k and $\mathcal{H}$ is the fixed point group of an involution σ of $\mathcal{G}$ defined over k. It is shown then that the natural representation $\mathcal{L}$ of G_k on the Hilbert space $L^2(\mathcal{H}_k\backslash\mathcal{G}_k)$ is multiplicity free if $\mathcal{H}$ is anisotropic. Next a criterion is derived that leads to multiplicity one also in the noncompact situation. Finally, in the nonarchimedean case, a general procedure is given that might lead to showing that a pair $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair. Here $\mathcal{G}$ and $\mathcal{H}$ are suitable algebraic groups defined over k.

Keywords: Hilbert subspace, Distribution vector, Spherical distribution, Plancherel formula

20G15, 20G20, 22E15, 22E46:

1. Introduction

Let G be a locally compact group and H be a closed subgroup. Now one is interested in unitary representations of G related to the homogeneous space $X := H\backslash G$. If both groups are unimodular, then inducing unitary representations ρ from H to G gives you an ample variety of examples of unitary representations $\mathcal{R}_\rho$ of G.

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In the paper [10] we presented for a second countable group G and for irreducible representations ρ with closed image a decomposition of this induced representation in Hilbert subspaces of a certain space of distributions. The representations relevant for this decomposition were shown to be determined by an extension of the notion of spherical distribution, which leads to a description of the decomposition on the level of these distributions.

The present paper is concerned with the uniqueness of the decomposition presented in [10]. It can be characterized by a number of equivalent properties. They hold for certain classes of semi-direct products. In the case that H is compact, the multiplicity free decomposition is shown to be equivalent with the commutativity of a suitable convolution algebra. As an example, one takes for X a symmetric k-variety $\mathcal{H}_k\backslash\mathcal{G}_k$, with k a locally compact field of characteristic not equal to two. Here $\mathcal{G}$ is a reductive algebraic group defined over k and $\mathcal{H}$ is the fixed point group of an involution σ of $\mathcal{G}$ defined over k. It is shown then that the natural representation $\mathcal{L}$ of $\mathcal{G}_k$ on the Hilbert space $L^2(\mathcal{H}_k\backslash\mathcal{G}_k)$ is multiplicity free as soon as $\mathcal{H}$ is anisotropic. Next a criterion is derived that leads to multiplicity one also in the noncompact situation. Finally a general procedure is given in the nonarchimedean case that might lead to showing that a pair $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair. Here $\mathcal{G}$ and $\mathcal{H}$ are suitable algebraic groups defined over k.

The precise content of the various subsections is as follows: the first subsection recalls the type of homogeneous spaces $H\backslash G$ one will work with, it describes the class of geometric representations $\mathcal{R}_\rho$ that will be decomposed and the realization that will be used. The second subsection presents the decomposition of the representation in Hilbert subspaces of a space of distributions. There one also finds the ingredients of the theory of Hilbert subspaces and that of conical integrals necessary to discuss the uniqueness. The Hilbert subspaces that are candidates for the components in the decomposition can be described in terms of certain distribution vectors or as a class of distributions on the group possessing certain invariance properties. Both descriptions are recalled in subsection 4. The properties characterizing multiplicity free decompositions can be found in subsection 5. The next subsection is devoted to the compact case. It is shown there that multiplicity one is equivalent to the commutativity of a specific convolution algebra. As an example the symmetric k-varieties $\mathcal{H}_k\backslash\mathcal{G}_k$ with anisotropic $\mathcal{H}$ are shown to be multiplicity free. The final section contains first of all the criterion that yields multiplicity one also in noncompact situation. The paper finishes with showing how this criterion can be used for spaces $\mathcal{H}_k\backslash\mathcal{G}_k$ with a nonarchimedean locally compact k.

2. The representations

In this paper all locally compact groups will without any further mentioning be assumed to be second countable in order to be able to apply the results from e.g. [23] and [25]. Let G be a unimodular locally compact group and consider a closed unimodular subgroup H of G. On G resp. H we have Haar measures dg resp. dh. It is well-known then that the homogeneous space $X := H\backslash G$ possesses a positive right G-invariant measure dx such that for all f in the space $C_c(G)$ of continuous functions on G with compact support

$$\int_G f(g)dg = \int_X \left\{ \int_H f(hx)dh \right\} dx. \tag{1}$$

A standard example of this setting is usually referred to as the *"group case"*.

Example 2.1. One starts out with a unimodular locally compact group G_1 and chooses G equal to the product group $G_1 \times G_1$. For H we take the diagonal subgroup

$$H = \{(g, g) | g \in G_1\}.$$

Clearly, both H and the variety X are isomorphic to G_1 and under this identification one can take for dx the Haar measure dg_1. The action of the group G on the space X consists then of the left and right translations from G_1.

A slight variation of this example for the same group G is that one chooses for H the group

$$H_{Z(G_1)} = \{gz, g) \mid g \in G_1, z \in Z(G_1)\},$$

where $Z(G_1)$ denotes the center of G_1. Then X is clearly isomorphic to the group $G_1/Z(G_1)$ and for dx one can choose its Haar measure.

A wide variety of examples is furnished by the following setting:

Example 2.2. Consider an affine algebraic group $\mathcal{G}$ defined over a locally compact field k. Then $G := \mathcal{G}_k$, the group of k-rational points of $\mathcal{G}$, is the candidate locally compact group, which is unimodular e.g. if $\mathcal{G}$ is reductive or unipotent. Usually one takes H equal to $\mathcal{H}_k$, where $\mathcal{H}$ is a suitable algebraic subgroup of $\mathcal{G}$ that is defined over k. In this situation it is customary to denote the homogeneous space as $X_k = \mathcal{H}_k\backslash\mathcal{G}_k$. For the class of reductive algebraic groups $\mathcal{G}$ defined over a field k of characteristic not equal to two, let $\sigma : \mathcal{G} \to \mathcal{G}$ be an involution of $\mathcal{G}$ and choose $\mathcal{H}$ equal to $\mathcal{G}_\sigma$, the group of fixed points under σ. According to [11], the group $\mathcal{H}$ is defined over k if and only if σ is defined over k.

So, for the involutions defined over k one can consider the group $\mathcal{H}_k$. As the fixed point set of an automorphism of finite order of a reductive algebraic group is reductive, see [21], the choice $H = \mathcal{H}_k$ gives you an unimodular subgroup. Analogous to the real situation, one calls the variety X_k a symmetric k-variety.

A concrete example of this case is the following: consider a finite dimensional vector space V over k. Let Q be a non-degenerate quadratic form on V and let B be the associated symmetric non-degenerate bilinear form on V, i.e.

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)), \qquad \text{for } x \text{ and } y \in V.$$

For $g \in \mathrm{Gl}(V)$, one denotes the adjoint of g w.r.t. B by g^T. As the involution σ of $\mathrm{Gl}(V)$ one takes now $\sigma(g) = (g^T)^{-1}$. Then the group of fixed points of σ is the orthogonal group associated with Q, O_Q, and the space X_k maps into the non-degenerate matrices that are symmetric w.r.t. B by the mapping $gH_k \to g^T g$.

Another class of examples can be found inside the category of semi-direct products.

Example 2.3. Let G be the semi-direct product $L \rtimes M$ of the locally compact groups L and M. The action of an element $l \in L$ on M is denoted by $\alpha(l)$. If dl and dm are left invariant Haar measures on L resp. M, then $dldm$ is a left invariant Haar measure on G. From this one concludes that G is unimodular if and only if M is unimodular and $\Delta_L(l){=}|\alpha(l)|$ for all $l \in L$. This last property holds e.g. if L is compact or equal to its commutator subgroup. As the subgroup H one chooses a subgroup of the form $L(0) \rtimes M(0)$ with $L(0)$ a closed subgroup of L that normalizes the closed subgroup $M(0)$ of M. It is unimodular under the same conditions.

A typical example of this setting is the Heisenberg group $A(\mathbf{G})$ associated to any abelian locally compact group $\mathbf{G}$, see [29]. If S^1 is the unit circle in the complex plane and if $< .|. >$ denotes the natural pairing between $\mathbf{G}$ and its dual $\hat{\mathbf{G}}$, then the group $A(\mathbf{G})$ is the topological space $\mathbf{G} \times \hat{\mathbf{G}} \times S^1$ with the multiplication

$$(g_1, d_1, t_1)(g_2, d_2, t_2) := (g_1 g_2, d_1 d_2, t_1 t_2 < d_1|g_2 >). \tag{2}$$

The group $A(\mathbf{G})$ is the semi-direct product of the groups $\mathbf{G}$ and $\hat{\mathbf{G}} \times S^1$, where these groups are viewed as being embedded into $A(\mathbf{G})$ as

$$\{(g, 1, 1)|g \in \mathbf{G}\} \text{ resp. } \{(1, d, t)|\ d \in \hat{\mathbf{G}}, t \in S^1\}. \tag{3}$$

The Heisenberg group $A(\mathbf{G})$ is unimodular since it is two-step nilpotent. As $L(0)$ one can choose any closed subgroup $\mathbf{H}$ of $\mathbf{G}$ and, if the group of characters that are trivial on $\mathbf{H}$ is denoted by $\mathbf{H}^\perp$, then the choice $M(0) = \mathbf{H}^\perp \times S^1$ is appropriate. The group $H = \mathbf{H} \times \mathbf{H}^\perp \times S^1$ is then namely a maximal commutative subgroup of $A(\mathbf{G})$ and in particular it is unimodular.

Natural geometric objects related to the homogeneous spaces X are complex vector bundles $\mathcal{V}$ over them. Here only vector bundles are considered for which the structure group reduces to the unitary group. The representation of G on the square-integrable global sections of this bundle is then a natural unitary representation. Concretely this means that one has a unitary representation ρ of H on the finite dimensional space of fibers V_ρ. Global sections correspond then bijectively with functions $f : G \to V_\rho$ such that

$$f(hg) = \rho(h)(f(g)) \tag{4}$$

Consider now the space $L^2(\rho, X, dx)$ of classes of measurable functions $f : G \to V_\rho$ satisfying the condition (4) and

$$\int_X < f(x), f(x) >_\rho dx \quad < \infty, \tag{5}$$

where $< \, . \, , \, . \, >_\rho$ denotes the inner product on V_ρ. On this space $L^2(\rho, X, dx)$ one has the inner product

$$< f, g >= \int_X < f(x), f(x) >_\rho dx. \tag{6}$$

The group G acts on this space by right translations and this representation is denoted by $\mathcal{R}_\rho$. Thanks to the right G-invariance of dx, this representation is unitary.

Since the representation ρ is completely reducible, it suffices for the decomposition of the space $L^2(\rho, X, dx)$ to consider irreducible ρ and this assumption is made from now on.

Let H_0 be the kernel of ρ and let dh_0 be a Haar measure on H_0. The group H_0 is also unimodular, being a normal subgroup of an unimodular group. Hence also the homogeneous manifold $X_0 := H_0\backslash G$ possesses a positive right G-invariant measure that is denoted by dx_0 and is related to dg and dh_0 by a formula like (1). One denotes the inner product of φ_1 and φ_2 in $L^2(X_0, dx_0)$ by $< \varphi_1, \varphi_2 >_0$ and the action of G by right translations on $L^2(X_0, dx_0)$ by $\mathcal{R}$.

Like in [10] one makes throughout the rest of this paper the following additional assumption:

Property 2.4. The group H/H_0 is compact.

Clearly this condition does not always hold, but it enables one to exploit the representation theory of the group H/H_0 and the condition was surely satisfied in the examples treated in [28] and [20].

Example 2.5. Consider the semi-direct products like in example (2.3). With the notations used there, let (ρ_1, V_1) resp. (ρ_2, V_2) be finite dimensional irreducible unitary representations of $L(0)$ resp. $M(0)$ that satisfy property (2.4) and assume that $L(0)$ centralizes ρ_2, then $\rho = \rho_1 \otimes \rho_2$ is a well-defined irreducible unitary representation of $H = L(0) \rtimes M(0)$ that satisfies this property too.

In the case of the Heisenberg group $A(\mathbf{G})$ and the subgroup $H = \mathbf{H} \times \mathbf{H}^\perp \times S^1$ concrete examples of such representations are the

$$\rho_m((h,k,t)) = t^m, m \in \mathbb{Z},$$

the case $m = 1, \mathbf{H} = \mathbf{G}$ being the *Schrödinger representation* of this group. Note that for $|m| = 1$ the groups $H = \mathbf{H} \times \mathbf{H}^\perp \times S^1$ are the centralizers in $A(\mathbf{G})$ of the character ρ_m. Hence, according to a theorem of Mackey, see [15], the representations $\mathcal{R}_{\rho_m}$ for these m are irreducible. For $|m| > 1$, let U_m be the group of $|m|$-th roots of unity in the complex plane and let $\mathbf{H}_m$ be the subgroup of $\mathbf{G}$ defined by

$$\mathbf{H}_m = \{g \in G \mid < \tilde{h} | g > \in U_m \text{ for all } \tilde{h} \in \mathbf{H}^\perp\}.$$

Let $\mathbf{H}(m)$ be the quotient group $\mathbf{H}_m \backslash \mathbf{H}$. Then for each character ψ in $\hat{\mathbf{H}}(m)$ one can define the one dimensional representation $\psi \otimes \rho_m$ of $\mathbf{H}_m \times \mathbf{H}^\perp \times S^1$ by

$$\psi \otimes \rho_m((h_m,k,t)) = \psi(h_m) t^m,\ h_m \in \mathbf{H}_m,\ k \in \mathbf{H}^\perp \text{ and } t \in S^1.$$

They are exactly the components of the representation obtained by inducing ρ_m from H to $\mathbf{H}_m \times \mathbf{H}^\perp \times S^1$. Inducing the $\psi \otimes \rho_m$ to the group $A(\mathbf{G})$ gives you irreducible representations of $A(\mathbf{G})$ that are inequivalent for different ψ. Thus the representation $\mathcal{R}_{\rho_m}$ decomposes into irreducible representations that each occur only once. This phenomenon occurs for a wide class of semi-direct products as one will see.

Note that in some of the extreme cases i.e. where $L = L(0)$ or $M = M(0)$, one can realize the representation as functions on one of the components. If L equals $L(0)$, the group L is unimodular and $|\alpha(l)| = 1$ for all $l \in L$, then the right M-invariant measure $d\tilde{m}$ on $X = M(0)\backslash M = H\backslash G$ is a right G-invariant measure on X. With this choice, restricting to M gives you an isomorphism between $L^2(\rho, X, dx)$ and $L^2(1 \otimes \rho_2, M(0)\backslash M, d\tilde{m})$. The action of M on this last space is simply by right translations and that of $l \in L$ is given by

$$\mathcal{R}_\rho(l)(f)(m) = \rho_1(l) \otimes 1(f(\alpha(l^{-1})(m))) \tag{7}$$

Assume now that $M = M(0)$ and that L and $L(0)$ are unimodular. The right L-invariant measure $d\tilde{l}$ on $X = L(0)\backslash L$ is also right G-invariant.

Like in the other extreme case, taking the restriction to L gives an isomorphism between $L^2(\rho, X, dx)$ and $L^2(\rho_1 \otimes 1, L(0)\backslash L, d\tilde{l})$. On the last space L acts by right translation and the group M by

$$\mathcal{R}_\rho(m)(f)(l) = 1 \otimes \rho_2(\alpha(l)(m))(f(l)). \tag{8}$$

Thus, the representation of $A(\mathbf{G})$ obtained by inducing ρ_m from $H = \mathbf{G}^\perp \times S^1$ can be realized on $L^2(\mathbf{G})$ and that from $H = \mathbf{G} \times S^1$ on $L^2(\mathbf{G}^\perp)$.

For the general case it is convenient to have another realization of $\mathcal{R}_\rho$ that is briefly described next. The group H acts also by left translations on the space X_0 and by transposition on the functions on X_0. Since dg is also left G-invariant, one sees from relation (1) that

$$\mathcal{L}(h)(f)(x) := f(h^{-1}x), \tag{9}$$

defines a unitary representation of H on $L^2(X_0, dx_0)$ that factorizes over H/H_0. Let $d\tilde{h}$ denote the normalized Haar measure on H/H_0. Then there is an algebra morphism from the convolution algebra of continuous functions on H/H_0 to the bounded linear operators on $L^2(X_0, dx_0)$. If φ is continuous on H/H_0, it is defined by

$$\mathcal{L}(\varphi)(f)(x_0) = \int_{H/H_0} \varphi(\tilde{h})\mathcal{L}(\tilde{h})(f)(x_0)d\tilde{h} =: \varphi * f(x_0). \tag{10}$$

For $u, v \in V_\rho$, let $e_{v,u}$ be the matrix coefficient of the representation ρ of H/H_0 given by

$$e_{v,u}(h) = d_\rho < \rho(h)(u), v >_\rho, \tag{11}$$

where $d_\rho = \dim(V_\rho)$. Let $\{f_i \mid 1 \leq i \leq d_\rho\}$ be an orthonormal basis of the space V_ρ. For each i and j, one denotes the function e_{f_i,f_j} by e_{ij}. Inside $L^2(X_0, dx_0)$ consider the following G-invariant closed subspace

$$L^2(e_{11}, X_0, dx_0) = \{\varphi \mid \varphi \in L^2(X_0, dx_0), e_{11} * \varphi = \varphi\}. \tag{12}$$

By using the orthogonality relations for the e_{ij} one shows that for each $\varphi \in L^2(e_{11}, X_0, dx_0)$ the function

$$A(\varphi) := \frac{1}{\sqrt{d_\rho}} \sum_{j=1}^{d_\rho} (e_{j1} * \varphi)\, f_j \tag{13}$$

satisfies equation (4). If the measures dx and dx_0 are chosen such that $d\tilde{h}dx = dx_0$, then the map $A : L^2(e_{11}, X_0, dx_0) \to L^2(\rho, X, dx)$ is a norm preserving bijection that commutes with the right G-action on both spaces.

3. Hilbert subspaces of distributions

Recall that Bruhat, see [2], has introduced for each locally compact group G_1 and each homogeneous space $F\backslash G_1$, where F is a closed subgroup of G_1, the spaces of test functions $\mathcal{D}(G_1)$ and $\mathcal{D}(F\backslash G_1)$ with an appropriate topology. It unifies the cases that G_1 is a Lie group, where it equals the space of C^∞-functions with compact support, and that of totally disconnected spaces, in which case it consists of the locally constant functions with compact support. The elements of their continuous antilinear duals are called distributions on G_1 resp. $F\backslash G_1$ and these spaces are denoted by $\mathcal{D}^1(G_1)$ and $\mathcal{D}^1(F\backslash G_1)$.

The group G acts on $\mathcal{D}(X_0)$ by right translation and it leaves the subspace

$$\mathcal{D}(e_{11}, X_0) = \{\phi \in \mathcal{D}(X_0) \mid e_{11} * \phi = \phi\}. \tag{14}$$

invariant. By transposing this representation $\mathcal{R}_\infty$ of G on $\mathcal{D}(X_0)$ one arrives at the representation $\mathcal{R}_{-\infty}$ of G on $\mathcal{D}^1(X_0)$, i.e. for $T \in \mathcal{D}^1(X_0)$

$$\mathcal{R}_{-\infty}(g)(T)(\varphi) = T(\mathcal{R}_\infty(g^{-1})\varphi).$$

Likewise one can dualize the left H-action on $\mathcal{D}(X_0)$ to a representation $\mathcal{L}_{-\infty}$ of H on $\mathcal{D}^1(X_0)$ and one verifies directly that the antilinear dual of the subspace $\mathcal{D}(e_{11}, X_0)$ can be identified with

$$\mathcal{D}^1(e_{11}, X_0) = \{T \in \mathcal{D}^1(X_0), \int_{H/H_0} e_{11}(\tilde{h})\mathcal{L}_{-\infty}(\tilde{h})(T)d\tilde{h} = T\}. \tag{15}$$

Hence, if we take an $f \in L^2(e_{11}, X_0, dx_0)$ and consider the distribution $T = f(x)dx$ on X_0, then it belongs to $\mathcal{D}^1(e_{11}, X_0)$ and

$$\mathcal{R}_{-\infty}(g)(f(x)dx) = f(xg)dx.$$

In other words the embedding $j : f(x) \mapsto f(x)dx$ of $L^2(e_{11}, X_0, dx_0)$ into $\mathcal{D}^1(e_{11}, X_0)$ is a G-morphism. Therefore $L^2(e_{11}, X_0, dx_0)$ is a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. Let $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ be the collection of G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$. The following partial ordering can be put on $\mathcal{D}^1(e_{11}, X_0)$: if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$, then one writes $\mathcal{H}_1 \leq \mathcal{H}_2$, if $\mathcal{H}_1 \subset \mathcal{H}_2$ and

$$\| x \|_{\mathcal{H}_2} \leq \| x \|_{\mathcal{H}_1}, \text{ for all } x \in \mathcal{H}_1. \tag{16}$$

For each $\lambda \in \mathbb{R}, \lambda > 0$, and each $\mathcal{H}$ belonging to $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$, let $\lambda\mathcal{H}$ be the Hilbert subspace $\mathcal{H}$ with the inner product

$$(x, y)_{\lambda\mathcal{H}} := \frac{1}{\lambda}(x, y)_{\mathcal{H}} \tag{17}$$

and for $\lambda = 0$ let $\lambda\mathcal{H}$ be the Hilbert subspace $\{0\}$. This choice is such that for all λ_1 and λ_2 with $\lambda_1 \leq \lambda_2$ and each $\mathcal{H}$ in $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$, there holds $\lambda_1\mathcal{H} \leq \lambda_2\mathcal{H}$.

Since convolution from the left with e_{11} projects $\mathcal{D}(X_0)$ onto $\mathcal{D}(e_{11}, X_0)$ and the antilinear dual of $\mathcal{D}^1(X_0)$ is $\mathcal{D}(X_0)$, one sees that also the antilinear dual of the space $\mathcal{D}^1(e_{11}, X_0)$ can be identified with $\mathcal{D}(e_{11}, X_0)$. Hence, if $j : \mathcal{H} \hookrightarrow \mathcal{D}^1(e_{11}, X_0)$ is the continuous embedding of $\mathcal{H}$ into $\mathcal{D}^1(e_{11}, X_0)$, then one has an adjoint map $j^* : \mathcal{D}(e_{11}, X_0) \hookrightarrow \mathcal{H}$ defined by

$$< j(h), \varphi >= (h, j^*(\varphi))_{\mathcal{H}}, \text{ for all } h \in \mathcal{H} \text{ and all } \varphi \in \mathcal{D}(e_{11}, X_0). \quad (18)$$

The map $\mathcal{K}_{\mathcal{H}} := jj^* : \mathcal{D}(e_{11}, X_0) \hookrightarrow \mathcal{D}^1(e_{11}, X_0)$ is called the *reproducing kernel* of $\mathcal{H}$ and is determined by

$$< \mathcal{K}_{\mathcal{H}}(\varphi), \psi >=< jj^*(\varphi), \psi >= (j^*\varphi, j^*\psi)_{\mathcal{H}}, \quad (19)$$

for all φ and $\psi \in \mathcal{D}(e_{11}, X_0)$. Here $(\cdot,\cdot)_{\mathcal{H}}$ is the inner product on $\mathcal{H}$ and $< \cdot, \cdot >$ is the natural pairing between $\mathcal{D}^1(e_{11}, X_0)$ and its antilinear dual $\mathcal{D}(e_{11}, X_0)$. Equation (19) implies a number of properties. First of all, the kernel $\mathcal{K}_{\mathcal{H}}$ is hermitian

$$< \mathcal{K}_{\mathcal{H}}(\varphi), \psi >= \overline{< \mathcal{K}_{\mathcal{H}}(\psi), \varphi >} \quad (20)$$

It is also *positive*, i.e. for all $\varphi \in \mathcal{D}(e_{11}, X_0)$

$$< \mathcal{K}_{\mathcal{H}}(\varphi), \varphi >\geq 0. \quad (21)$$

The G-invariance of the embedding j translates into

$$< \mathcal{K}_{\mathcal{H}}(\mathcal{R}_\infty(g)(\varphi)), \mathcal{R}_\infty(g)(\psi) >=< \mathcal{K}_{\mathcal{H}}(\varphi), \psi > \quad (22)$$

or in terms of the operator $\mathcal{K}_{\mathcal{H}}$: for all $g \in G$,

$$\mathcal{R}_{-\infty}(g^{-1}) \circ \mathcal{K}_{\mathcal{H}} \circ \mathcal{R}_\infty(g) = \mathcal{K}_{\mathcal{H}}. \quad (23)$$

Finally, the fact that kernel $\mathcal{K}_{\mathcal{H}}$ relates to a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, not just of $\mathcal{D}^1(X_0)$, leads to

$$< \mathcal{K}_{\mathcal{H}}(e_{11} * \varphi), e_{11} * \psi >=< \mathcal{K}_{\mathcal{H}}(\varphi), \psi > . \quad (24)$$

Let $\mathcal{F}$ be the real vector space of all linear maps $K : \mathcal{D}(e_{11}, X_0) \to \mathcal{D}^1(e_{11}, X_0)$ that are hermitian, i.e. they satisfy equation (20), that are G-invariant, i.e. they satisfy equation (22), and are equivariant for left convolution with e_{11}, i.e. they satisfy equation (24). On $\mathcal{F}$ one can put various suitable topologies of which one chooses here that of uniform convergence on bounded subsets. The subset of $\mathcal{F}$ consisting of the positive operators, i.e. that satisfy condition (21), is denoted by Γ_G and forms a closed convex cone in $\mathcal{F}$. The cone Γ_G is also proper, i.e. $\Gamma_G \cap -\Gamma_G = \{0\}$. For the relation $< K(\varphi), \varphi >= 0$, for all $\varphi \in \mathcal{D}(e_{11}, X_0)$ combined with property (20), implies namely that $< K(\varphi) \mid \psi >$ is purely imaginary for all φ and $\psi \in \mathcal{D}(e_{11}, X_0)$. Hence K is equal to zero.

The cone Γ_G can be used to put a partial order on $\mathcal{F}$. It is defined by

$$K_1 \leq K_2 \Leftrightarrow K_2 - K_1 \in \Gamma_G. \tag{25}$$

As we have seen the reproducing kernels are examples of elements in Γ_G. According to Schwartz, see [18], the map $\mathcal{H} \to \mathcal{K}_{\mathcal{H}}$ is a bijection between $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ and Γ_G that is compatible with the cone structure and the partial order on both sides. In particular, there holds for all $\lambda \geq 0$ and all $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ that $\mathcal{K}_{\lambda\mathcal{H}} = \lambda\mathcal{K}_{\mathcal{H}}$. The order relation $\mathcal{H}_1 \leq \mathcal{H}_2$ between two Hilbert subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{D}^1(e_{11}, X_0)$ can be expressed purely in terms of the corresponding kernels, see [18]. For, if $\mathcal{K}_{\mathcal{H}_2} = j \circ j^*$, then one has

$$\mathcal{K}_{\mathcal{H}_1} = j \circ B \circ j^*, \text{ with } B : \mathcal{H}_2 \to \mathcal{H}_2, \ \ 0 \leq B \leq \mathrm{Id}_{\mathcal{H}_2}. \tag{26}$$

Next one takes the G-invariance (23) of both $\mathcal{H}_1$ and $\mathcal{H}_2$ into account and obtains then

$$\begin{aligned} j \circ B \circ j^* = \mathcal{K}_{\mathcal{H}_1} &= \mathcal{R}_{-\infty}(g^{-1}) \circ \mathcal{K}_{\mathcal{H}_1} \circ \mathcal{R}_{\infty}(g) \qquad (27)\\ &= j \circ \mathcal{R}_{-\infty}(g^{-1})|_{\mathcal{H}_2} \circ B \circ \mathcal{R}_{-\infty}(g)|_{\mathcal{H}_2} \circ j^*. \end{aligned}$$

As the map j is injective and the image of j^* is dense, one may conclude that B and all the $\mathcal{R}_{-\infty}(g)|_{\mathcal{H}_2}$ commute.

Let $\mathrm{ext}(\Gamma_G)$ be the set of extremal rays of Γ_G. Those are the reproducing kernels K that satisfy

$$0 \leq K^1 \leq K, K^1 \in \Gamma_G \Rightarrow K^1 = \alpha K. \tag{28}$$

The relevance of $\mathrm{ext}(\Gamma_G)$ follows from

Theorem 3.1. *Let* $(\pi, \mathcal{H}_\pi)$ *be a* G*-invariant Hilbert subspace of* $\mathcal{D}^1(e_{11}, X_0)$ *and let* $K_\pi \in \Gamma_G$ *be the corresponding distribution. Then there holds*

$$(\pi, \mathcal{H}_\pi) \ \textit{ is irreducible } \ \Leftrightarrow K_\pi \ \textit{ is extremal}$$

Proof. This theorem holds in wide generality, see [14]. Since it is short, the proof for the present setting is included here. Let $\mathcal{K}_{\mathcal{H}} \in \mathrm{ext}(\Gamma_G)$. Consider a G-invariant closed subspace $\mathcal{H}_1$ of $\mathcal{H}$. By definition there holds $\mathcal{H}_1 \leq \mathcal{H}$. Since $\mathcal{K}_{\mathcal{H}}$ is on an extremal ray, one has $\mathcal{H}_1 = \lambda\mathcal{H}$. However the norm on $\mathcal{H}_1$ is equal to that on $\mathcal{H}$ so that $\lambda = 0$ or 1.

Hence the only closed G-invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$ itself. Reversely, let the representation of G on $\mathcal{H}$ be irreducible and consider a $\mathcal{H}_1 \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ with $\mathcal{H}_1 \leq \mathcal{H}$. As we saw above its kernel has the form $\mathcal{K}_{\mathcal{H}_1} = j \circ B \circ j^*$ with $B : \mathcal{H} \to \mathcal{H}$ commuting with the G-action on $\mathcal{H}$. By Schur's lemma, $B = \lambda \, \mathrm{Id}$ and $\mathcal{K}_{\mathcal{H}_1} = \lambda \mathcal{K}_{\mathcal{H}}$. This proves the claim in the theorem. □

From the representation theoretic point of view one is interested in decomposing the unitary G-modules corresponding to the kernels in Γ_G into irreducible representations. Now integral decompositions of elements in convex cones in extremal elements are known starting with the work of Choquet for cones with a compact base, see [4], and its generalization later by Thomas, see [25], to conuclear cones. Hence, thanks to 3.1, this can be used to realize the representation theoretic goal. This setting has moreover the great advantage that the matter of uniqueness is well-defined. It requires a number of notions that will be recalled.

Let $\mathcal{F}'$ be the space of continuous linear forms on $\mathcal{F}$. By $h(\mathcal{F})$ one denotes the space of Choquet test functions, consisting of functions $h : \mathcal{F} \mapsto \mathbb{R}$ of the form

$$h = \sup_{i \in I} v_i - \sup_{j \in J} w_j,$$

where $\{v_i \mid i \in I\}$ and $\{w_j \mid j \in J\}$ are finite collections of elements in $\mathcal{F}'$. Following Choquet [5], a *conical measure* is a linear form $\mu : h(\mathcal{F}) \mapsto \mathbb{R}$ such that $\mu(h) \geq 0$ for all $h \in h(\mathcal{F})$ such that $h \geq 0$. A *localization* of a conical measure μ is a measure m on the Borel sets of $\mathcal{F}_* = \{f \in \mathcal{F} \mid f \neq 0\}$ that is first of all inner regular, i.e.

$$m(A) = \sup\{m(K) \mid K \subset \mathcal{F}_*, K \text{ compact}\},$$

it should moreover satisfy

$$\int |v(x)| dm(x) < +\infty, \text{ for all } v \in \mathcal{F}'$$

and finally its relation to μ should be given by

$$\mu(h) = \int h(x) dm(x), \text{ for all } h \in h(\mathcal{F}).$$

A conical measure with a localization is called a *conical integral.* The space of all conical integrals is denoted by $M^+_{loc}(\mathcal{F})$. Note that there are infinitely many localizations of a conical integral μ. Let, namely, $D_\lambda, \lambda > 0$, be the dilation $x \mapsto \frac{1}{\lambda} x$ of $\mathcal{F}_*$. Then the $\frac{1}{\lambda} D_\lambda(m), \lambda > 0$, are all localizations of μ, because of

$$\mu(h) = \frac{1}{\lambda}\int h(\lambda x)dm(x), \text{ for all } h \in h(\mathcal{F}).$$

To get rid of this ambiguity, one replaces a localization by one that is concentrated on a *section* S of $\mathcal{F}$ that is a subset of $\mathcal{F}_*$ meeting each ray of $\mathcal{F}_*$ in precisely one point. According to [25] there exists a function $p : \mathcal{F}_* \mapsto (0, +\infty)$ satisfying $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$ and all $x \in \mathcal{F}_*$, which is universally measurable, see also [19]. The set

$$S = \{x \in \mathcal{F}_* \mid p(x) = 1\}$$

is then a section of $\mathcal{F}$ on which every conical integral can be localized. Consider namely the map $\varrho : \mathcal{F}_* \mapsto \mathcal{F}_*$ defined by $\varrho(x) = \frac{x}{p(x)}$. For a conical integral μ with localization m, let $\tilde{m} = \varrho(pm)$ be the image under ϱ of the measure pm. It is clearly concentrated on S and a localization of μ because one has for all $h \in h(\mathcal{F})$

$$\int h(y)d\tilde{m}(y) = \int h(\varrho(x))p(x)dm(x) = \int h(x)dm(x) = \mu(h).$$

In particular, this construction yields for all the $\frac{1}{\lambda}D_\lambda(m), \lambda > 0$, the localization $\tilde{m}$.

Likewise, one defines for any cone Γ in $\mathcal{F}$ the notion of a section S of Γ. It is called *admissible* if the function $p_S(\lambda s) = \lambda$ for $s \in S$ and $\lambda \geq 0$ is universally measurable from Γ to $[0, +\infty)$. One speaks of an *admissible parametrization* of the cone Γ if one has a continuous injection $\gamma : T \mapsto \Gamma$ such that $S := \gamma(T)$ is an admissible section of Γ and the inverse map $S \mapsto T$ is universally measurable. Here T is some Hausdorff space.

Recall that the resultant $r(\mu)$ of a conical measure μ is the element of the weak completion of $\mathcal{F}$ corresponding to

$$v \mapsto \mu(v), \text{ with } v \in \mathcal{F}^{'}.$$

If μ has a localization m, then this linear form is described by

$$r(\mu)(v) = \int v(x)dm(x),$$

in which case one also uses the notation

$$r(\mu) = \int xdm(x).$$

For any cone $\Gamma \subset \mathcal{F}$ one is interested in the conical integrals that are concentrated on Γ and for which the resultant belongs to $\mathcal{F}$, i.e.

$$\mathcal{M}^+(\Gamma) = \{\mu \in M^+_{loc}(\mathcal{F}), \mu \text{ concentrated on } \Gamma, r(\mu) \in \mathcal{F}\}.$$

If Γ is closed and convex, like Γ_G, then the resultant of a $\mu \in \mathcal{M}^+(\Gamma)$ is easily seen to belong to Γ. If Γ is proper, i.e. $\Gamma \cap -\Gamma = \{0\}$, then one defines the set of its extreme generators, $\mathrm{ext}(\Gamma)$, as in the case of Γ_G. The set $\mathrm{ext}(\Gamma)$ is a subcone of Γ and one sees that for proper, closed, convex cones Γ one has a natural map $\mu \mapsto r(\mu)$ from $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ to Γ. For such cones Γ with a compact base, it is known, see [5], that this map is surjective. Now the antilinear dual of $\mathcal{D}^1(e_{11}, X_0)$ is the space $\mathcal{D}(e_{11}, X_0)$. Since the group G is second countable, one sees that $\mathcal{D}^1(e_{11}, X_0)$ is the dual of a nuclear barrelled space. Hence, according to [23], the cone Γ_G is the union of its caps. Recall that a *cap* of a cone Γ is a compact convex subset $K \subset \Gamma$ such that $\Gamma \backslash K$ is convex. Note that the compactness of K and the fact that $\Gamma \backslash K$ is convex, imply that the intersection of K and the ray through γ has the form $\{\lambda\gamma \mid \lambda \in [0, b] \text{ with } b \geq 0\}$. Let $k \in K$ be an extremal element of K, i.e. for each $k_1 \in K$ such that $k_1 \leq k$ there holds: $k_1 = \lambda k$. Then k belongs to $\mathrm{ext}(\Gamma_G)$, for assume that $\gamma_1 \in \Gamma_G \backslash K$ satisfies $\gamma_1 \leq k$. Then one has $k = \gamma_1 + \gamma_2$, with $\gamma_2 \in \Gamma_G$. If $\gamma_2 \in \Gamma_G \backslash K$, then $\frac{k}{2}$ would belong to K and $\Gamma_G \backslash K$, which is a contradiction. On the other hand, if $\gamma_2 \in K$, then one has that $\gamma_2 \leq k$ and this implies that γ_2 is a multiple of k. The same holds then for γ_1. By applying Choquet's result to each subcone generated by a cap, one gets that for Γ_G the map $\mu \mapsto r(\mu)$ is surjective. Combining this result with those of the admissible parametrizations and the theory of disintegration of spectralmeasures from [22], one obtains

Theorem 3.2. *Let* $\gamma : T \mapsto \mathrm{ext}(\Gamma_G)$ *be an admissible parametrization of* $\mathrm{ext}(\Gamma_G)$ *and let* $K_{\gamma(t)}$ *be the kernel corresponding to* $\gamma(t)$*. For each* $\mathcal{K}_{\mathcal{H}} \in \mathrm{ext}(\Gamma_G)$ *there is a Radon measure* m *on* T *such that*

$$\mathcal{K}_{\mathcal{H}} = \int_T K_{\gamma(t)} dm(t).$$

Moreover, let $\mathcal{H}_{\gamma(t)}$ *be the Hilbert subspace corresponding to* $K_{\gamma(t)}$*. Then the* $\{\mathcal{H}_{\gamma(t)} \mid t \in T\}$ *form a* m*-summable family of Hilbert subspaces and one has the direct integral decomposition*

$$\mathcal{H} = \int_T^{\oplus} \mathcal{H}_{\gamma(t)} dm(t).$$

In particular for the Hilbert subspace $L^2(e_{11}, X_0, dx_0)$ this theorem gives you a decomposition of $L^2(e_{11}, X_0, dx_0)$ in minimal unitary G-models.

Another way to decompose a Hilbert subspace $\mathcal{H}$ of $\mathcal{D}^1(e_{11}, X_0)$ is to consider a maximal commutative von Neumann algebra $\mathcal{A}$ in the commutant of all the

$$\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}} \mid g \in G\}.$$

and a weakly dense C^*-subalgebra $\mathcal{C}$ of $\mathcal{A}$ containing the identity. Let Z be the spectrum of $\mathcal{C}$. According to Gelfand's theorem, one has a $*$-algebra isomorphism $f \mapsto T_f$ between the space $C(Z)$ of complex valued continuous functions on Z and $\mathcal{C}$. For all $x, y \in \mathcal{H}$ the map $f \mapsto (T_f(x), y)_{\mathcal{H}}$ is continuous on $C(Z)$. Hence, there is a unique measure $\nu_{x,y}$ on Z such that

$$(T_f(x), y)_{\mathcal{H}} = \int_Z f(z) d\nu_{x,y}(z). \tag{29}$$

According to [6], one can choose $\mathcal{C}$ such that there exists a basic measure ν on Z, i.e. a measure such that $Y \subset Z$ is ν-negligible if and only if Y is $\nu_{x,x}$-negligible for all $x \in \mathcal{H}$. In that case each spectral measure $\nu_{x,y}$ has a density $h_{x,y}$ w.r.t. ν. i.e. $\nu_{x,y} = h_{x,y}\nu$. The sesquilinear form $(x, y) \mapsto h_{x,y}(z)$ defines an inner product on the quotient space $\mathcal{H}\backslash\mathcal{N}_z$, where

$$\mathcal{N}_z = \{n \in Z \mid h_{n,y}(z) = 0 \text{ for all } y \in Z\}.$$

Its completion is denoted by $\mathcal{H}_z$. This defines a ν-measurable family of Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$. By relation (29) the map $f \mapsto T_f$ extends to an algebra homomorphism from the bounded ν-measurable functions to $\mathcal{A}$. Denote for each subset Y of Z its characteristic function by 1_Y. Then for each ν-measurable $Y \subset Z$ the operator T_{1_Y} is an orthogonal projection. This defines a spectral measure $P : Y \mapsto T_{1_Y}$ that is absolutely continuous w.r.t. ν and the $\{\mathcal{H}_z \mid z \in Z\}$ are a disintegration of P, i.e.

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}_z d\nu(z). \tag{30}$$

This decomposition is direct, since $T_{1_{Y(1)}} T_{1_{Y(2)}} = 0$ as soon as $Y(1)$ and $Y(2)$ have an empty intersection. The maximality of $\mathcal{A}$ implies according to Mautner's theorem, see [16], that apart from a ν-negligible set N all $\mathcal{H}_z$, $z \in Z\backslash N$, are irreducible unitary representations of G. Thus one obtains in a different fashion the core of the decomposition in theorem 3.2.

In view of Theorem 3.2 it is important to have an idea of which representations can be realized as a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. In the case of a trivial ρ, these are the representations that possess a non-zero cyclic H-invariant distribution vector. A similar notion will be introduced in the present setting together with a useful characterization.

4. C^∞-vectors and distribution vectors

Let $(\pi, \mathcal{H}_\pi)$ be a continuous representation of G on the Hilbert space $\mathcal{H}_\pi$. In [10], the space $\mathcal{H}_\pi^\infty$ of C^∞-vectors of $\mathcal{H}_\pi$ was introduced as

$$\mathcal{H}_\pi^\infty = \varinjlim_{\mathbb{K}_n} \mathcal{H}_\pi(\mathbb{K}_n), \tag{31}$$

where $\{\mathbb{K}_n\}$ is a sequence of compact normal subgroups in an open Yamabe subgroup of G, that converges to the identity, and, where the spaces $\mathcal{H}_\pi(\mathbb{K}_n)$ are given by

$$\mathcal{H}_\pi(\mathbb{K}_n) = \{v \in \mathcal{H}_\pi \mid \pi(\mathbb{K}_n)v = v \text{ and } g \to \pi(g)v \in C^\infty(G/\mathbb{K}_n, \mathcal{H}_\pi)\},$$

It has a natural Fréchet topology and the natural embedding $\mathcal{H}_\pi^\infty \to \mathcal{H}_\pi$ is continuous and has a dense image. For more details we refer to [10].

Example 4.1. According to example 2.5 the Schrödinger representation of the group $A(\mathbf{G})$ can be realized on $L^2(\mathbf{G})$. It is shown in [13] for $\mathbf{G} = \mathbb{R}^n$ that the space of C^∞-vectors of this representation is exactly the space $\mathcal{S}(\mathbf{G})$ of Schwarz-Bruhat functions on $\mathbf{G}$. By combining this with the general structure of locally compact abelian groups, one shows that this holds for general $\mathbf{G}$.

The topological antilinear dual of $\mathcal{H}_\pi^\infty$ is called the space of *distribution vectors* of $(\pi, \mathcal{H}_\pi)$ and is denoted by $\mathcal{H}_\pi^{-\infty}$. Since $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$, we get a continuous embedding $\mathcal{H}_\pi \hookrightarrow \mathcal{H}_\pi^{-\infty}$. The space $\mathcal{H}_\pi^\infty$ is G-invariant and the restriction of π to $\mathcal{H}_\pi^\infty$ is also denoted by π_∞.

By transposition we have a representation $\pi_{-\infty}$ of G on $\mathcal{H}_\pi^{-\infty}$, i.e.

$$\langle \pi_{-\infty}(g)T, v\rangle = \langle T, \pi_\infty(g^{-1})v\rangle \tag{32}$$

Examples of C^∞-vectors are easily obtained. For each $\varphi \in \mathcal{D}(G)$ and each $v \in \mathcal{H}_\pi$ the vector $\pi(\varphi)v$ belongs to $\mathcal{H}_\pi^\infty$. Moreover the space $G(\mathcal{H}_\pi)$ spanned by all these vectors is dense in $\mathcal{H}_\pi^\infty$. It is called the *Gårding space* of $(\pi, \mathcal{H}_\pi)$.

The foregoing fact enables you to define for each $\varphi \in \mathcal{D}(G)$ and each $T \in \mathcal{H}_\pi^{-\infty}$ the distribution vector $\pi_{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^{-\infty}$ by

$$\begin{aligned}\langle \pi_{-\infty}(\varphi)(T), v\rangle &= \int_{G_k} \varphi(g)\langle \pi_{-\infty}(g)T, v\rangle dg \\ &= \langle T, \pi_\infty(\overset{\vee}{\varphi}_0)(v)\rangle.\end{aligned} \tag{33}$$

for all $v \in \mathcal{H}_\pi^\infty$. Here $\check{\varphi}$ is defined by $\check{\varphi}(g) = \varphi(g^{-1})$. As in the Lie group case there holds

Lemma 4.2. *The distribution vector $\pi_{-\infty}(\varphi)(T)$ for $\varphi \in \mathcal{D}(G)$ and $T \in \mathcal{H}_\pi^{-\infty}$ belongs to $\mathcal{H}_\pi^\infty$.*

Hence each $T \in \mathcal{H}_\pi^{-\infty}$ defines a linear map $A_T : \mathcal{D}(G) \to \mathcal{H}_\pi$ by $A_T(\varphi) = \pi_{-\infty}(\check{\varphi})(T)$. By reduction to the Lie group case one shows that it is continuous. With respect to left translations on $\mathcal{D}(G)$ the map A_T behaves as follows

$$A_T(\varepsilon_g * \varphi) = \pi_{-\infty}(\check{\varphi})\pi_{-\infty}(g^{-1})T, \tag{34}$$

for all $g \in G$. The map A_T also intertwines the action of G by right translation on $\mathcal{D}(G)$ and by the representation π on $\mathcal{H}_\pi$, i.e. for all $g \in G$ and all $\varphi \in \mathcal{D}(G)$

$$A_T(\varphi * \varepsilon_{g^{-1}}) = \pi(g)(A_T(\varphi)). \tag{35}$$

All continuous maps from $\mathcal{D}(G)$ to $\mathcal{H}_\pi$ with the property (35) have this form, for there holds analogous to the Lie group case:

Theorem 4.3. *Let $A : \mathcal{D}(G) \to \mathcal{H}_\pi$ be a continuous map that satisfies for all $g \in G$ and all $\varphi \in \mathcal{D}(G)$, $A(\varphi * \varepsilon_{g^{-1}}) = \pi(g)(A_T(\varphi))$. Then there is a unique distribution vector $T \in \mathcal{H}_\pi^{-\infty}$ such that $A = A_T$.*

Now that the action of G on $\mathcal{H}_\pi^{-\infty}$ has been defined, consider

$$(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \left\{ T \in \mathcal{H}_\pi^{-\infty} \;\middle|\; \begin{array}{l} \pi_{-\infty}(h)T = T \text{ for all } h \in H_0, \\ \pi_{-\infty}(\check{e}_{11})T = T \end{array} \right\}.$$

Note that, if H is compact, then $\pi(e_{11})$ is a well-defined orthogonal projection of the space $\mathcal{H}_\pi$ and the conditions on a $T \in (\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ simply mean that it factorizes over $\pi(\check{e_{11}})$. Hence

Lemma 4.4. *For compact H, one has $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \pi(\check{e_{11}})(\mathcal{H}_\pi)^{-\infty}$.*

Clearly in the noncompact case the operator $\pi(\check{e_{11}})$ can not be given a sense and that is why one has to proceed more carefully. Before coming to the characterization of $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ in terms of distribution vectors one introduces still the notion

Definition 4.5. A distribution vector T in $\mathcal{H}_\pi^{-\infty}$ is called *cyclic* if the space

$$\{\pi_{-\infty}(\varphi)(T) \mid \varphi \in \mathcal{D}(G)\}$$

is lying dense in $\mathcal{H}_\pi$.

With the help of this notion, one can see from the space $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ if a unitary representation is a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, for there holds

Theorem 4.6. *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Then the set of non-zero cyclic elements of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ is in bijective correspondence with the continuous G-equivariant embeddings $j : \mathcal{H}_\pi \hookrightarrow \mathcal{D}^1(e_{11}, X_0)$.*

A proof can be found in [10]. One calls the nonzero cyclic elements of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ the (ρ, H)-*spherical distribution vectors* of $(\pi, \mathcal{H}_\pi)$ or spherical distribution vectors of class ρ.

If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G and $j : \mathcal{H}_\pi \to \mathcal{D}^1(e_{11}, X_0)$ a continuous G-equivariant embedding, then one denotes the to j corresponding non-zero cyclic element of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ by T. As in the real case one can associate with T a special distribution σ_T on G. For $\varphi \in \mathcal{D}(G)$ one knows from Lemma 4.2 that $\pi_{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^\infty$ and defines then $\sigma_T \in \mathcal{D}^1(G)$ by

$$\langle \sigma_T, \varphi \rangle = \langle T, \pi_{-\infty}(\varphi)(T) \rangle.$$

Remark 4.7. If H is compact and T corresponds to a cyclic vector $v \in \pi(\check{e_{11}})(\mathcal{H}_\pi)$, then the distribution σ_T equals

$$\langle \sigma_T, \varphi \rangle = \int_G \overline{\varphi(g)} (v, \pi(g)(v))_\pi dg.$$

In other words, one has $\sigma_T = (v, \pi(g)(v))_\pi dg$. Following the terminology of the invariant context, this last function is called *the spherical function of class* ρ of the representation.

Now the distribution σ_T on G is nonzero, positive definite and bi-H_0-invariant. Moreover it satisfies

$$\check{e_{11}} * \sigma_T = \sigma_T * \check{e_{11}} = \sigma_T. \qquad (36)$$

Reversely, let σ be such a distribution on G. Then it is shown in [10] that σ determines a G-invariant Hilbert subspace $\mathcal{H}_\sigma$ of $\mathcal{D}^1(e_{11}, X_0)$.

The class of positive definite bi-H_0-invariant distributions σ on G, satisfying equation (36), is called that of (ρ, H)-*spherical distributions or spherical distributions of class* ρ. The foregoing can be summarized as follows

Theorem 4.8. *The map* $\sigma \mapsto \mathcal{H}_\sigma$ *that associates with each* (ρ, H)-*spherical distribution the unitary* G-*module* $\mathcal{H}_\sigma$, *is a bijection between this class of distributions on* G *and the collection of* G-*invariant Hilbert subspaces of* $\mathcal{D}^1(e_{11}, X_0)$.

5. Multiplicity one

In this section the uniqueness of the integral decomposition in Theorem 3.2 is being discussed. In the simplest topological situation, namely that of a finite group G, then a splitting of $L^2(\rho, X, dx)$ into nonequivalent irreducible components implies by Schur's lemma that the algebra of G-endomorphisms of $L^2(\rho, X, dx)$ is commutative. Reversely, if this splitting contains equivalent irreducible components then the algebra of G-endomorphisms of $L^2(\rho, X, dx)$ is easily seen to contain a subalgebra of 2×2-matrices. The commutativity of suitable algebras comes back also in the characterization of uniqueness for the general situation.

As usual in the context of proper, closed, convex cones in some topological vector space, one speaks of a *unique* integral representation for the elements of the cone Γ_G if the map $\mu \mapsto r(\mu)$ from $\mathcal{M}^+(\mathrm{ext}(\Gamma_G))$ to Γ_G is a bijection. It is known, see [5] and [25], that in convex analytic terms the injectivity of the map r is equivalent to Γ_G being a lattice cone, i.e. for each pair of points (γ_1, γ_2) on the cone there exists a least upper bound and a highest lower bound on the cone.

In the case of the trivial representation ρ, Thomas has given a number of equivalent criteria characterizing unique integral decompositions, see [24]. They extend to the present context. Their proof follows the same line as in [24] with the proper extension of the results from [22] and [23] to the present setting

Theorem 5.1. *Let G, H and ρ be as in section 2 and let $\gamma : T \mapsto \mathrm{ext}(\Gamma_G)$ be an admissible parameterization of $\mathrm{ext}(\Gamma_G)$ as in theorem 3.2. The following properties are equivalent:*

(1) The cone Γ_G is a lattice cone.

(2) Assume $\mathcal{H}_1$ and $\mathcal{H}_2$ are minimal G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$, which differ as linear subspaces. Then the irreducible unitary representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$ are inequivalent.

(3) If $(\pi, \mathcal{H}_\pi)$ is any G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, then the commutant of $\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}_\pi} \mid g \in G\}$ is abelian.

(4) For each $K \in \Gamma_G$ there exists a unique Radon measure μ on T such that

$$K = \int_T K_{\gamma(t)} d\mu(t).$$

(5) For every G-invariant Hilbert subspace $\mathcal{H}$ of $\mathcal{D}(e_{11}, X_0)$ there exists a unique Radon measure μ on T such that

$$\mathcal{H} = \int_T^{\oplus} \mathcal{H}_t d\mu(t).$$

Definition 5.2. The space $\mathcal{D}^1(e_{11}, X_0)$ is said to decompose *multiplicity free* if one of the equivalent properties from theorem (5.1) holds. It is also customary to say that *multiplicity one* holds for this space or the triple (G,H,ρ).

A wide variety of examples of multiplicity free representations for trivial ρ can be found in [14] and [26]. Here a number of examples will be discussed, where ρ can be non trivial.

Example 5.3. The first example is that of a commutative G. For example criterion (2) of Theorem 5.1 is then clearly fulfilled, since the minimal G-invariant subspaces are characters of G.

Example 5.4. Consider the semi-direct product $G = L \rtimes M$ like in example (2.3) with the subgroup $H = L \rtimes M(0)$. Let (ρ_1, V_1) resp. (ρ_2, V_2) be finite dimensional irreducible unitary representations of L resp. $M(0)$ as in example 5.4. Since M acts by right translations on $L^2(1\otimes\rho_2, M(0)\backslash M, d\tilde{m})$, this space decomposes w.r.t. the M-action as the direct sum of $\dim(V_1)$ copies of $L^2(\rho_2, M(0)\backslash M, d\tilde{m})$. If C_2 denotes the commutant of the M-action on $L^2(\rho_2, M(0)\backslash M, d\tilde{m})$, then the commutant of the M-action on $L^2(1\otimes\rho_2, M(0)\backslash M, d\tilde{m})$ is $End(V_1)\otimes C_2$. To get the commutant for the G-action the operators in $End(V_1)\otimes C_2$ still have to commute with the L-action from (7). Since ρ_1 is irreducible, one can conclude that the G-commutant is abelian as soon as C_2 is. Thanks to the second criterion in theorem 5.1, one knows that multiplicity one holds for this triple (G,H,ρ). Thanks to the foregoing example this holds for sure if M is abelian.

The foregoing is an extension of the result that was observed already for the Heisenberg group. In this last case one concludes from the Stone-van Neumann theorem, see e.g. [12], that the Schrödinger representation is unitarily equivalent with the representation obtained by inducing ρ_1 from any $H = \mathbf{H}\times\mathbf{H}^\perp\times S^1$. This implies in particular that upto a constant the Schrödinger representation has a unique $\mathbf{H}\times\mathbf{H}^\perp$-fixed distribution vector and it is given on $\mathcal{S}$, its space of C^∞-vectors, by

$$\varphi \to \int_H \overline{\varphi(h)}dh. \tag{37}$$

If one takes for $\mathbf{G} = \mathbb{R}^n$ and $\mathbf{H} = \mathbb{Z}^n$ a realization of the Schrödinger representation on a space of entire functions on $\mathbb{C}^n$, the so called Fock representation, then a direct computation shows that this distribution vector corresponds with a theta function and its uniqueness reflects characterizing transformation properties of this function, see also [3]. This distribution vector gives also rise to an important class of automorphic forms on the metaplectic group , see [29].

Likewise one can consider the other extreme case, namely $M = M(0)$. The L-action on $L^2(\rho_1\otimes 1, L(0)\backslash L, d\tilde{l})$ is by right translations and therefore this space decomposes in $\dim(V_2)$ copies of $L^2(\rho_1, L(0)\backslash L, d\tilde{l})$.

If C_1 denotes the commutant of the L-action in this last space, then the G-commutant consists again of the operators in $C_1 \otimes End(V_2)$ that commute with the M-action from 8. In particular if C_2 is commutative, then we have again reduced multiplicity one for (G, H, ρ) to that of $(L, L(0), \rho_1)$. In view of example 5.3, this holds for sure, if L is abelian.

Example 5.5. The third example of a multiplicity free decomposition is the variant on the group case from example 2.1. Here one proves readily the criterion (3) of Theorem 5.1. Let ρ be a unitary character of $Z(G_1)$ satisfying condition 2.4. By extending it trivially on the diagonal group it determines a representation ρ of the group $H_{Z(G_1)}$. The space $L^2(\rho, H_{Z(G_1)}, G_1 \times G_1)$ can be identified directly with $L^2(\rho, Z(G_1), G_1)$. In that case the group $G_1 \times G_1$ acts on $\mathcal{D}^1(e_{11}, X_0)$ by means of left and right translation

$$U(g_1, g_2) = L(g_1)R(g_2).$$

If one denotes the commutant of a representation with an accent and one writes $\mathcal{N}(L)$ for the van Neumann algebra generated by the left translations from G_1, then one knows from the commutativity theorem of Segal-Godement, see [16], that $R' = \mathcal{N}(L)$. Hence there holds

$$U' = L' \cap R' = L' \cap \mathcal{N}(L).$$

As $\mathcal{N}(L) \cap L'$ is clearly a commutative algebra this proves the result.

By combining the description of the G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$ in Theorem 4.6 with the second characterization from Theorem 5.1 of a multiplicity free decomposition, one obtains:

Corollary 5.6. *If for all irreducible unitary representations* $(\pi, \mathcal{H}_\pi)$ *of* G*, the dimension of the space* $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ *is maximally one, then the integral decomposition of* $\mathcal{D}^1(e_{11}, X_0)$ *in Theorem 3.2 is unique.*

6. The compact case

The criterion in corollary 5.6 applies well in the case that H is compact. For, then one knows from Lemma 4.4 that $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \pi(\check{e}_{11})(\mathcal{H}_\pi)^{-\infty}$. On the subspace $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ one has a natural action of the subalgebra

$$L^1(\check{e}_{11}, X_0) = \{f \mid f \in L^1(G), f = \check{e}_{11} * f * \check{e}_{11}\}$$

of the convolution Banach algebra $L^1(G)$. The subalgebra $L^1(\check{e}_{11}, X_0)$ is also involutive, for if one denotes for an $f \in L^1(G)$ the element

$f^* \in L^1(G)$ by $f^*(g) = \overline{f(g^{-1})}$, then f^* belongs to $L^1(\check{e}_{11}, X_0)$ as soon as f does. Since for each $f \in L^1(\check{e}_{11}, X_0)$ the operator $\pi(f)$ satisfies

$$\pi(f) = \int_G f(g)\pi(g)dg = \pi(\check{e}_{11})\pi(f)\pi(\check{e}_{11}),$$

it is clear that $\pi(f)$ maps $\mathcal{H}_\pi$ to $\pi(\check{e}_{11})(\mathcal{H}_\pi)$. Therefore the map

$$f \mapsto \pi(f)|_{\pi(\check{e}_{11})(\mathcal{H}_\pi)}$$

defines a $*$-representation of $L^1(\check{e}_{11}, X_0)$ onto $\pi(\check{e}_{11})(\mathcal{H}_\pi)$. Hereby irreducibility is preserved, for

Lemma 6.1. *If $(\pi, \mathcal{H}_\pi)$ is an irreducible unitary representation of G such that $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is non-zero, then the representation of $L^1(\check{e}_{11}, X_0)$ on $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is topologically irreducible.*

Proof. Choose a non-zero vector $v \in \pi(\check{e}_{11})(\mathcal{H}_\pi)$. Now the G-module $\mathcal{H}_\pi$ is irreducible if and only if the $L^1(G)$-module $\mathcal{H}_\pi$ is irreducible, see [7]. Therefore the subspace $\{\pi(h)(v) \mid h \in L^1(G)\}$ is a dense subspace of $\mathcal{H}_\pi$. Consequently its image under $\pi(\check{e}_{11})$ is dense in $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ and that equals

$$\begin{aligned}\{\pi(\check{e}_{11})(\pi(h)(v)) \mid h \in L^1(G)\} &= \{\pi(\check{e}_{11})\pi(h)\pi(\check{e}_{11})(v) \mid h \in L^1(G)\} \\ &= \{\pi(\check{e}_{11} * h * \check{e}_{11})(v) \mid h \in L^1(G)\} \\ &= \{\pi(f)(v) \mid f \in L^1(\check{e}_{11}, X_0)\}.\end{aligned}$$

In other words, every non-zero vector in $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is cyclic for the action of $L^1(\check{e}_{11}, X_0)$. This proves the claim of the lemma. □

So, in the case of a compact H, one has a multiplicity-free decomposition if and only if $\dim(\pi(\check{e}_{11})(\mathcal{H}_\pi)) \leq 1$ for all irreducible unitary representations of G. If the Banach $*$-algebra $L^1(\check{e}_{11}, X_0)$ is commutative, then it follows from the work of Gelfand and Naimark that every topologically irreducible $*$-representation of $L^1(\check{e}_{11}, X_0)$ is one dimensional. The foregoing Lemma gives you then the desired estimate. However, also the reverse holds, for there holds

Proposition 6.2. *If H is compact, then $\mathcal{D}^1(e_{11}, X_0)$ decomposes multiplicity free if and only if $L^1(\check{e}_{11}, X_0)$ is commutative.*

Proof. One merely has to show the necessity still. Here one uses the fact that the non degenerate irreducible $*$-representations of the algebra $L^1(G)$ separate the points of $L^1(G)$, see [17]. In particular, for each $f \neq 0$ in $L^1(G)$, there is a non degenerate irreducible $*$-representation

$(\pi, \mathcal{H}_\pi)$ of $L^1(G)$ such that $\pi(f) \neq 0$. According to [7] each non degenerate irreducible $*$-representation of $L^1(G)$ corresponds bijectively to an irreducible unitary representation of G. Take any two elements f_1 and f_2 in $L^1(\check{e}_{11}, X_0)$ and consider $f = f_1 * f_2 - f_2 * f_1$. If f is always zero, then we have the desired result. Assume that there is a non-zero f. Then there is an irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G such that $\pi(f) \neq 0$. As $\pi(f)$ maps $\mathcal{H}_\pi$ to $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ this implies that $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is non-zero. By assumption, the space $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is one-dimensional and spanned by a scalar λ_h on v:

$$\pi(h)(v) = \lambda_h(v)$$

and, since $\pi(f)$ is non-zero the scalar λ_f has to be non-zero. However, $f = f_1 * f_2 - f_2 * f_1$, so that there holds

$$\pi(f)(v) = \pi(f_1)\pi(f_2)(v) - \pi(f_2)\pi(f_1)(v) = (\lambda_{f_1}\lambda_{f_2} - \lambda_{f_2}\lambda_{f_1})(v) = 0.$$

This contradiction is due to the assumption f is non zero. Therefore $L^1(\check{e}_{11}, X_0)$ can not be but commutative. □

Remark 6.3. If H is compact and the algebra $L^1(H\backslash G/H)$ is commutative, then the pair (G, H) is called *a Gelfand pair*, see e.g. [8]. Still for a compact H, but for a not necessarily trivial ρ one can speak of *a Gelfand pair of class* ρ if the algebra $L^1(\check{e}_{11}, X_0)$ is commutative. In case that H is no longer compact it is therefore natural to speak of *a generalized Gelfand pair of class* ρ if one of the properties (2)–(5) of Theorem 5.1 holds. If ρ is trivial, then the adjective "of class ρ" is left out.

The commutativity of algebras like $L^1(\check{e}_{11}, X_0)$ is often proved by showing that an anti algebra morphism is in fact an algebra morphism. This will be illustrated at a subclass of the symmetric varieties $X_k = \mathcal{H}_k\backslash\mathcal{G}_k$. The subclass under consideration are the symmetric varieties for which the group $\mathcal{H}$ is anisotropic over k and with the representation ρ equal to a character χ, with $\chi^2 = 1$. This yields in the real situation the Riemannian symmetric spaces that are known to be multiplicity free. In full generality, there holds:

Theorem 6.4. *Let $X_k = \mathcal{H}_k\backslash\mathcal{G}_k$ be a symmetric variety defined over a locally compact field of characteristic not equal to two and assume that $\mathcal{H}$ is anisotropic over k. Let, furthermore, χ be a character of $\mathcal{H}_k$, with $\chi^2 = 1$. Then the following properties hold:*

(a) The group H_k is compact.

(b) Every class in the double coset space $\mathcal{H}_k\backslash\mathcal{G}_k\backslash\mathcal{H}_k$ can be represented by an element x in $\mathcal{G}_k$ such that $\sigma(x) = x^{-1}$.

(c) The pair $(\mathcal{G}_k, \mathcal{H}_k)$ *is a Gelfand pair of type* χ*.*

Proof. Part (a) is a general fact for the k-rational points of an anisotropic group defined over a locally compact field k.

As for (b) note that by [11, 6.7] the space $\mathcal{H}_k\backslash\mathcal{G}_k \simeq Q_k = \{g\sigma(g)^{-1} \mid g \in \mathcal{G}_k\}$ and $\mathcal{H}_k\backslash\mathcal{G}_k\backslash\mathcal{H}_k$ corresponds with the set of $\mathcal{H}_k$-conjugacy classes in Q_k, which clearly have a representative in Q_k.

The property 6.4 (b) is used to show that $L^1(\chi, (\mathcal{H}_k)_0\backslash\mathcal{G}_k)$ is commutative. For a function f on $\mathcal{G}_k$ we define the functions f^σ and $f^\vee$ by

$$f^\sigma(x) = f(\sigma(x)) \text{ and } f^\vee(x) = f(x^{-1}).$$

Since $\chi^2 = 1$ and the elements f of $L^1(\chi, (\mathcal{H}_k)_0\backslash\mathcal{G}_k)$ satisfy $f = \chi * f * \chi$ the maps $f \mapsto f^\sigma$ and $f \mapsto f^\vee$ are bijections of $L^1(\chi, (\mathcal{H}_k)_0\backslash\mathcal{G}_k)$. Because of property (b) they are equal. On the other hand $f \to f^\sigma$ is an algebra-isomorphism of $L^1(\chi, (\mathcal{H}_k)_0\backslash\mathcal{G}_k)$ for

$$\begin{aligned}(f_1 * f_2)^\sigma(x) &= \int f_1(\sigma(x)y)f_2(y^{-1})dy \\ &= \int f_1(\sigma(x)\sigma^2(y))f_2(\sigma^2(y)^{-1})dy \\ &= \int f_1^\sigma(x\sigma(y))f_2^\sigma(\sigma(y)^{-1})dy \\ &= \int f_1^\sigma(xt)f_2^\sigma(t^{-1})dt \\ &= f_1^\sigma * f_2^\sigma(x),\end{aligned}$$

but $f \to f^\vee$ on the contrary is an anti-algebra homomorphism:

$$\begin{aligned}(f_1 * f_2)^\vee(x) &= \int_{\mathcal{G}_k} f_1(x^{-1}y)f_2(y^{-1})dy \\ &= \int_{\mathcal{G}_k} f_1(x^{-1}t^{-1})f_2(t)dt \\ &= \int_{\mathcal{G}_k} f_2(tx^{-1})f_1(t^{-1})dt \\ &= \int_{\mathcal{G}_k} f_2^\vee(xt^{-1})f_1^\vee(t)dt \\ &= \int_{\mathcal{G}_k} f_2^\vee(xu)f_2^\vee(u^{-1})du \\ &= f_2^\vee * f_1^\vee(x).\end{aligned}$$

Hence we have for all f_1 and $f_2 \in L^1(\chi, (\mathcal{H}_k)_0\backslash\mathcal{G}_k)$ that $f_1^\vee * f_2^\vee = f_2^\vee * f_1^\vee$. In other words this convolution algebra is commutative. This concludes the proof.

□

Example 6.5. If Q is an anisotropic quadratic form over k, then O_Q is anisotropic. Take for χ the trivial character or the determinant of elements in $O_Q(k)$. Then the pair $(\mathrm{Gl}_n(k), O_Q(k))$ from example 2.2 is a Gelfand pair of type χ .

7. A general criterion

In the case of a non compact H, the convolution algebra $L^1(\check{e}_{11}, X_0)$ makes no sense and hence its commutativity is no longer a means to obtain multiplicity free decompositions. As a substitute of the trick used in theorem 6.4, one has the following useful criterion, which is the extension to the present setting of one by Thomas, see [24]

Theorem 7.1. *Let J be an anti-automorphism of $\mathcal{D}^1(e_{11}, X_0)$ such that $J\mathcal{H} = \mathcal{H}$ for all minimal G-invariant Hilbert subspaces $\mathcal{H}$ of $\mathcal{D}^1(e_{11}, X_0)$. Then the pair (G, H) is a generalized Gelfand pair of class ρ.*

Proof. From the integral decomposition in Theorem 3.2 it follows that as soon as each minimal subspace in $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ is invariant under J, the same holds for any $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$. Take any $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ and let $\mathcal{A}$ be the commutant in $\mathcal{L}(\mathcal{H})$ of the $\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}} \mid g \in G\}$. This is a von Neumann algebra.

Just as for linear operators one defines the antilinear automorphism J^* of $\mathcal{D}^1(e_{11}, X_0)$ by

$$\langle T, J^*\varphi \rangle = \langle J(T), \varphi \rangle.$$

Let $j : \mathcal{H} \to \mathcal{D}^1(e_{11}, X_0)$ be the embedding and $K = jj^*$ be the corresponding kernel. Since $J(\mathcal{H}) = \mathcal{H}$, this implies that the kernel K satisfies

$$JKJ^* = K.$$

As J is antilinear, this relation gives you that $J|_{\mathcal{H}}$ is anti-unitary and in particular $(J|_{\mathcal{H}}) = (J|_{\mathcal{H}})^*$. In the von Neumann algebra $\mathcal{A}$ the spectral components of a Hermitian $A_1 \in \mathcal{A}$, belong again to $\mathcal{A}$, see [6]. Since A_1 commutes with all the $\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}} \mid g \in G\}$, the corresponding kernel K_1 given by

$$\langle K_1(\varphi), \psi \rangle := \langle A_1(j^*(\varphi)), j^*(\psi) \rangle$$

is again G-invariant and corresponds to a G-invariant Hilbert subspace $\mathcal{H}_1 \hookrightarrow \mathcal{H}$. This subspace $\mathcal{H}_1$ satisfies again $J(\mathcal{H}_1) = \mathcal{H}_1$ and thus there holds

$$JK_1J^* = JjA_1j^*J^* = jA_1j^*.$$

This relation in its turn implies that

$$J|_{\mathcal{H}} A_1 J|_{\mathcal{H}}^{-1} = A_1 = A_1^*.$$

Since the orthogonal projections in $\mathcal{A}$ generate the algebra $\mathcal{A}$ and the map $J|_{\mathcal{H}}$ is antilinear, one sees that for all $A \in \mathcal{A}$ there holds

$$J|_{\mathcal{H}} A J|_{\mathcal{H}}^{-1} = A^*.$$

Applying this formula to the product of two operators A and B in $\mathcal{A}$ gives

$$A^*B^* = J|_{\mathcal{H}}AJ|_{\mathcal{H}}^{-1}J|_{\mathcal{H}}BJ|_{\mathcal{H}}^{-1} = (AB)^* = B^*A^*.$$

In other words the algebra $\mathcal{A}$ is commutative. This shows that the third characterization of a multiplicity-free decomposition holds. □

Example 7.2. Let G be commutative. Then one knows that for each $\varphi \in \mathcal{D}(e_{11}, X_0)$ the function $\check{\overline{\varphi}}$ belongs again to $\mathcal{D}(e_{11}, X_0)$ and the formula

$$J(T)(\varphi) = \overline{T(\check{\overline{\varphi}})}, \text{ for } T \in \mathcal{D}^1(e_{11}, X_0),$$

defines then an antilinear automorphism of $\mathcal{D}^1(e_{11}, X_0)$. Now the distribution T_ψ that correspond to a character ψ of G has the form

$$T_\psi(\varphi) = \int_G \overline{\varphi}(g)\psi(g^{-1})dg.$$

Those relevant for $\mathcal{D}^1(e_{11}, X_0)$ are the ones such that the restriction of ψ to H is equal to ρ. It is a direct verification that they are invariant under J and thus one sees once again that we have multiplicity one. For several of the classes of semi-direct products one can define similar J. This is left to the reader.

For real symmetric spaces this criterion was e.g. used to show that $(SL_n(\mathbb{R}), GL_{n-1}(\mathbb{R}))$ for $n \geq 3$ is a generalized Gelfand pair, see [27]. To illustrate its potential in another direction, one considers from now on the homogeneous spaces $X_k = \mathcal{H}_k \backslash \mathcal{G}_k$, with k a nonarchimedean local field. It will be shown that theorem 7.1 can be applied as soon as one has sufficiently many $\mathcal{H}_k$-invariant distribution vectors coming from the construction that is described next.

The general construction that renders $\mathcal{H}_k$-invariant distribution vectors starts from decent $\mathcal{H}_k$-invariant functions on X_k. Assume Y is a $\mathfrak{p}$-adic variety and $P : X_k \to Y$ is a submersion such that $P(hx) = P(x)$ for all $h \in H_k$ and $x \in X_k$. Let dy be a volume form on Y. According to [9] there is a surjective linear mapping M_P from $\mathcal{D}(X_k)$ to $\mathcal{D}(Y)$ such that

$$\int_{X_k} \varphi(x)\alpha(P(x))dx = \int_Y M_P(\varphi)(y)\alpha(y)dy \tag{38}$$

for all $\varphi \in \mathcal{D}(X_k)$ and all $\alpha \in \mathcal{D}(Y)$. Let $M_P^* : \mathcal{D}^1(Y) \to \mathcal{D}^1(X_k)$ be the dual mapping. Then we have

Lemma 7.3. *The elements of $M_P^*(\mathcal{D}^1(Y))$ are H_k-invariant distribution vectors.*

Proof. By definition we have $M_P^*(T) = T \circ M_P$ and $\mathcal{L}_{-\infty}(h)(M_P^*(T)) = T \circ M_P \circ \mathcal{L}_\infty(h^{-1})$. From the left $\mathcal{H}_k$-invariant of P we conclude for all $\alpha \in \mathcal{D}(Y)$ and $\varphi \in \mathcal{D}(X_k)$

$$\begin{aligned} &\int_{X_k} \varphi(x)\alpha(P(x))dx = \int_{X_k} \varphi(hx)\alpha(P(hx))dx \\ = &\int_{X_k} \varphi(hx)\alpha(P(x))dx = \int_Y M_P(\mathcal{L}_\infty(h^{-1})(\varphi))(y)\alpha(y)dy \\ = &\int_Y M_P(\varphi)(y)\alpha(y)dy. \end{aligned}$$

Hence $M_P \circ \mathcal{L}_\infty(h^{-1}) = M_P$ and this proves the desired property. □

Remark 7.4. In case that the analytic map P is only a submersion on a dense subset X_k^1 of X_k, then we can still obtain H_k-invariant distribution vectors by this construction, if the map M_P can be extended from $\mathcal{D}(X_k^1)$ to $\mathcal{D}(X_k)$ in such a way that formula (38) holds. The image of this extended map M_P can of course be wider than $\mathcal{D}(Y)$ in that situation. We mention a few examples.

Example 7.5. Take for $\mathcal{G}$ the group SL_n. Let J be the matrix

$$J = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Consider the involution $\sigma(g) = JgJ^{-1}$ of $\mathcal{G}$. Then one verifies directly that

$$\mathcal{H} = \left\{ \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{pmatrix} \in \mathrm{SL}_n \right\} = \left\{ \begin{pmatrix} \det(g)^{-1} & 0 \\ 0 & g \end{pmatrix} \mid g \in \mathrm{GL}_{n-1} \right\}$$

Let Z be the $\mathfrak{p}$-adic manifold given by

$$Z = \{x \in M_n(k) \mid \mathrm{rank}(x) = 1, \mathrm{trace}(x) = 1\}.$$

The group $\mathcal{G}_k$ acts on Z by conjugation. In Z we take the element

$$y_0 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then one verifies by direct computation that the stabilizer of y_0 is $\mathcal{H}_k$ and that $\mathcal{G}_k$ acts transitively on Z, in other words $\mathcal{H}_k \backslash \mathcal{G}_k \simeq Z$.

One defines an analytic map $P : X_k \to k$ by

$$P(x) = \mathrm{Tr}(xy_0).$$

This map satisfies first of all for all $h \in \mathcal{H}_k$

$$P(hxh^{-1}) = \mathrm{Tr}(hxh^{-1}y_0) = Tr(hxh^{-1}y_0hh^{-1} = \mathrm{Tr}(xy_0) = P(x),$$

since $hy_0h^{-1} = y_0$. A second property is that for all $g \in \mathcal{G}_k$

$$P(g^{-1}y_0g) = Tr(g^{-1}y_0gy_0) = Tr(gy_0g^{-1}y_0) = P(gy_0g^{-1}).$$

The map P is a submersion on a dense open submanifold and in [1] one can find the proof that the map M_P extends to $\mathcal{D}(X_k)$.

Example 7.6. Let J_n be the $2n \times 2n$-matrix given by

$$J_n = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

For $\mathcal{G}$ one takes the group

$$\mathrm{Sp}(n) = \{g \in \mathrm{GL}_{2n} \mid {}^t g J_n g = J_n\}.$$

As usual E_{ij} denotes the $2n \times 2n$-matrix with a 1 at the (i,j)-th entry and zeros elsewhere. If one defines $J = \mathrm{Id} - 2E_{nn} - 2E_{2n2n}$, then one has an involution σ of $\mathcal{G}$ defined by $\sigma(g) = JgJ$. A direct computation shows

$$\mathcal{H} = \left\{ \begin{pmatrix} & 0 & & 0 \\ g_1 & \vdots & g_2 & \vdots \\ & 0 & & 0 \\ 0\ \dots\ 0 & a & 0\ \dots\ 0 & b \\ & 0 & & 0 \\ g_3 & \vdots & g_4 & \vdots \\ & 0 & & 0 \\ 0\ \dots\ 0 & c & 0\ \dots\ 0 & d \end{pmatrix} \;\middle|\; \begin{array}{ll} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} & \in \mathrm{Sp}(n-1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \in \mathrm{Sp}(1) \end{array} \right\}.$$

The space X_k can be realized as

$$X_k = \{A \in M_{2n}(k) \mid J_n A J_n^{-1} = {}^t A, \mathrm{rank}(A) = \mathrm{trace}(A) = 2\}$$

on which $\mathcal{G}_k$ acts transitively by conjugation. Thus $X_k = \mathcal{G}_k \cdot y_0$ with $y_0 = E_{nn} + E_{2n2n}$.

As in the foregoing example one defines the analytic map $P : X_k \to k$ by

$$P(x) = \mathrm{Tr}(xy_0)$$

and one shows that it satisfies

$$P(hxh^{-1}) = P(x) \text{ and } P(g^{-1}y_0g) = P(gy_0g^{-1}).$$

Also this map has the required properties, see [1].

Let μ be an automorphism of $\mathcal{G}_k$ such that for all $h \in \mathcal{H}_k$ we have $\mu(h) = h$. Note that $\mu = Id$ is always an option but there can be others. In the case of the symmetric varieties e.g. one can also take $\mu = \sigma$. To μ is associated a natural anti-automorphism J_μ of $\mathcal{D}^1(X_k)$. The map $J_\mu : \mathcal{D}^1(X_k) \to \mathcal{D}^1(X_k)$ is defined by

$$\langle J_\mu(T), \varphi \rangle = \langle \overline{T, \overline{\varphi}^\mu} \rangle, \tag{39}$$

where $\varphi^\mu \in \mathcal{D}(X_k)$ is defined as $\varphi^\mu(x) = \varphi(\mu(x))$.

Assume now that one has an $\mathcal{H}_k$-invariant analytic map $P : X_k \to Y$ as in remark 7.4 that satisfies for all $g \in \mathcal{G}_k$

$$P(\mathcal{H}_k \mu(g)) = P(\mathcal{H}_k g^{-1}). \tag{40}$$

Then there holds

Proposition 7.7. *For all τ in the image of M_P^*, there holds $J_\mu(\tau) = \tau$.*

Proof. Let τ be $\tau = M_P^*(\xi) = \xi \circ M_P$ in $\mathcal{D}^1(X_k)^{\mathcal{H}_k}$. If $P_{\mathcal{H}_k} : \mathcal{D}(\mathcal{G}_k) \to \mathcal{D}(X_k)$ is the natural projection given by

$$P_{\mathcal{H}_k}(\varphi)(x) = \int_{\mathcal{H}_k} \varphi(hx)dh$$

Then one writes ξ for the element $\tau \circ P_{\mathcal{H}_k}$ in $\mathcal{D}^1(\mathcal{G}_k)$. Likewise one sees P as an analytic map: $\mathcal{G}_k \to Y$ and then formula (38) gives for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$

$$\int_{\mathcal{G}_k} \varphi(g)\alpha \circ P(g)dg = \int_Y M_P \circ P_{\mathcal{H}_k}(\varphi)(y)\alpha(y)dy.$$

By equation (40) the left hand side equals

$$\begin{aligned}
\int_{\mathcal{G}_k} \varphi(\mu(g))\alpha \circ P(\mu(g))dg &= \int_{\mathcal{G}_k} \varphi^\mu(g)\alpha \circ P(g^{-1})dg \\
&= \int_{\mathcal{G}_k} \overset{\vee}{\varphi}{}^\mu(g)\alpha \circ P(g)dg \\
&= \int_Y M_P \circ P_{\mathcal{H}_k}(\overset{\vee}{\varphi}{}^\mu)(y)\alpha(y)dy \\
&= \int_Y M_P \circ P_{\mathcal{H}_k}(\varphi)(y)\alpha(y)dy.
\end{aligned}$$

Since α is arbitrary, we may conclude $M_P \circ P_{\mathcal{H}_k}(\varphi) = M_P \circ P_{\mathcal{H}_k}(\overset{\vee}{\varphi}{}^{\mu})$. Thus one gets that for all φ in $\mathcal{D}(\mathcal{G}_k)$

$$\xi(\varphi) = \xi(\overset{\vee}{\varphi}{}^{\mu}).$$

Therefore it suffices to show that for all $\varphi \in \mathcal{D}(\mathcal{G}_k), \overline{\xi(\overline{\varphi})} = \xi(\overset{\vee}{\varphi})$ and this is a general property. For $\varphi \to \overline{\xi(\overline{\varphi})}$ is clearly a positive definite bi-$\mathcal{H}_k$-invariant distribution on $\mathcal{G}_k$. As, for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$

$$\overline{\widetilde{\varphi}} * \overline{\varphi}(x) = \widetilde{\varphi} * \varphi(x^{-1}),$$

we see that

$$\xi((\widetilde{\varphi} * \varphi)^{\vee}) = \xi(\overline{\widetilde{\varphi}} * \overline{\varphi}) = \overline{\xi(\overline{\widetilde{\varphi}} * \overline{\varphi})} \geq 0 \tag{41}$$

so that $\varphi \to \xi(\overset{\vee}{\varphi})$ is also a positive definite bi-$\mathcal{H}_k$-invariant distribution G_k. From (41) follows that for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$ and all $\psi \in \mathcal{D}(\mathcal{G}_k)$

$$\xi((\varphi * \psi)^{\vee}) = \overline{\xi(\overline{(\varphi * \psi)})}.$$

As the elements $\varphi * \psi$ span $\mathcal{D}(\mathcal{G}_k)$, one gets for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$, $\overline{\xi(\overline{\varphi})} = \xi(\overset{\vee}{\varphi})$. This is the desired result. □

For τ as in proposition 7.7, let $\mathcal{H}_\tau$ be the Hilbert subspace of $\mathcal{D}^1(X_k)$. From the J_μ-invariance of τ follows that $J_\mu \mathcal{H}_\tau = \mathcal{H}_\tau$.

Hence one may conclude now

Theorem 7.8. *Let* P_i, $i \in I$, *be analytic maps that satisfy first of all the requirements in remark 7.4 and equation* (40) *for the same* μ. *If the images of the* $M^*_{P_i}$ *span* $\mathcal{D}^1(X_k)^{\mathcal{H}_k}$, *then* $(\mathcal{G}_k, \mathcal{H}_k)$ *is a generalized Gelfand pair.*

Example 7.9. One returns to the examples 7.5 and 7.6. With the P's from these examples, Bosman proved following the procedure set out above that in both cases $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair for $n \geq 4$, see [1].

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