MULTIPLICITY ONE FOR REPRESENTATIONS CORRESPONDING TO SPHERICAL DISTRIBUTION VECTORS OF CLASS ρ

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ABSTRACT. In this paper one considers a unimodular second countable locally compact group G and the homogeneous space $X := H/G$, where H is a closed unimodular subgroup of G. Over X complex vector bundles are considered such that H acts on the fibers by a unitary representation ρ with closed image. The natural action of G on the space of square integrable sections is unitary and possesses an integral decomposition in so-called spherical distributions of class ρ. The uniqueness of this decomposition can be characterized by a number of equivalent properties. Uniqueness is shown to hold for a class of semidirect products. In the case that H is compact, the multiplicity free decomposition is shown to be equivalent with the commutativity of a suitable convolution algebra. As an example, one takes for X a symmetric k-variety $\mathcal{H}_k/\mathcal{G}_k$, with k a locally compact field of characteristic not equal to two. Here $\mathcal{G}$ is a reductive algebraic group defined over k and $\mathcal{H}$ is the fixed point group of an involution σ of $\mathcal{G}$ defined over k. It is shown then that the natural representation $\mathcal{L}$ of G_k on the Hilbert space $L^2(\mathcal{H}_k/\mathcal{G}_k)$ is multiplicity free if $\mathcal{H}$ is anisotropic. Next a criterion is derived that leads to multiplicity one also in the noncompact situation. Finally, in the nonarchimedean case, a general procedure is given that might lead to showing that a pair $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair. Here $\mathcal{G}$ and $\mathcal{H}$ are suitable algebraic groups defined over k.

1. INTRODUCTION

Let G be a locally compact group and H be a closed subgroup. Now one is interested in unitary representations of G related to the homogeneous space $X := H/G$. If both groups are unimodular, then inducing unitary representations ρ from H to G gives you an ample variety of examples of unitary representations $\mathcal{R}_\rho$ of G.

In the paper [HH02] we presented for a second countable group G and for irreducible representations ρ with closed image a decomposition of this induced representation in Hilbert subspaces of a certain space of distributions. The representations relevant for this decomposition were shown to be determined by an extension of the notion of spherical distribution, which leads to a description of the decomposition on the level of these distributions.

The present paper is concerned with the uniqueness of the decomposition presented in [HH02]. It can be characterized by a number of equivalent properties. They hold for certain classes of semi-direct products. In the case that H is compact, the multiplicity free decomposition is shown to be equivalent with the commutativity of a suitable convolution algebra. As an example, one takes for X a symmetric k-variety $\mathcal{H}_k/\mathcal{G}_k$, with k a locally compact field of characteristic not equal to two. Here $\mathcal{G}$ is a reductive algebraic group defined over k and $\mathcal{H}$ is the fixed point group of an involution σ of $\mathcal{G}$ defined over k. It is shown then that the

1991 *Mathematics Subject Classification.* 20G15, 20G20, 22E15, 22E46.

Key words and phrases. Spherical distributions, Multiplicity free decompositions, Symmetric varieties.

Partially supported by N.S.A. Grant MDA904-97-1-0092.

natural representation $\mathcal{L}$ of $\mathcal{G}_k$ on the Hilbert space $L^2(\mathcal{H}_k/\mathcal{G}_k)$ is multiplicity free as soon as $\mathcal{H}$ is anisotropic. Next a criterion is derived that leads to multiplicity one also in the noncompact situation. Finally a general procedure is given in the nonarchimedean case that might lead to showing that a pair $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair. Here $\mathcal{G}$ and $\mathcal{H}$ are suitable algebraic groups defined over k.

The precise content of the various subsections is as follows: the first subsection recalls the type of homogeneous spaces H/G we will work with, describes the class of geometric representations $\mathcal{R}_\rho$ that will be decomposed and the realization that we will use. The second subsection presents the decomposition of the representation, the uniqueness of which forms the topic of this paper. There one also finds the ingredients of the theory of Hilbert subspaces necessary to discuss the uniqueness. The Hilbert subspaces that are candidates for the components in the decomposition can be described in terms of certain distribution vectors or as a class of distributions on the group possessing certain invariance properties. Both descriptions are recalled in subsection 4. The properties characterizing multiplicity free decompositions can be found in subsection 5. The next section is devoted to the compact case. It is shown then that multiplicity one is equivalent to the commutativity of a specific convolution algebra. As an example the symmetric k-varieties $\mathcal{H}_k/\mathcal{G}_k$ with anisotropic $\mathcal{H}$ are shown to be multiplicity free. The final section contains first of all the criterion that yields multiplicity one also in noncompact situation. The paper finishes with showing how this criterion can be used for spaces $\mathcal{H}_k/\mathcal{G}_k$ with a nonarchimedean locally compact k.

2. The representations

In this paper all locally compact groups will without any further mentioning be assumed to be second countable in order to be able to apply the results from e.g. [Tho78]. Let G be a unimodular locally compact group and consider a closed unimodular subgroup H of G. On G resp. H we have Haar measures dg resp. dh. It is well-known then that the homogeneous space $X := H/G$ possesses a positive right G-invariant measure dx such that for all f in the space $C_c(G)$ of continuous functions on G with compact support

$$\int_G f(g)dg = \int_X \left\{ \int_H f(hx)dh \right\} dx. \tag{2.0.1}$$

A standard example of this setting is usually referred to as the *"group case"*.

Example 2.1. One starts out with a unimodular locally compact group G_1 and chooses G equal to the product group $G_1 \times G_1$. For H we take the diagonal subgroup

$$H = \{(g, g) | g \in G_1\}.$$

Clearly, both H and the variety X are isomorphic to G_1 and under this identification one can take for dx the Haar measure dg_1. The action of the group G on the space X consists then of the left and right translations from G_1.

A wide variety of examples of this situation is furnished by the following setting:

Example 2.2. Consider an affine algebraic group $\mathcal{G}$ defined over a locally compact field k. Then $G := \mathcal{G}_k$, the group of k-rational points of $\mathcal{G}$, is the candidate locally compact group, which is unimodular e.g. if $\mathcal{G}$ is reductive or unipotent. Usually one takes H equal to $\mathcal{H}_k$, where $\mathcal{H}$

is a suitable algebraic subgroup of $\mathcal{G}$ that is defined over k. In this situation it is customary to denote the homogeneous space as $X_k = \mathcal{H}_k/\mathcal{G}_k$. For the class of reductive algebraic groups defined over a field k of characteristic not equal to two, let $\sigma : G \to G$ be an involution of G and choose $\mathcal{H}$ equal to $\mathcal{G}_\sigma$, the group of fixed points under σ. According to [HW93], the group $\mathcal{H}$ is defined over k if and only if σ is defined over k. So, for the involutions defined over k we can consider the group $\mathcal{H}_k$. As the fixed point set of an automorphism of finite order of a reductive algebraic group is reductive, see [Ste68], the choice $H = \mathcal{H}_k$ gives you an unimodular subgroup. Analogous to the real situation, we call the variety X_k a symmetric k-variety.

A concrete example of this case is the following: consider a finite dimensional vector space V over k. Let Q be a non-degenerate quadratic form on V and let B be the associated symmetric non-degenerate bilinear form on V, i.e.

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)), \qquad \text{for } x \text{ and } y \in V.$$

For $g \in \mathrm{Gl}(V)$, we denote the adjoint of g w.r.t. B by g^T. As the involution σ of $\mathrm{Gl}(V)$ we take now $\sigma(g) = (g^T)^{-1}$. Then the group of fixed points of σ is the orthogonal group associated with Q, O_Q, and the space X_k maps into the non-degenerate matrices that are symmetric w.r.t. B by the mapping $gH_k \to g^T g$.

Another class of examples can be found inside the category of semi-direct products.

Example 2.3. Let G be the semi-direct product $L \ltimes M$ of the locally compact groups L and M. The action of an element $l \in L$ on M is denoted by $\alpha(l)$. If dl and dm are left invariant Haar measures on L resp. M, then $dldm$ is a left invariant Haar measure on G. From this one concludes that G is unimodular if and only if M is unimodular and $\Delta_L(l)=|\alpha(l)|$ for all $l \in L$. This last property holds e.g. if L is compact or equal to its commutator subgroup. As the subgroup H one chooses a subgroup of the form $L(0) \ltimes M(0)$ with $L(0)$ a closed subgroup of L that normalizes the closed subgroup $M(0)$ of M. It is unimodular under the same conditions. A concrete nontrivial example that will occur in the sequel is the Heisenberg group $A(\mathbf{G})$ associated to any abelian locally compact group $\mathbf{G}$, see [Wei64]. If S^1 denotes the unit circle in the complex plane and $< .|. >$ denotes the natural pairing between $\mathbf{G}$ and its dual $\hat{\mathbf{G}}$, then the group $A(\mathbf{G})$ is the topological space $\mathbf{G} \times \hat{\mathbf{G}} \times S^1$ with the multiplication

$$(g_1, d_1, t_1)(g_2, d_2, t_2) := (g_1 g_2, d_1 d_2, t_1 t_2 < d_1|g_2 >). \tag{2.3.1}$$

The group $A(\mathbf{G})$ is the semi-direct product of the groups $\mathbf{G}$ and $\hat{\mathbf{G}} \times S^1$, where these groups are viewed as embedded in $A(\mathbf{G})$ as

$$\{(g, 1, 1)|g \in \mathbf{G}\} \text{ resp. } \{(1, d, t)|\ d \in \hat{\mathbf{G}}, t \in S^1\}. \tag{2.3.2}$$

It is unimodular since it is two-step nilpotent. For each closed subgroup $\mathbf{H}$ of $\mathbf{G}$, the group of characters that are trivial on $\mathbf{H}$ is denoted by $\mathbf{H}^\perp$. The group $H = \mathbf{H} \times \mathbf{H}^\perp \times S^1$ is a maximal commutative subgroup of $A(\mathbf{G})$ and in particular it is unimodular.

Natural geometric objects related to the homogeneous spaces X are complex vector bundles $\mathcal{V}$ over them. Here only vector bundles are considered for which the structure group reduces to the unitary group. The representation of G on the square-integrable global sections of this bundle is then a natural unitary representation. Concretely this means that one has a unitary representation ρ of H on the finite dimensional space of fibers V_ρ. Global sections correspond then bijectively with functions $f : G \to V_\rho$ such that

$$f(hg) = \rho(h)(f(g)) \tag{2.3.3}$$

Consider now the space $L^2(\rho, X, dx)$ of classes of measurable $f : G \to V_\rho$ satisfying the condition (2.3.3) and

$$\int_X < f(x), f(x) >_\rho dx \ < \infty, \tag{2.3.4}$$

where $< . , . >_\rho$ denotes the inner product on V_ρ. On this space $L^2(\rho, X, dx)$ we have the inner product

$$< f, g >= \int_X < f(x), f(x) >_\rho dx. \tag{2.3.5}$$

The group G acts on this space by right translations and this representation we denote by $\mathcal{R}_\rho$. Thanks to the right G-invariance of dx, this representation is unitary.

Since the (unitary) representation ρ is completely reducible, it suffices for the decomposition of the space $L^2(\rho, X, dx)$ to consider irreducible ρ and this assumption we make from now on, unless otherwise stated.

Let H_0 be the kernel of ρ and let dh_0 be a Haar measure on H_0. The group H_0 is also unimodular, being a normal subgroup of an unimodular group. Hence also the homogeneous manifold $X_0 := H_0/G$ possesses a positive right G-invariant measure that is denoted by dx_0 and is related to dg and dh_0 by a formula like (2.0.1). We denote the inner product of φ_1 and φ_2 in $L^2(X_0, dx_0)$ by $< \varphi_1, \varphi_2 >_0$ and the action of G by right translations on $L^2(X_0, dx_0)$ by $\mathcal{R}$.

Like in [HH02] we assume throughout the rest of this paper the following :

Property 2.4. The group H/H_0 is compact.

Clearly this condition does not always hold, but it enables one to exploit the representation theory of the group H/H_0 and the condition was surely satisfied in the examples treated in [vDS00] and [Sha00].

Example 2.5. Let G be a semi-direct product like in example (2.3). Let (ρ_1, V_1) resp. (ρ_2, V_2) be finite dimensional irreducible unitary representations of $L(0)$ resp. $M(0)$ that satisfy property (2.4) and assume that $L(0)$ centralizes ρ_2, then $\rho = \rho_1 \otimes \rho_2$ is a well-defined irreducible unitary representation of $H = L(0) \rtimes M(0)$ that satisfies this property too.

In the case of the Heisenberg group $A(\mathbf{G})$ and the subgroup $H = \mathbf{H} \times \mathbf{H}^\perp \times S^1$ concrete examples of such representations are the $\rho_m((h, k, t)) = t^m, m \in \mathbb{Z}$, the case $m = 1, \mathbf{H} = \mathbf{G}$ being the *Schrödinger representation* of this group. Note that for $|m| = 1$ the groups $H = \mathbf{H} \times \mathbf{H}^\perp \times S^1$ are the centralizers in $A(\mathbf{G})$ of the character ρ_m. Hence, according to a theorem of Mackey, see [Mac68], the representations $\mathcal{R}_{\rho_m}$ for these m are irreducible. For $|m| > 1$, let U_m be the group of $|m|$-th roots of unity in the complex plane and let $\mathbf{H}_m$ be the subgroup of $\mathbf{G}$ defined by

$$\mathbf{H}_m = \{g \in G | < \tilde{h} | g > \in U_m \text{ for all } \tilde{h} \in \mathbf{H}^\perp\}.$$

Let $\mathbf{H}(m)$ be the quotient group $\mathbf{H}_m/\mathbf{H}$. Then for each character ψ in $\hat{\mathbf{H}}(m)$ one can define the one dimensional representation $\psi \otimes \rho_m$ of $\mathbf{H}_m \times \mathbf{H}^\perp \times S^1$ by

$$\psi \otimes \rho_m((h_m, k, t)) = \psi(h_m)t^m,\ h_m \in \mathbf{H}_m,\ k \in \mathbf{H}^\perp \text{ and } t \in S^1.$$

They are exactly the components of the representation obtained by inducing ρ_m from H to $\mathbf{H}_m \times \mathbf{H}^\perp \times S^1$. Inducing the $\psi \otimes \rho_m$ to the group $A(\mathbf{G})$ gives you irreducible representations of $A(\mathbf{G})$ that are inequivalent for different ψ, see [Hel]. Thus the representation $\mathcal{R}_{\rho_m}$ decomposes into irreducible representations that each occur only once. This phenomenon occurs for a wide class of semi-direct products as we will see.

We have a closer look at the extreme cases i.e. $L = L(0)$ and $M = M(0)$, where one can realize the representation as functions on one of the components. If $L = L(0)$, L is unimodular and $|\alpha(l)| = 1$ for all $l \in L$, then the right M-invariant measure $d\tilde{m}$ on $X = M(0)/M$ is a right G-invariant measure on X. With this choice, restricting to M gives you an isomorphism between $L^2(\rho, X, dx)$ and $L^2(1 \otimes \rho_2, M(0)/M, d\tilde{m})$. The action of M on this last space is simply by right translations and that of $l \in L$ is given by

$$\mathcal{R}_\rho(l)(f)(m) = \rho_1(l) \otimes 1(f(\alpha(l^{-1}(m)))) \tag{2.5.1}$$

Assume now that $M = M(0)$ and that L and $L(0)$ are unimodular. The right L-invariant measure $d\tilde{l}$ on $X = L(0)/L$ is also right G-invariant. Like in the other extreme case, taking the restriction to L gives an isomorphism between $L^2(\rho, X, dx)$ and $L^2(\rho_1 \otimes 1, L(0)/L, d\tilde{l})$. On the last space L acts by right translation and the group M by

$$\mathcal{R}_\rho(m)(f)(l) = 1 \otimes \rho_2(\alpha(l)(m))(f(l)). \tag{2.5.2}$$

Thus, the representation of $A(\mathbf{G})$ obtained by inducing ρ_m from $H = \mathbf{G}^\perp \times S^1$ can be realized on $L^2(\mathbf{G})$ and that from $H = \mathbf{G} \times S^1$ on $L^2(\mathbf{G}^\perp)$.

For the general case we will use another realization of $\mathcal{R}_\rho$ that we will briefly describe next.

The group H acts by left translations on the space X_0 and by transposition on the functions on X_0. Since dg is also left G-invariant, one sees from relation (2.0.1) that

$$\mathcal{L}(h)(f)(x) := f(h^{-1}x), \tag{2.5.3}$$

defines a unitary representation of H on $L^2(X_0, dx_0)$. It clearly factorizes over H/H_0. Let $d\tilde{h}$ denote the normalized Haar measure on H/H_0. Then we have an algebra morphism from the convolution algebra of continuous functions on H/H_0 to the bounded linear operators on $L^2(X_0, dx_0)$. It is defined by

$$\mathcal{L}(\varphi)(f)(x_0) = \int_{H/H_0} \varphi(\tilde{h})\mathcal{L}(\tilde{h})(f)(x_0)d\tilde{h} =: \varphi * f(x_0), \tag{2.5.4}$$

with φ continuous on H/H_0.

For $u, v \in V_\rho$, let $e_{v,u}$ be the matrix coefficient of the representation ρ of H/H_0 given by

$$e_{v,u}(h) = d_\rho < \rho(h)(u), v >_\rho, \tag{2.5.5}$$

where $d_\rho = \dim(V_\rho)$. Let $\{f_i \mid 1 \le i \le d_\rho\}$ be an orthonormal basis of the space V_ρ. For simplicity, we denote for each i and j the function e_{f_i, f_j} by e_{ij}. Inside $L^2(X_0, dx_0)$ consider the following G-invariant closed subspace

$$L^2(e_{11}, X_0, dx_0) = \{\varphi \mid \varphi \in L^2(X_0, dx_0), e_{11} * \varphi := \mathcal{L}(e_{11})(\varphi) = \varphi\}. \tag{2.5.6}$$

By using the orthogonality relations for the e_{ij} one shows that for each $\varphi \in L^2(e_{11}, X_0, dx_0)$ the function

$$A(\varphi) := \frac{1}{\sqrt{d_\rho}} \sum_{j=1}^{d_\rho} \left(e_{j1} * \varphi\right) f_j \tag{2.5.7}$$

satisfies (2.3.3). If the measures dx and dx_0 are chosen such that $d\tilde{h}dx = dx_0$, then one can show that the map $A : L^2(e_{11}, X_0, dx_0) \to L^2(\rho, X, dx)$ is a norm preserving bijection that commutes with the right G-action on both spaces. Therefore we will work from now on with $(\mathcal{R}, L^2(e_{11}, X_0, dx_0))$ instead of $(\mathcal{R}_\rho, L^2(\rho, X, dx))$.

3. Hilbert subspaces of distributions

Recall that Bruhat, see [Bru61], has introduced for each locally compact group G_1 and each homogeneous space F/G_1, where F is a closed subgroup of G_1, the spaces of test functions $\mathcal{D}(G_1)$ and $\mathcal{D}(F/G_1)$ with an appropriate topology. It unifies the cases that G_1 is a Lie group, where it equals the space of C^∞-functions with compact support, and that of totally disconnected spaces, in which case it consists of the locally constant functions with compact support. Therefore the notations $C_c^\infty(G_1)$ respectively $C_c^\infty(F/G_1)$ are also common in this last setting. The elements of their continuous antilinear duals are called distributions on G_1 resp. F/G_1 and these spaces are denoted by $\mathcal{D}^1(G_1)$ and $\mathcal{D}^1(F/G_1)$.

The group G acts on $\mathcal{D}(X_0)$ by right translation and it leaves the subspace

$$\mathcal{D}(e_{11}, X_0) = \{\phi \in \mathcal{D}(X_0) \mid e_{11} * \phi = \phi\}. \tag{3.0.8}$$

invariant. By transposing this representation $\mathcal{R}_\infty$ of G on $\mathcal{D}(X_0)$ one arrives at the representation $\mathcal{R}_{-\infty}$ of G on $\mathcal{D}^1(X_0)$, i.e. for $T \in \mathcal{D}^1(X_0)$

$$\mathcal{R}_{-\infty}(g)(T)(\varphi) = T(\mathcal{R}_\infty(g^{-1})\varphi).$$

Likewise one can dualize the left H-action on $\mathcal{D}(X_0)$ to a representation $\mathcal{L}_{-\infty}$ of H on $\mathcal{D}^1(X_0)$ and one verifies directly that the antilinear dual of the subspace $\mathcal{D}(e_{11}, X_0)$ can be identified with

$$\mathcal{D}^1(e_{11}, X_0) = \{T \in \mathcal{D}^1(X_0), \int_{H/H_0} e_{11}(\tilde{h})\mathcal{L}_{-\infty}(\tilde{h})(T)d\tilde{h} = T\}. \tag{3.0.9}$$

Hence, if we take an $f \in L^2(e_{11}, X_0, dx_0)$ and consider the distribution $T = f(x)dx$ on X_0, then it belongs to $\mathcal{D}^1(e_{11}, X_0)$ and

$$\mathcal{R}_{-\infty}(g)(f(x)dx) = f(xg)dx.$$

In other words the embedding $j : f(x) \mapsto f(x)dx$ of $L^2(e_{11}, X_0, dx_0)$ into $\mathcal{D}^1(e_{11}, X_0)$ is a G-morphism. Therefore $L^2(e_{11}, X_0, dx_0)$ is a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. Let $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ be the collection of G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$. It is well-known, see [Sch64a], that the Hilbert subspaces of $\mathcal{D}^1(X_0)$ are completely determined by their reproducing kernel $jj^* : \mathcal{D}(X_0) \to \mathcal{D}^1(X_0)$, where j^* is the adjoint map of the embedding $j : \mathcal{H} \hookrightarrow \mathcal{D}^1(X_0)$. From the Schwartz kernel theorem [Sch64b] one sees that each such a Hilbert subspace $\mathcal{H}$ of $\mathcal{D}^1(X_0)$ corresponds bijectively to a distribution $K \in \mathcal{D}^1(X_0 \times X_0)$ defined by

$$K(\varphi \otimes \overline{\psi}) =< jj^*(\varphi), \psi >= (j^*\varphi, j^*\psi)_{\mathcal{H}}. \tag{3.0.10}$$

Here $(\cdot,\cdot)_{\mathcal{H}}$ is the inner product on $\mathcal{H}$ and $< \cdot, \cdot >$ is the natural pairing between $\mathcal{D}(X_0)$ and its antilinear dual. From this relation one sees directly that K is a distribution of positive type

$$K(\varphi \otimes \overline{\varphi}) = (j^*(\varphi) \mid j^*(\varphi))_{\mathcal{H}} \geq 0.$$

The G-invariance of the corresponding Hilbert subspace translates into

$$K(\mathcal{R}(g)\varphi \otimes \mathcal{R}(g)\psi) = K(\varphi \otimes \psi).$$

Finally, one has to require of the distribution K that it renders you a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, not just of $\mathcal{D}^1(X_0)$. Therefore it has to satisfy still the following relation

$$K(e_{11} * \varphi \otimes \overline{e_{11} * \psi}) = K(\varphi \otimes \overline{\psi}). \tag{3.0.11}$$

Let $\mathcal{F}$ be the space of all G-invariant distributions in $\mathcal{D}^1(X_0 \times X_0)$ that satisfy property (3.0.11) and are hermitian, i.e. for all φ and $\psi \in \mathcal{D}(X_0)$ there holds

$$K(\varphi \otimes \overline{\psi}) = \overline{K(\psi \otimes \overline{\varphi})}.$$

Then $\mathcal{F}$ is a real vector subspace and we denote the subset of those of positive type by Γ_G. It corresponds bijectively to $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$. In particular, for each $K \in \Gamma_G$ that corresponds to $H_K \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ and each $\alpha \geq 0$, the distribution αK corresponds to the space αH_K consisting of the space H_K with the inner product

$$(h_1, h_2)_{H_K,\alpha} = \frac{1}{\alpha}(h_1, h_2)_{H_K}.$$

the subset Γ_G forms a convex cone in $\mathcal{F}$ that is used to put a partial order on $\mathcal{F}$. It is defined by

$$K_1 \leq K_2 \Leftrightarrow K_1 - K_2 \in \Gamma_G. \tag{3.0.12}$$

Let $\mathrm{ext}(\Gamma_G)$ be the set of extremal rays of Γ_G. Those are the distributions K that satisfy

$$0 \leq K^1 \leq K, K^1 \in \Gamma_G \Rightarrow K^1 = \alpha K. \tag{3.0.13}$$

The relevance of $\mathrm{ext}(\Gamma_G)$ follows from

Theorem 3.1. *Let $(\pi, \mathcal{H}_\pi)$ be a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$ and let $K_\pi \in \Gamma_G$ be the corresponding distribution. Then there holds*

$$(\pi, \mathcal{H}_\pi) \text{ is irreducible } \Leftrightarrow K_\pi \text{ is extremal}$$

A proof of this theorem can be found in [Kla79] or in [Tho79]. Since the group G is second countable, we know that $\mathcal{D}^1(e_{11}, X_0)$ is the dual of a nuclear barrelled space. Hence, according to [Tho84], there exists a Hausdorff topological space S and an admissible parametrization of $\mathrm{ext}(\Gamma_G)$, $s \to K_s$, independent of $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$, such that, if $\mathcal{H}_s$ is the Hilbert space corresponding to K_s, then there holds

Theorem 3.2. *For every $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ there exists a Radon measure m on S such that*

$$\mathcal{H} = \int_S^{\oplus} \mathcal{H}_s dm(s). \tag{3.2.1}$$

In particular for the Hilbert subspace $L^2(e_{11}, X_0, dx_0)$ this theorem gives you a decomposition of $L^2(e_{11}, X_0, dx_0)$ in minimal unitary G-models. In view of Theorem 3.2 it is important to have an idea of which representations can be realized as a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. In the real case, these are the representations that possess a non-zero cyclic H-invariant distribution vector. A similar notion will be introduced in the present setting together with a useful characterization.

4. C^∞-vectors and distribution vectors

Let $(\pi, \mathcal{H}_\pi)$ be a continuous representation of G on the Hilbert space $\mathcal{H}_\pi$. In [HH02], we introduced the space $\mathcal{H}_\pi^\infty$ of C^∞-vectors of $\mathcal{H}_\pi$ as

$$\mathcal{H}_\pi^\infty = \varinjlim_{\mathbb{K}_n} \mathcal{H}_\pi(\mathbb{K}_n) \tag{4.0.2}$$

where $\{\mathbb{K}_n\}$ is a sequence of compact normal subgroups in an open Yamabe subgroup of G, that converges to the identity, and, where the spaces $\mathcal{H}_\pi(\mathbb{K}_n)$ are given by

$$\mathcal{H}_\pi(\mathbb{K}_n) = \{v \in \mathcal{H}_\pi \mid \pi(\mathbb{K}_n)v = v \text{ and } g \to \pi(g)v \in C^\infty(G/\mathbb{K}_n, \mathcal{H}_\pi)\},$$

It has a natural Fréchet topology and the natural embedding $\mathcal{H}_\pi^\infty \to \mathcal{H}_\pi$ is continuous and has a dense image. For more details we refer to [HH02].

Example 4.1. According to example 2.5 the Schrödinger representation of the group $A(\mathbf{G})$ can be realized on $L^2(\mathbf{G})$. It is shown in [Hel] that the space of C^∞-vectors of this representation is exactly the space $\mathcal{S}(\mathbf{G})$ of Schwarz-Bruhat functions on $\mathbf{G}$.

The topological antilinear dual of $\mathcal{H}_\pi^\infty$ is called the space of *distribution vectors* of $(\pi, \mathcal{H}_\pi)$ and is denoted by $\mathcal{H}_\pi^{-\infty}$. Since $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$, we get a continuous embedding $\mathcal{H}_\pi \hookrightarrow \mathcal{H}_\pi^{-\infty}$. The space $\mathcal{H}_\pi^\infty$ is G-invariant and the restriction of π to $\mathcal{H}_\pi^\infty$ is also denoted by π_∞.

By transposition we have a representation $\pi_{-\infty}$ of G on $\mathcal{H}_\pi^{-\infty}$, i.e.

$$\langle \pi_{-\infty}(g)T, v\rangle = \langle T, \pi_\infty(g^{-1})v\rangle \tag{4.1.1}$$

Examples of C^∞-vectors are easily obtained. For each $\varphi \in \mathcal{D}(G)$ and each $v \in \mathcal{H}_\pi$ the vector $\pi(\varphi)v$ belongs to $\mathcal{H}_\pi^\infty$. Moreover the space $G(\mathcal{H}_\pi)$ spanned by all these vectors is dense in $\mathcal{H}_\pi^\infty$. It is called the *Gårding space* of $(\pi, \mathcal{H}_\pi)$.

The foregoing fact enables you to define for each $\varphi \in \mathcal{D}(G)$ and each $T \in \mathcal{H}_\pi^{-\infty}$ the distribution vector $\pi_{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^{-\infty}$ by

$$\begin{aligned} \langle \pi_{-\infty}(\varphi)(T), v\rangle &= \int_{G_k} \varphi(g)\langle \pi_{-\infty}(g)T, v\rangle dg \\ &= \langle T, \pi_\infty(\overset{\vee}{\varphi_0})(v)\rangle. \end{aligned} \tag{4.1.2}$$

for all $v \in \mathcal{H}_\pi^\infty$. Here $\check{\varphi}$ is defined by $\check{\varphi}(g) = \varphi(g^{-1})$. As in the Lie group case there holds

Lemma 4.2. *The distribution vector* $\pi_{-\infty}(\varphi)(T)$ *for* $\varphi \in \mathcal{D}(G)$ *and* $T \in \mathcal{H}_\pi^{-\infty}$ *belongs to* $\mathcal{H}_\pi^{\infty}$.

Hence each $T \in \mathcal{H}_\pi^{-\infty}$ defines a linear map $A_T : \mathcal{D}(G) \to \mathcal{H}_\pi$ by $A_T(\varphi) = \pi_{-\infty}(\check{\varphi})(T)$. By reduction to the Lie group case one shows that it is continuous. With respect to left translations on $\mathcal{D}(G)$ the map A_T behaves as follows

$$A_T(\varepsilon_g * \varphi) = \pi_{-\infty}(\check{\varphi})\pi_{-\infty}(g^{-1})T, \tag{4.2.1}$$

for all $g \in G$. The map A_T also intertwines the action of G by right translation on $\mathcal{D}(G)$ and by the representation π on $\mathcal{H}_\pi$, i.e. for all $g \in G$ and all $\varphi \in \mathcal{D}(G)$

$$A_T(\varphi * \varepsilon_{g^{-1}}) = \pi(g)(A_T(\varphi)). \tag{4.2.2}$$

All continuous maps from $\mathcal{D}(G)$ to $\mathcal{H}_\pi$ with the property (4.2.2) have this form, for there holds analogous to the Lie group case:

Theorem 4.3. *Let* $A : \mathcal{D}(G) \to \mathcal{H}_\pi$ *be a continuous map that satisfies for all* $g \in G$ *and all* $\varphi \in \mathcal{D}(G)$, $A(\varphi * \varepsilon_{g^{-1}}) = \pi(g)(A_T(\varphi))$. *Then there is a unique distribution vector* $T \in \mathcal{H}_\pi^{-\infty}$ *such that* $A = A_T$.

Now that we have the action of G on $\mathcal{H}_\pi^{-\infty}$ we define

$$(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \left\{ T \in \mathcal{H}_\pi^{-\infty} \;\middle|\; \begin{array}{l} \pi_{-\infty}(h)T = T \text{ for all } h \in H_0, \\ \pi_{-\infty}(\check{e}_{11})T = T \end{array} \right\}.$$

Note that, if H is compact, then $\pi(e_{11})$ is a well-defined orthogonal projection of the space $\mathcal{H}_\pi$ and the conditions on a $T \in (\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ simply mean that it factorizes over $\pi(\check{e}_{11})$. Hence

Lemma 4.4. *For compact* H, *we have* $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \pi(\check{e}_{11})(\mathcal{H}_\pi)^{-\infty}$.

Clearly in the noncompact case the operator $\pi(\check{e}_{11})$ can not be given a sense and that is why one has to proceed more carefully. Before coming to the characterization of $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ in terms of distribution vectors we introduce still

Definition 4.5. A distribution vector T in $\mathcal{H}_\pi^{-\infty}$ is called *cyclic* if the space

$$\{\pi_{-\infty}(\varphi)(T) \mid \varphi \in \mathcal{D}(G)\}$$

is lying dense in $\mathcal{H}_\pi$.

With the help of this notion, one can see from the space $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ if a unitary representation is a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, for there holds

Theorem 4.6. *Let* $(\pi, \mathcal{H}_\pi)$ *be a unitary representation of* G. *Then the set of non-zero cyclic elements of* $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ *is in bijective correspondence with the continuous* G*-equivariant embeddings* $j : \mathcal{H}_\pi \hookrightarrow \mathcal{D}^1(e_{11}, X_0)$.

For a proof we refer to [HH02]. We will call the nonzero cyclic elements of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ the (ρ, H)-*spherical distribution vectors* of $(\pi, \mathcal{H}_\pi)$ or spherical distribution vectors of class ρ.

If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G and $j : \mathcal{H}_\pi \to \mathcal{D}^1(e_{11}, X_0)$ a continuous G-equivariant embedding, then we denote the to j corresponding non-zero cyclic element of

$(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ by T. As in the real case we can associate with T a special distribution σ_T on G. For $\varphi \in \mathcal{D}(G)$ we know from Lemma 4.2 that $\pi_{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^\infty$ and then we define $\sigma_T \in \mathcal{D}^1(G)$ by

$$\langle \sigma_T, \varphi \rangle = \langle T, \pi_{-\infty}(\varphi)(T) \rangle.$$

Remark 4.7. If H is compact and T corresponds to a cyclic vector $v \in \pi(\check{e_{11}})(\mathcal{H}_\pi)$, then the distribution σ_T equals

$$\langle \sigma_T, \varphi \rangle = \int_G \overline{\varphi(g)}(v, \pi(g)(v))_\pi dg.$$

In other words, we have $\sigma_T = (v, \pi(g)(v))_\pi dg$. Following the terminology of the invariant context, this last function is called *the spherical function of class* ρ of the representation.

Now the distribution σ_T on G is nonzero, positive definite and bi-H_0-invariant. Moreover it satisfies

$$\check{e_{11}} * \sigma_T = \sigma_T * \check{e_{11}} = \sigma_T. \tag{4.7.1}$$

Reversely, let σ be such a distribution on G. Then it is shown in [HH02] that σ determines a G-invariant Hilbert subspace $\mathcal{H}_\sigma$ of $\mathcal{D}^1(e_{11}, X_0)$. We call the class of positive definite bi-H_0-invariant distributions σ on G, satisfying equation (4.7.1) that of (ρ, H)-*spherical distributions or spherical distributions of class* ρ. The foregoing can be summarized as follows

Theorem 4.8. *The map $\sigma \mapsto \mathcal{H}_\sigma$ that associates with each (ρ, H)-spherical distribution the unitary G-module $\mathcal{H}_\sigma$, is a bijection between this class of distributions on G and the collection of G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$.*

Example 4.9. Our main interest is in the Hilbert subspace $L^2(e_{11}, X_0, dx_0)$ of $\mathcal{D}^1(e_{11}, X_0)$. The positive definite bi-H_0-invariant distribution in this case is

$$\tau_0(\varphi) = e_{11} * P_{H_0}(\overline{\varphi})(e) = \{\int_{H/H_0} e_{11}(\tilde{t})\{\int_{H_0} \overline{\varphi}(\tilde{t}^{-1}h_0)dh_0\}d\tilde{t}\}, \tag{4.9.1}$$

where e is the point H_0 of X_0. If we combine the theorems 3.2 and 4.6, then we get a decomposition

$$\tau_0 = \int_S^\oplus \sigma_s dm(s), \tag{4.9.2}$$

where the σ_s are the (ρ, H)-spherical distributions corresponding to the irreducible G-modules $\mathcal{H}_s$, $s \in S$.

5. Multiplicity one

In this section we want to discuss the uniqueness of the integral decomposition in Theorem 3.2. Hereby we do not want to make a distinction between

$$H = \int^\oplus H_s d\mu(s) \text{ and } H = \int^\oplus (\alpha H_s)\frac{d\mu(s)}{\alpha}, \text{ for } \alpha > 0.$$

So, given an admissible parametrization S of $\mathrm{ext}(\Gamma_G)$, we call an integral decomposition of a $H \in \Gamma$ in extremal generators unique, if for 2 Radon measures μ_1 and μ_2 on S there holds

$$H = \int^{\oplus} H_s d\mu_1(s) = \int^{\oplus} H_s d\mu_2(s) \to \quad \mu_1 = \mu_2.$$

Thomas has given a number of equivalent criteria for multiplicity-free decomposition in the case of the trivial representation ρ, see [Tho84]. They extend to the present context

Theorem 5.1. *Let G, H and ρ be as in section 2 and let S be the Hausdorff topological space that gives an admissible parameterization of* $\mathrm{ext}(\Gamma_G)$ *as in theorem 3.2. We have the following equivalent properties*

(1) *If H_1 and H_2 are minimal G-invariant Hilbert subspaces of $\mathcal{D}(e_{11}, X_0)$, which differ as linear subspaces, then the irreducible unitary representations of G on H_1 and H_2 are inequivalent.*
(2) *If $(\pi, \mathcal{H}_\pi)$ is any G-invariant Hilbert subspace of $\mathcal{D}(e_{11}, X_0)$, then the commutant of $\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}_\pi} \mid g \in G\}$ is abelian.*
(3) *The convex cone Γ_G of all G-invariant positive kernels on $X_0 \times X_0$ that satisfy the condition (3.0.11) is a lattice cone, i.e. for each pair of points (γ_1, γ_2) on the cone there exists a least upper bound and a highest lower bound on the cone.*
(4) *For each $K \in \Gamma_G$ there exists a unique Radon measure μ on S such that*

$$K = \int K_s d\mu(s).$$

(5) *For every G-invariant Hilbert subspace H of $\mathcal{D}(e_{11}, X_0)$ there exists a unique Radon measure μ on S such that*

$$H = \int^{\oplus} H_s d\mu(s).$$

Proof. To show that property 1 implies 2, consider a maximal commutative van Neumann algebra A_1 in the commutant C of the $\{\mathcal{R}_{-\infty}(g)|H,\ g \in G\}$. If we can show that all the unitary operators in C belong to A_1, then $C = A_1$, since the unitary operators in a van Neumann algebra span the algebra. Let U be a unitary operator in C, then $A_2 = UA_1U^{-1}$ is another maximal commutative van Neumann algebra with the same spectrum S. Let μ be a positive basic measure on S, then the space H decomposes in two ways, corresponding to the diagonalization of A_1 resp. A_2

$$H = \int_S^{\oplus} H_s(1) d\mu(s) = \int_S^{\oplus} H_s(2) d\mu(s).$$

Since the operators from C commute with the $\{\mathcal{R}_{-\infty}(g)|H,\ g \in G\}$, it follows that the spectral measures $\mu_i(x, y) = k_i(x, y, s) d\mu(s)$, x and $y \in H$, i=1,2, satisfy

(5.1.1) $$k_i(\mathcal{R}_{-\infty}(g)(x), \mathcal{R}_{-\infty}(g)(y), s) = k_i(x, y, s) \text{ for almost all } s.$$

Now the functions k_i are used to build the inner product on the spaces $H_s(i)$, so that almost all the $H_s(i)$ are G-invariant. From the construction one sees moreover that they are Hilbert subspaces of $\mathcal{D}(e_{11}, X_0)$. Since both A_1 and A_2 are maximal commutative in C, there follows from Mautner's theorem, see [Mau68], that almost all the $H_s(i)$ are irreducible G-modules.

From $UT_2(f)U^{-1} = T_1(f)$ for all $f \in L^{\infty}(S, d\mu)$, follows that U is a decomposable operator, see [Dix96], i.e.

$$U = \int_S^{\oplus} U_s, \text{ with } U_s \in End(H_s(1), H_s(2)).$$

From the fact that for almost all s

$$k_2(x, y, s) = k_1(U^{-1}(x), U^{-1}(y), s), \tag{5.1.2}$$

one sees that almost all U_s are unitary. By combining the relations 5.1.1 and 5.1.2 one gets that almost all the U_s also intertwine the G-actions on $H_s(1)$ and $H_s(2)$. Due to property 1 one may conclude now that for almost all $s \in S$ the spaces $H_s(1)$ and $H_s(2)$ are a multiple of each other and thus the U_s a multiple of the identity, thanks to Schur's lemma. This implies that U belongs to A_1 and hence C is commutative.

The equivalence of the properties 2 and 3 holds more generally for a group G of automorphisms of a quasi-complete locally convex space E and can be found in [Tho79]. There and in [Tho78] one finds that the properties 3 and 4 are the same as soon as your space E is the strong dual of a barrelled nuclear space. In particular, it holds for $\mathcal{D}(e_{11}, X_0)$. The implication 4 to 5 being immediate, one is left to show that 5 implies 1. This argument holds again in fair generality and can be found in [Kla79] □

Definition 5.2. The space $\mathcal{D}(e_{11}, X_0)$ is said to decompose *multiplicity free* if one of the equivalent properties from theorem (5.1) holds. It is also customary to say that *multiplicity one* holds for this space or the triple (G, H, ρ).

Example 5.3. The first example is that of a commutative G. For example criterion (1) of Theorem 5.1 is then clearly fulfilled, since the minimal G-invariant subspaces are characters of G. As soon as one takes G not necessarily commutative, then multiplicity free decompositions are not the rule as the example of the left regular representation of a finite nonabelian group shows.

Example 5.4. Consider the semi-direct product $G = L \rtimes M$ like in example (2.3) with the subgroup $H = L \rtimes M(0)$. Let (ρ_1, V_1) resp. (ρ_2, V_2) be finite dimensional irreducible unitary representations of L resp. $M(0)$ as in example 5.4. Since M acts by right translations on $L^2(1 \otimes \rho_2, M(0)/M, d\tilde{m})$, this space decomposes w.r.t. the M-action as the direct sum of dim(V_1) copies of $L^2(\rho_2, M(0)/M, d\tilde{m})$. If C_2 denotes the commutant of the M-action on $L^2(\rho_2, M(0)/M, d\tilde{m})$, then the commutant of the M-action on $L^2(1 \otimes \rho_2, M(0)/M, d\tilde{m})$ is $End(V_1) \otimes C_2$. To get the commutant for the G-action the operators in $End(V_1) \otimes C_2$ still have to commute with the L-action from (2.5.1). Since ρ_1 is irreducible, one can conclude that the G-commutant is abelian as soon as C_2 is. Thanks to the second criterion in theorem 5.1, one knows that multiplicity one holds for this triple (G, H, ρ). Thanks to the foregoing example this holds for sure if M is abelian. This is an extension of the result that was observed already for the Heisenberg group. In this last case one concludes from the Stone-van Neumann theorem, see e.g. [Hel], that the Schrödinger representation is unitarily equivalent with the representation obtained by inducing ρ_1 from any $H = \mathbf{H} \times \mathbf{H}^{\perp} \times S^1$. This implies in particular that up to a constant the Schrödinger representation has a unique $\mathbf{H} \times \mathbf{H}^{\perp}$-fixed distribution vector and it is given on $\mathcal{S}$, its space of C^∞-vectors, by

$$\varphi \to \int_H \overline{\varphi(h)} dh. \tag{5.4.1}$$

If one takes for $\mathbf{G} = \mathbb{R}^n$ and $\mathbf{H} = \mathbb{Z}^n$ a realization of the Schrödinger representation on a space of entire functions on $\mathbb{C}^n$, the so called Fock representation, then this distribution vector corresponds with a theta function and its uniqueness reflects characterizing transformation properties of this function. For details, we refer to [Hel]. This distribution vector gives also rise to an important class of automorphic forms on the metaplectic group , see [Wei64].

Likewise one can consider the other extreme case, namely $M = M(0)$. The L-action on $L^2(\rho_1 \otimes 1, L(0)/L, d\tilde{l})$ is by right translations and therefore this space decomposes in dim(V_2) copies of $L^2(\rho_1, L(0)/L, d\tilde{l})$. If C_1 denotes the commutant of the L-action in this last space, then the G-commutant consists again of the operators in $C_1 \otimes End(V_2)$ that commute with the M-action from 2.5.2. In particular if C_2 is commutative, then we have again reduced multiplicity one for (G, H, ρ) to that of $(L, L(0), \rho_1)$. In view of example 5.3, this holds for sure, if L is abelian.

Example 5.5. The third example of a multiplicity free decomposition is the group case. Here one proves readily the criterion (2) of Theorem 5.1. For in that case $X = G_1$ and $G_1 \times G_1$ acts on $\mathcal{D}^1(G_1)$ by means of left and right translation

$$U(g_1, g_2) = L(g_1)R(g_2).$$

If one denotes the commutant of a representation with an accent and one writes $\mathcal{N}(L)$ for the van Neumann algebra generated by the left translations from G_1, then we know from the theorem of Segal-Godement that $R' = \mathcal{N}(L)$. Hence we have

$$U' = L' \cap R' = L' \cap \mathcal{N}(L).$$

As $\mathcal{N}(L) \cap L'$ is clearly a commutative algebra this proves the result. Various other examples of multiplicity free representations for trivial ρ can be found in [Kla79] and [vD94].

By combining the description of the G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$ in Theorem 4.6 with the first characterization from Theorem 5.1 of a multiplicity free decomposition, one obtains:

Corollary 5.6. *If for all irreducible unitary representations $(\pi, \mathcal{H}_\pi)$ of G, the dimension of the space $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ is maximally one, then the integral decomposition of $\mathcal{D}^1(e_{11}, X_0)$ in Theorem 3.2 is unique.*

6. The compact case

The criterion in 5.6 applies well in the case that H is compact. For, then we know from Lemma 4.4 that $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \pi(\check{e}_{11})(\mathcal{H}_\pi)$. On this subspace we have a natural action of the subalgebra

$$L^1(\check{e}_{11}, X_0) = \{ f \mid f \in L^1(G),\ f = \check{e}_{11} * f * \check{e}_{11} \}$$

of the convolution Banach algebra $L^1(G)$. The subalgebra $L^1(\check{e}_{11}, X_0)$ is also involutive, for if we denote for an $f \in L^1(G)$ the element $f^* \in L^1(G)$ by $f^*(g) = \overline{f(g^{-1})}$, then f^* belongs to $L^1(\check{e}_{11}, X_0)$ as soon as f does. Since for each $f \in L^1(\check{e}_{11}, X_0)$ the operator $\pi(f)$ satisfies

$$\pi(f) = \int_G f(g)\pi(g)dg = \pi(\check{e}_{11})\pi(f)\pi(\check{e}_{11}),$$

it is clear that $\pi(f)$ maps $\mathcal{H}_\pi$ to $\pi(\check{e}_{11})(\mathcal{H}_\pi)$. Hence the map $f \mapsto \pi(f)|\pi(\check{e}_{11})(\mathcal{H}_\pi)$ defines a $*$-representation of $L^1(\check{e}_{11}, X_0)$ onto $\pi(\check{e}_{11})(\mathcal{H}_\pi)$. Hereby irreducibility is preserved, for

Lemma 6.1. *If $(\pi, \mathcal{H}_\pi)$ is an irreducible unitary representation of G such that $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is non-zero, then the representation of $L^1(\check{e}_{11}, X_0)$ on $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is topologically irreducible.*

Proof. Choose a non-zero vector $v \in \pi(\check{e}_{11})(\mathcal{H}_\pi)$. Now the G-module $\mathcal{H}_\pi$ is irreducible if and only if the $L^1(G)$-module $\mathcal{H}_\pi$ is irreducible, see [Dix94]. Therefor the subspace $\{\pi(h)(v) \mid h \in L^1(G)\}$ is a dense subspace of $\mathcal{H}_\pi$. Consequently its image under $\pi(\check{e}_{11})$ is dense in $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ and that equals

$$\begin{aligned}\{\pi(\check{e}_{11})(\pi(h)(v)) \mid h \in L^1(G)\} &= \{\pi(\check{e}_{11})\pi(h)\pi(\check{e}_{11})(v) \mid h \in L^1(G)\}\\ &= \{\pi(\check{e}_{11} * h * \check{e}_{11})(v) \mid h \in L^1(G)\}\\ &= \{\pi(f)(v) \mid f \in L^1(\check{e}_{11}, X_0)\}.\end{aligned}$$

In other words, every non-zero vector in $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is cyclic for the action of $L^1(\check{e}_{11}, X_0)$. This proves the claim of the lemma. □

So, in the case of a compact H, we have a multiplicity-free decomposition if and only if $\dim(\pi(\check{e}_{11})(\mathcal{H}_\pi)) \leq 1$ for all irreducible unitary representations of G. If the Banach $*$-algebra $L^1(\check{e}_{11}, X_0)$ is commutative, then it follows from the work of Gelfand and Naimark that every topologically irreducible $*$-representation of $L^1(\check{e}_{11}, X_0)$ is one dimensional. The foregoing Lemma gives you then the desired estimate. However, also the reverse holds, for we have

Proposition 6.2. *If H is compact, then $\mathcal{D}^1(e_{11}, X_0)$ decomposes multiplicity free if and only if $L^1(\check{e}_{11}, X_0)$ is commutative.*

Proof. We merely have to show the necessity still. Here we use the fact that the non degenerate irreducible $*$-representations of the algebra $L^1(G)$ separate the points of $L^1(G)$, see [Nm70]. In particular, for each $f \neq 0$ in $L^1(G)$, there is a non degenerate irreducible $*$-representation $(\pi, \mathcal{H}_\pi)$ of $L^1(G)$ such that $\pi(f) \neq 0$. According to [Dix94] each non degenerate irreducible $*$-representation of $L^1(G)$ corresponds bijectively to an irreducible unitary representation of G. Take any two elements f_1 and f_2 in $L^1(\check{e}_{11}, X_0)$ and consider $f = f_1 * f_2 - f_2 * f_1$. If f is always zero, then we have the desired result. Assume that there is a non-zero f. Then there is an irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G such that $\pi(f) \neq 0$. As $\pi(f)$ maps $\mathcal{H}_\pi$ to $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ this implies that $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is non-zero. By assumption, the space $\pi(\check{e}_{11})(\mathcal{H}_\pi)$ is one-dimensional and spanned by a scalar λ_h on v:

$$\pi(h)(v) = \lambda_h(v)$$

and, since $\pi(f)$ is non-zero the scalar λ_f has to be non-zero. However, $f = f_1 * f_2 - f_2 * f_1$, so that there holds

$$\pi(f)(v) = \pi(f_1)\pi(f_2)(v) - \pi(f_2)\pi(f_1)(v) = (\lambda_{f_1}\lambda_{f_2} - \lambda_{f_2}\lambda_{f_1})(v) = 0.$$

This contradiction is due to the assumption f is non zero. Therefore $L^1(\check{e}_{11}, X_0)$ can not be but commutative. □

Remark 6.3. If H is compact and the algebra $L^1(H\backslash G/H)$ is commutative, then the pair (G, H) is called *a Gelfand pair*. Still for a compact H, but for a not necessarily trivial ρ one can speak of *a Gelfand pair of class* ρ if the algebra $L^1(\check{e}_{11}, X_0)$ is commutative. In case that H is no longer compact it is therefore natural to speak of *a generalized Gelfand pair of class* ρ if one of the properties (1)–(5) of Theorem 5.1 holds. If ρ is trivial, then the adjective “of class ρ” is left out.

The commutativity of algebras like $L^1(\check{e}_{11}, X_0)$ is often proved by showing that an anti algebra morphism is in fact an algebra morphism. We illustrate this with a subclass of the symmetric varieties $X_k = \mathcal{H}_k/\mathcal{G}_k$. The subclass that we consider are the symmetric varieties for which the group $\mathcal{H}$ is anisotropic over k. This yields in the real situation the Riemannian symmetric spaces that are known to be multiplicity free. In full generality, there holds:

Theorem 6.4. *Let $X_k = \mathcal{H}_k/\mathcal{G}_k$ be a symmetric variety defined over a locally compact field of characteristic not equal to two and assume that H is anisotropic over k. Then the following properties hold:*

(a) *The group H_k is compact.*
(b) *Every class in the double coset space $\mathcal{H}_k\backslash\mathcal{G}_k/\mathcal{H}_k$ can be represented by an element x in $\mathcal{G}_k$ such that $\sigma(x) = x^{-1}$.*
(c) *The pair $(\mathcal{G}_k, \mathcal{H}_k)$ is a Gelfand pair*

Proof. Part (a) is a general fact for the k-rational points of an anisotropic group defined over a locally compact field k.

As for (b) note that by [HW93, 6.7] $\mathcal{H}_k/\mathcal{G}_k \simeq Q_k = \{g\sigma(g)^{-1} \mid g \in \mathcal{G}_k\}$ and $\mathcal{H}_k\backslash\mathcal{G}_k/\mathcal{H}_k$ corresponds with the set of $\mathcal{H}_k$-conjugacy classes in Q_k, which clearly have a representative in Q_k. Now one can use property 6.4 (b) to show that $L^1(\mathcal{H}_k\backslash\mathcal{G}_k/\mathcal{H}_k)$ is commutative. For a function f on $\mathcal{G}_k$ we define the functions f^σ and $f^\vee$ by

$$f^\sigma(x) = f(\sigma(x)) \text{ and } f^\vee(x) = f(x^{-1}).$$

Because of property (b) we have for all $\mathcal{H}_k$-biinvariant $f \in L^1(\mathcal{G}_k)$ that $f^\sigma = f^\vee$. On the other hand $f \to f^\sigma$ is an algebra-isomorphism of $L^1(\mathcal{H}_k\backslash\mathcal{G}_k/\mathcal{H}_k)$ for

$$\begin{aligned}
(f_1 * f_2)^\sigma(x) &= \int f_1(\sigma(x)y)\, f_2(y^{-1})dy \\
&= \int f_1(\sigma(x)\sigma^2(y))\, f_2(\sigma^2(y)^{-1})dy \\
&= \int f_1^\sigma(x\sigma(y))\, f_2^\sigma(\sigma(y)^{-1})dy \\
&= \int f_1^\sigma(xt)\, f_2^\sigma(t^{-1})dt \\
&= f_1^\sigma * f_2^\sigma(x),
\end{aligned}$$

but $f \to f^\vee$ is an anti-algebra homomorphism:

$$\begin{array}{rcl} (f_1 * f_2)^\vee(x) & = & \int_{\mathcal{G}_k} f_1(x^{-1}y)\, f_2(y^{-1})dy \\ & = & \int_{\mathcal{G}_k} f_1(x^{-1}t^{-1})\, f_2(t)dt \\ & = & \int_{\mathcal{G}_k} f_2(tx^{-1})\, f_1(t^{-1})dt \\ & = & \int_{\mathcal{G}_k} f_2^\vee(xt^{-1})\, f_1^\vee(t)dt \\ & = & \int_{\mathcal{G}_k} f_2^\vee(xu)\, f_2^\vee(u^{-1})du \\ & = & f_2^\vee * f_1^\vee(x). \end{array}$$

Hence we have for all f_1 and $f_2 \in L^1(\mathcal{H}_k\backslash\mathcal{G}_k/\mathcal{H}_k)$ that $f_1^\vee * f_2^\vee = f_2^\vee * f_1^\vee$. In other words this convolution algebra is commutative. This concludes the proof.

□

Example 6.5. If Q is an anisotropic quadratic form over k, then O_Q is anisotropic and in that case the pair $(\mathrm{Gl}_n(k), O_Q(k))$ from example 2.2 is a Gelfand pair.

7. A general criterion

In the case of a non compact H, the commutativity of these convolution algebras is no longer a means to obtain multiplicity free decompositions. As a substitute, one has the following useful criterion, which is the extension to the present setting of one by Thomas, see [Tho84]

Theorem 7.1. *Let J be an anti-automorphism of $\mathcal{D}^1(e_{11}, X_0)$ such that $J\mathcal{H} = \mathcal{H}$ for all minimal G-invariant Hilbert subspaces $\mathcal{H}$ of $\mathcal{D}^1(e_{11}, X_0)$. Then the pair (G, H) is a generalized Gelfand pair of class ρ.*

Proof. From the integral decomposition in Theorem 3.2 it follows that as soon as each minimal subspace in $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ is invariant under J, the same holds for any $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$. Take any $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ and let $\mathcal{A}$ be the commutant in $\mathcal{L}(\mathcal{H})$ of the $\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}} \mid g \in G\}$. This is a von Neumann algebra. Just as for linear operators we define the antilinear automorphism J^* of $\mathcal{D}^1(e_{11}, X_0)$ by

$$\langle T, J^*\varphi\rangle = \langle J(T), \varphi\rangle.$$

Let $j : \mathcal{H} \to \mathcal{D}^1(e_{11}, X_0)$ be the embedding and $K = jj^*$ be the corresponding kernel. Since $J(\mathcal{H}) = \mathcal{H}$, this implies that the kernel K satisfies

$$JKJ^* = K.$$

As J is antilinear, this relation gives you that $J|_{\mathcal{H}}$ is anti-unitary and in particular $(J|_{\mathcal{H}}) = (J|_{\mathcal{H}})^*$. In the von Neumann algebra $\mathcal{A}$ the spectral components of a Hermitian $A_1 \in \mathcal{A}$, belong again to $\mathcal{A}$, see [Dix96]. Since A_1 commutes with all the $\{\mathcal{R}_{-\infty}(g)|_{\mathcal{H}} \mid g \in G\}$, the corresponding kernel K_1 given by

$$\langle K_1(\varphi), \psi\rangle := \langle A_1(j^*(\varphi)), j^*(\psi)\rangle$$

is again G-invariant and corresponds to a G-invariant Hilbert subspace $\mathcal{H}_1 \hookrightarrow \mathcal{H}$. This subspace $\mathcal{H}_1$ satisfies again $J(\mathcal{H}_1) = \mathcal{H}_1$ and thus there holds

$$JK_1J^* = JjA_1j^*J^* = jA_1j^*.$$

This relation in its turn implies that

$$J|_{\mathcal{H}} A_1 J|_{\mathcal{H}}^{-1} = A_1 = A_1^*.$$

Since the orthogonal projections in $\mathcal{A}$ generate the algebra $\mathcal{A}$ and the map $J|_{\mathcal{H}}$ is antilinear, we see that for all $A \in \mathcal{A}$ there holds

$$J|_{\mathcal{H}} A J|_{\mathcal{H}}^{-1} = A^*.$$

Applying this formula to the product of two operators A and B in $\mathcal{A}$ gives

$$A^* B^* = J|_{\mathcal{H}} A J|_{\mathcal{H}}^{-1} J|_{\mathcal{H}} B J|_{\mathcal{H}}^{-1} = (AB)^* = B^* A^*.$$

In other words the algebra $\mathcal{A}$ is commutative. This shows the second characterization of a multiplicity-free decomposition. □

Example 7.2. Let G be commutative. Then we know that for each $\varphi \in \mathcal{D}(e_{11}, X_0)$ the function $\check{\overline{\varphi}}$ belongs again to $\mathcal{D}(e_{11}, X_0)$ and the formula

$$J(T)(\varphi) = \overline{T(\check{\overline{\varphi}})}, \text{ for } T \in \mathcal{D}^1(e_{11}, X_0),$$

defines then an antilinear automorphism of $\mathcal{D}^1(e_{11}, X_0)$. Now the distribution T_ψ that correspond to a character ψ of G has the form

$$T_\psi(\varphi) = \int_G \overline{\varphi}(g)\psi(g^{-1})dg.$$

Those relevant for $\mathcal{D}^1(e_{11}, X_0)$ are the ones such that the restriction of ψ to H is equal to ρ. It is a direct verification that they are invariant under J and thus one sees once again that we have multiplicity one. For several of the classes of semi-direct products one can define similar J. This is left to the reader as an exercise.

For real symmetric spaces this criterion was e.g. used to show that $(SL_n(\mathbb{R}), GL_{n-1}(\mathbb{R}))$ for $n \geq 3$ is a generalized Gelfand pair, see [vDP86]. To illustrate its potential in another direction, one considers from now on the homogeneous spaces $X_k = \mathcal{H}_k/\mathcal{G}_k$, with k a nonarchimedean local field. It will be shown that theorem 7.1 can be applied as soon as one has sufficiently many $\mathcal{H}_k$-invariant distribution vectors coming from the construction that is described next.

The general construction that renders $\mathcal{H}_k$-invariant distribution vectors starts from decent $\mathcal{H}_k$-invariant functions on X_k. Assume Y is a $\mathfrak{p}$-adic variety and $P : X_k \to Y$ is a submersion such that $P(hx) = P(x)$ for all $h \in H_k$ and $x \in X_k$. Let dy be a volume form on Y. According to [HC70] there is a surjective linear mapping M_P from $\mathcal{D}(X_p)$ to $\mathcal{D}(Y)$ such that

$$\int_{X_k} \varphi(x)\alpha(P(x))dx = \int_Y M_P(\varphi)(y)\alpha(y)dy \tag{7.2.1}$$

for all $\varphi \in \mathcal{D}(X_k)$ and all $\alpha \in \mathcal{D}(Y)$. Let $M_P^* : \mathcal{D}^1(Y) \to \mathcal{D}^1(X_k)$ be the dual mapping. Then we have

Lemma 7.3. *The elements of $M_P^*(\mathcal{D}^1(Y))$ are H_k-invariant distribution vectors.*

Proof. By definition we have $M_P^*(T) = T \circ M_P$ and $\mathcal{L}_{-\infty}(h)(M_P^*(T)) = T \circ M_P \circ \mathcal{L}_\infty(h^{-1})$. From the left $\mathcal{H}_k$-invariant of P we conclude for all $\alpha \in \mathcal{D}(Y)$ and $\varphi \in \mathcal{D}(X_k)$

$$\begin{aligned} &\int_{X_k} \varphi(x)\alpha(P(x))dx = \int_{X_k} \varphi(hx)\alpha(P(hx))dx \\ = &\int_{X_k} \varphi(hx)\alpha(P(x))dx = \int_Y M_P(\mathcal{L}_\infty(h^{-1})(\varphi))(y)\alpha(y)dy \\ = &\int_Y M_P(\varphi)(y)\alpha(y)dy. \end{aligned}$$

Hence $M_P \circ \mathcal{L}_\infty(h^{-1}) = M_P$ and this proves the desired property. □

Remark 7.4. In case that the analytic map P is only a submersion on a dense subset X_k^1 of X_k, then we can still obtain H_k-invariant distribution vectors by this construction, if the map M_P can be extended from $\mathcal{D}(X_k^1)$ to $\mathcal{D}(X_k)$ in such a way that formula (7.2.1) holds. The image of this extended map M_P can of course be wider than $\mathcal{D}(Y)$ in that situation. We mention a few examples.

Example 7.5. Take for $\mathcal{G}$ the group SL_n. Let J be the matrix

$$J = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Consider the involution $\sigma(g) = JgJ^{-1}$ of $\mathcal{G}$. Then one verifies directly that

$$\mathcal{H} = \left\{ \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{pmatrix} \in \mathrm{SL}_n \right\} = \left\{ \begin{pmatrix} \det(g)^{-1} & 0 \\ 0 & g \end{pmatrix} \mid g \in \mathrm{GL}_{n-1} \right\}$$

Let Z be the $\mathfrak{p}$-adic manifold given by

$$Z = \{x \in M_n(k) \mid \mathrm{rank}(x) = 1, \mathrm{trace}(x) = 1\}.$$

The group $\mathcal{G}_k$ acts on Z by conjugation. In Z we take the element

$$y_0 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then one verifies by direct computation that the stabilizer of y_0 is $\mathcal{H}_k$ and that $\mathcal{G}_k$ acts transitively on Z, in other words $\mathcal{H}_k/\mathcal{G}_k \simeq Z$.

One defines an analytic map $P : X_k \to k$ by

$$P(x) = \mathrm{Tr}(xy_0).$$

This map satisfies first of all for all $h \in \mathcal{H}_k$

$$P(hxh^{-1}) = \mathrm{Tr}(hxh^{-1}y_0) = Tr(hxh^{-1}y_0hh^{-1} = \mathrm{Tr}(xy_0) = P(x),$$

since $hy_0h^{-1} = y_0$. A second property is that for all $g \in \mathcal{G}_k$

$$P(g^{-1}y_0g) = Tr(g^{-1}y_0gy_0) = Tr(gy_0g^{-1}y_0) = P(gy_0g^{-1}).$$

The map P is a submersion on a dense open submanifold and we refer to ([Bos92]) for the proof that the map M_P extends to $\mathcal{D}(X_k)$.

Example 7.6. Let J_n be the $2n \times 2n$-matrix given by

$$J_n = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

For $\mathcal{G}$ we take the group

$$\mathrm{Sp}(n) = \{g \in \mathrm{GL}_{2n} \mid {}^t g J_n g = J_n\}.$$

As usual E_{ij} denotes the $2n \times 2n$-matrix with a 1 at the (i, j)-th entry and zeros elsewhere. If we define $J = \mathrm{Id} - 2E_{nn} - 2E_{2n2n}$, then we have an involution σ of $\mathcal{G}$ defined by $\sigma(g) = JgJ$. A direct computation shows

$$\mathcal{H} = \left\{ \begin{pmatrix} & & & 0 & & & & 0 \\ & g_1 & & \vdots & & g_2 & & \vdots \\ & & & 0 & & & & 0 \\ 0 & \dots & 0 & a & 0 & \dots & 0 & b \\ & & & 0 & & & & 0 \\ & g_3 & & \vdots & & g_4 & & \vdots \\ & & & 0 & & & & 0 \\ 0 & \dots & 0 & c & 0 & \dots & 0 & d \end{pmatrix} \middle| \begin{array}{ll} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} & \in \mathrm{Sp}(n-1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \in \mathrm{Sp}(1) \end{array} \right\}.$$

The space X_k can be realized as

$$X_k = \{A \in M_{2n}(k) \mid J_n A J_n^{-1} = {}^t A, \mathrm{rank}(A) = \mathrm{trace}(A) = 2\}$$

on which $\mathcal{G}_k$ acts transitively by conjugation. Thus $X_k = \mathcal{G}_k \cdot y_0$ with $y_0 = E_{nn} + E_{2n2n}$.

As in the foregoing example one defines the analytic map $P : X_k \to k$ by

$$P(x) = \mathrm{Tr}(xy_0)$$

and one shows that it satisfies

$$P(hxh^{-1}) = P(x) \text{ and } P(g^{-1}y_0g) = P(gy_0g^{-1}).$$

Also this map has the required properties, see ([Bos92]).

Let μ be an automorphism of $\mathcal{G}_k$ such that for all $h \in \mathcal{H}_k$ we have $\mu(h) = h$. Note that $\mu = Id$ is always an option but there can be others. In the case of the symmetric varieties e.g. one can also take $\mu = \sigma$. To μ is associated a natural anti-automorphism J_μ of $\mathcal{D}^1(X_k)$. The map $J_\mu : \mathcal{D}^1(X_k) \to \mathcal{D}^1(X_k)$ is defined by

(7.6.1) $$\langle J_\mu(T), \varphi\rangle = \langle \overline{T, \overline{\varphi}^\mu}\rangle,$$

where $\varphi^\mu \in \mathcal{D}(X_k)$ is defined as $\varphi^\mu(x) = \varphi(\mu(x))$.

Assume now that one has an $\mathcal{H}_k$-invariant analytic map $P : X_k \to Y$ as in remark 7.4 that satisfies for all $g \in \mathcal{G}_k$

$$P(\mathcal{H}_k\mu(g)) = P(\mathcal{H}_k g^{-1}). \tag{7.6.2}$$

Then there holds

Proposition 7.7. *For all τ in the image of M_P^*, there holds $J_\mu(\tau) = \tau$.*

Proof. Let τ be $\tau = M_P^*(\xi) = \xi \circ M_P$ in $\mathcal{D}^1(X_k)^{\mathcal{H}_k}$. If $P_{\mathcal{H}_k} : \mathcal{D}(\mathcal{G}_k) \to \mathcal{D}(X_k)$ is the natural projection given by

$$P_{\mathcal{H}_k}(\varphi)(x) = \int_{\mathcal{H}_k} \varphi(hx)dh$$

Then we write ξ for the element $\tau \circ P_{\mathcal{H}_k}$ in $\mathcal{D}^1(\mathcal{G}_k)$. Likewise we see P as an analytic map: $\mathcal{G}_k \to Y$ and then formula (7.2.1) gives for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$

$$\int_{\mathcal{G}_k} \varphi(g)\alpha \circ P(g)dg = \int_Y M_P \circ P_{\mathcal{H}_k}(\varphi)(y)\alpha(y)dy.$$

By equation (7.6.2) the left hand side equals

$$\begin{aligned}
\int_{\mathcal{G}_k} \varphi(\mu(g))\alpha \circ P(\mu(g))dg &= \int_{\mathcal{G}_k} \varphi^\mu(g)\alpha \circ P(g^{-1})dg \\
&= \int_{\mathcal{G}_k} \overset{\vee}{\varphi}{}^\mu(g)\alpha \circ P(g)dg \\
&= \int_Y M_P \circ P_{\mathcal{H}_k}(\overset{\vee}{\varphi}{}^\mu)(y)\alpha(y)dy \\
&= \int_Y M_P \circ P_{\mathcal{H}_k}(\varphi)(y)\alpha(y)dy.
\end{aligned}$$

Since α is arbitrary, we may conclude $M_P \circ P_{\mathcal{H}_k}(\varphi) = M_P \circ P_{\mathcal{H}_k}(\overset{\vee}{\varphi}{}^\mu)$. Thus we get that for all φ in $\mathcal{D}(\mathcal{G}_k)$

$$\xi(\varphi) = \xi(\overset{\vee}{\varphi}{}^\mu).$$

Therefore it suffices to show that for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$, $\overline{\xi(\overline{\varphi})} = \xi(\overset{\vee}{\varphi})$ and this is a general property. For $\varphi \to \overline{\xi(\overline{\varphi})}$ is clearly a positive definite bi-$\mathcal{H}_k$-invariant distribution on $\mathcal{G}_k$. As, for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$

$$\overline{\widetilde{\varphi}} * \overline{\varphi}(x) = \widetilde{\varphi} * \varphi(x^{-1}),$$

we see that

$$\xi((\widetilde{\varphi} * \varphi)^\vee) = \xi(\overline{\widetilde{\varphi}} * \overline{\varphi}) = \overline{\xi(\overline{\widetilde{\varphi}} * \overline{\varphi})} \geq 0 \tag{7.7.1}$$

so that $\varphi \to \xi(\overset{\vee}{\varphi})$ is also a positive definite bi-$\mathcal{H}_k$-invariant distribution G_k. From (7.7.1) follows that for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$ and all $\psi \in \mathcal{D}(\mathcal{G}_k)$

$$\xi((\varphi * \psi)^\vee) = \overline{\xi(\overline{(\varphi * \psi)})}.$$

As the elements $\varphi * \psi$ span $\mathcal{D}(\mathcal{G}_k)$, we get for all $\varphi \in \mathcal{D}(\mathcal{G}_k)$, $\overline{\xi(\overline{\varphi})} = \xi(\overset{\vee}{\varphi})$. This is the desired result. □

For τ as in proposition 7.7, let $\mathcal{H}_\tau$ be the Hilbert subspace of $\mathcal{D}^1(X_k)$. From the J_μ-invariance of τ follows that $J_\mu \mathcal{H}_\tau = \mathcal{H}_\tau$.

Hence we may conclude now

Theorem 7.8. *Let P_i, $i \in I$, be analytic maps that satisfy first of all the requirements in remark 7.4 and equation* (7.6.2) *for the same μ. If the images of the $M^*_{P_i}$ span $\mathcal{D}^1(X_k)^{\mathcal{H}_k}$, then $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair.*

Example 7.9. We return to the examples 7.5 and 7.6. With the P's from these examples, Bosman proved following the procedure set out above that in both cases $(\mathcal{G}_k, \mathcal{H}_k)$ is a generalized Gelfand pair for $n \geq 4$, see [Bos92].

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