CLASSIFICATION OF INVOLUTIONS OF $\mathbf{SL(2,}\ \boldsymbol{k}\mathbf{)}$

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ABSTRACT. In this paper we give a simple characterization of the isomorphy classes of involutions of $\mathrm{SL}(2,k)$ with k any field of characteristic not 2. We also classify the isomorphy classes of involutions for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. We determine in which cases the corresponding fixed point group H is k-anisotropic. In those cases the corresponding symmetric k-variety consists of semisimple elements.

INTRODUCTION

Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k, H a k-open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. These varieties occur in many problems in representation theory, geometry and singularity theory. To study these symmetric k-varieties one needs first a classification of the related k-involutions. A characterization of the isomorphy classes of the k-involutions was given in [Hel00] essentially using the following 3 invariants:

(1) classification of admissible (Γ,θ)-indices.
(2) classification of the G_k-isomorphy classes of k-involutions of the k-anisotropic kernel of G.
(3) classification of the G_k-isomorphy classes of k-inner elements of G.

For more details, see [Hel00]. The admissible (Γ,θ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was included in [Hel00] as well. For k algebraically closed or k the real numbers the full classification can be found in [Hel88]. For other fields a classification of the remaining two invariants is still lacking. In particular the case of symmetric k-varieties over the $\mathfrak{p}$-adic numbers is of interest. We note that the above characterization only holds for k a perfect field.

To classify the remaining two invariants we start in this paper with a full classification of the case that $G = \mathrm{SL}(2,k)$. This case will be fundamental

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in the analysis of the general case. We will first give a simple characterization of the isomorphy classes of k-involutions, which does not depend on any of the results in [Hel00]. Also these results hold for any field of characteristic not 2, not only perfect fields. Next we classify the possible isomorphy classes for k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. Finally we determine the fixed point groups and determine which are k-anisotropic. For k the $\mathfrak{p}$-adic numbers the symmetric k-varieties G_k/H_k with H_k k-anisotropic have a similar structure as the Riemannian symmetric spaces and therefore these cases are of particular interest for studying their representations. The results in this paper will play a fundamental role in the classification of the isomorphy classes of involutions of $\mathrm{SL}(n,k)$. This will be discussed in a forthcoming paper.

1. CHARACTERIZATION OF THE ISOMORPHY CLASSES OF INVOLUTIONS

Our basic reference for reductive groups will be the papers of Borel and Tits [BT65, BT72] and also the books of Borel [Bor91], Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations and terminology. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups.

1.1. **Notation.** Throughout this paper let k be a field of characteristic not 2, k_1 an extension field of k, $G = \mathrm{SL}(2,k)$ and $G_1 = \mathrm{SL}(2,k_1)$.

1.2. **Automorphisms of *G*.** For $A \in G_1$ let $\mathrm{Int}(A) = I_A$ denote the inner automorphism defined by $I_A(X) = A^{-1}XA$, $X \in G_1$. Let $\mathrm{Aut}(G)$ denote the set of automorphisms of G and $\mathrm{Int}(G) = \{\mathrm{Int}(x) \mid x \in G\}$ the set of inner automorphisms of G. In this subsection we characterize the group $\mathrm{Aut}(G)$. First we recall the following:

Definition 1. $\theta, \phi \in \mathrm{Aut}(G)$ are said to be k_1-conjugate or k_1-isomorphic if and only if there is a $\chi \in \mathrm{Int}(G_1)$, such that $\chi^{-1}\theta\chi = \phi$. In the case that $k = k_1$ we will also say that they are conjugate or isomorphic.

We recall the following results, which can be found in [Bor91]:

Lemma 1. *If k is an algebraically closed field, then we have* $\mathrm{Aut}(G) = \mathrm{Int}(G)$.

Remark 1. For $\theta \in \mathrm{Aut}(G)$ and k is not algebraically closed, there always exists an extension field k_1 of k and $\tau \in \mathrm{Int}(G_1)$ such that $\tau|G = \theta$, i.e. there exists a 2×2-matrix $A \in G_1$, such that $\theta = \tau|G = I_A|G$.

Lemma 2. *Suppose* $A \in \mathrm{GL}(2,k_1)$. *If* $I_A|G = \mathrm{Id}$, *then* $A = p\,\mathrm{Id}$ *for some* $p \in k_1$.

Proof. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in k_1$. Since $I_A|G = \mathrm{Id}$, we have for all $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in G$, $I_A(X) = AXA^{-1} = X$, i.e. $AX = XA$. So

$$\begin{pmatrix} ax_1 + bx_3 & ax_2 + bx_4 \\ cx_1 + dx_3 & cx_2 + dx_4 \end{pmatrix} = \begin{pmatrix} ax_1 + cx_2 & bx_1 + dx_2 \\ ax_3 + cx_4 & bx_3 + dx_4 \end{pmatrix}.$$

Since X is arbitrary, we have $a = d$ and $b = c = 0$, i.e. $A = a\,\mathrm{Id}$. □

This result enables us to determine when an element of $\mathrm{Int}(G_1)$ keeps G invariant:

Lemma 3. *$I_A \in \mathrm{Int}(G_1)$ keeps G invariant if and only if $A = pB$ for some $p \in k_1$, $B \in \mathrm{GL}(2,k)$.*

Proof. Write $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $a_i \in k_1$, $i = 1, 2, 3, 4$. So $A^{-1} = (\det A)^{-1} \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$.
For all $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in G$, we have $I_A(X) = A^{-1}XA =$

$$(\det A)^{-1} \begin{pmatrix} a_1a_4x_1 + a_3a_4x_2 - a_1a_2x_3 - a_2a_3x_4 & a_2a_4x_1 + a_4^2x_2 - a_2^2x_3 - a_2a_4x_4 \\ -a_1a_3x_1 - a_3^2x_2 + a_1^2x_3 + a_3a_4x_4 & -a_2a_3x_1 - a_3a_4x_2 + a_1a_2x_3 + a_1a_4x_4 \end{pmatrix}.$$

Since X is arbitrary we have

$$\begin{aligned} I_A(X) \in G &\Longleftrightarrow \frac{a_ia_j}{(a_1a_4 - a_2a_3)} \in k, \ \forall i, j = 1, 2, 3, 4 \\ &\Longleftrightarrow \frac{a_i}{a_j} \in k, \forall i, j = 1, 2, 3, 4, \end{aligned}$$

provided $a_j \neq 0$, i.e. $A = pB$ for some $p \in k_1$, $B \in \mathrm{GL}(2,k)$. □

Remark 2. Since for all $p \in k_1$ and $B \in \mathrm{GL}(2,k_1)$, we always have $I_{pB} = I_B$. It follows that every automorphism over $\mathrm{SL}(2,k)$ can be written as the restriction to $\mathrm{SL}(2,k)$ of a inner automorphism of $\mathrm{GL}(2,k)$.

1.3. Now let's turn to involutions on $G = \mathrm{SL}(2,k)$. Suppose $\theta \in \mathrm{Aut}(G)$ is an involution, i.e. $\theta^2 = \mathrm{Id}$. By Lemma 1, there is a field $k_1 \supset k$ and an $A \in G_1 = \mathrm{SL}(2,k_1)$ such that $\theta = I_A|G$ and by Lemma 2 there exists a matrix $A \in \mathrm{GL}(2,k)$, such that $\theta = I_A|G$.

Lemma 4. *Suppose $\theta \in \mathrm{Aut}(G)$ is an involution. Then there exists a matrix $A \in \mathrm{GL}(2,k)$, such that $\theta = I_A|G$ and A is conjugate to $\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$, for some $p, q \in k$.*

Proof. Since θ is an involution, we know that there is a matrix $A \in \mathrm{GL}(2,k)$ such that $I_A^2 = \theta^2 = \mathrm{Id}$, i.e. $I_{A^2} = \mathrm{Id}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^2 = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{pmatrix}$. By Lemma 2, we have $A^2 = p\,\mathrm{Id}$, for some $p \in k$, i.e. $a^2+bc = d^2+bc$ and $(a+d)c = (a+d)b = 0$.

1. If $a+d = 0$, then we have two subcases:
   - (a) If we also have $b = c = 0$, then we get $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

     Let $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$.
   - (b) If $a+d = 0$ and both b and c are not equal to 0, then without loss of generality, we may assume that $b \neq 0$. Let $P = \begin{pmatrix} 1 & 0 \\ -a/b & 1 \end{pmatrix}$, then we have $P^{-1}AP = \begin{pmatrix} 0 & b \\ a^2/b+c & 0 \end{pmatrix}$.
2. If $a+d \neq 0$, then $a = d \neq 0$, $b = c = 0$, i.e. $A = a\,\mathrm{Id}$ with $a \in k$. It follows that θ is the identity, which is not a involution.

□

Since by remark 2, we can multiply A by any constant without changing the involution, we get:

Corollary 1. *All the k-isomorphy classes of involutions over G can be represented by the matrices* $\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \in \mathrm{GL}(2,k)$.

Lemma 5. *Let* $A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \in G$. *Then* I_A *is conjugate to* I_B *if and only if there is a matrix* $X \in G$ *and a constant* $c \in k$, *such that* $X^{-1}AX = cB$.

Proof. The result follows from the following equivalent statements:

- I_A is conjugate to I_B.
- there is a matrix $X \in G$, such that $I_{X^{-1}} I_A I_X = I_B$.
- $I_{X^{-1}AXB^{-1}} = \mathrm{Id}$.
- there is $c \in k$, such that $X^{-1}AXB^{-1} = c\,\mathrm{Id}$ (Lemma 2, $k_1 = k$).
- $X^{-1}AX = cB$.

□

Theorem 1. *Suppose* $\theta, \phi \in \mathrm{Aut}(G)$ *involutions of which the corresponding matrices in G are* $A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$. *Then* θ *is conjugate to* ϕ *if and only if* q/p *is a square in* k^*.

Proof. ($\Rightarrow$) By the result of Lemma 5, and by taking determinants on both sides, we get $-p=-c^2q$, i.e. $c^2{=}p/q$.

($\Leftarrow$) Suppose $c^2{=}q/p$, $c\in k$ and $X=\begin{pmatrix}0 & 1\\ pc & 0\end{pmatrix}$. Then $X^{-1}BX=cA$. By Lemma 5, we're done. ☐

For a field k let k^* denote the product group of all the nonzero elements and $(k^*)^2=\{a^2 \mid a\in k^*\}$. Then $(k^*)^2$ is a normal subgroup of k^*.

Corollary 2. *The number of isomorphy classes of involutions over G equals the order of* $k^*/(k^*)^2$.

Remark 3. One easily verifies that the isomorphy classes of involutions of GL$(2,k)$ are the same as those of SL$(2,k)$. So the results in this paper give a characterization for GL$(2,k)$ as well.

Remark 4. For k a perfect field the results in this section could also have been derived from the characterization of the k-involutions in [Hel00]. The diagonal subgroup T is a maximal k-split torus, hence there is no k-anisotropic kernel. Since the root system $\Phi(T)$ of T with respect to G is of type A_1, there is only one nontrivial involution of $\Phi(T)$. So in this case one only needs to check the third invariant mentioned in the introduction. Let $\theta\in\operatorname{Aut}(G)$ be the involution defined by $\theta(g)={}^Tg^{-1}$, $g\in G$. Note that $\theta=I_A$ with $A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Then any other involution of G is isomorphic to $\theta\operatorname{Int}(x)$ for some $x\in T$. It suffices to check then the isomorphy classes of these involutions. It is easy to check that for the choice of x one can restrict to the elements $x=\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}$ with a a representative of $k^*/(k^*)^2$. Then finally one has to check that none of these involutions are isomorphic to each other.

2. CLASSIFICATION OF THE ISOMORPHY CLASSES OF INVOLUTIONS

In this section we give a classification of the isomorphy classes of involutions for k algebraically closed, the real numbers, finite fields and the $\mathfrak{p}$-adic numbers.

2.1. Throughout this section let $\theta=I_A$ with $A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $\tau=I_B$, with $B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Note that $\theta(g)=I_A(g)={}^Tg^{-1}$.

(1). **Algebraically closed fields.** In this case $|k^*/(k^*)^2|=1$, so there is only one isomorphy class of involutions, represented by θ.

(2). **Real numbers** $\mathbb{R}$**.** In this case $\mathbb{R}^*/(\mathbb{R}^*)^2\simeq\mathbb{Z}_2$, so there are two isomorphy classes of involutions and θ and τ are representatives.

(3). **Rational numbers $\mathbb{Q}$.** In this case $|k^*/(k^*)^2| = \infty$, so there are infinitely many isomorphy classes of involutions.

(4). **Finite fields ($\mathbb{F}_p$, $p \neq 2$).** In this case $\mathbb{F}_p^*/\mathbb{F}_p^{*2} \simeq \mathbb{Z}_2$, so there are only two isomorphy classes of involutions. This can be seen as follows. Let $\phi : \mathbb{F}_p \to \mathbb{F}_p$ be the map defined by $\phi(x) = x^2$. Then $\phi(\mathbb{F}_p^*) = \mathbb{F}_p^{*2}$ is a normal subgroup of $\mathbb{F}_p^*$ and $|\mathbb{F}_p^*/\mathbb{F}_p^{*2}| = |\operatorname{Ker}(\phi)| = 2$. Let $1, 2, \ldots, p-1$ be representatives of $\mathbb{F}_p^*$ and N_p be the "smallest number" in $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$. Then 1 and N_p are the representatives of $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$ and I_B and I_C with $C = \begin{pmatrix} 0 & 1 \\ N_p & 0 \end{pmatrix}$ are representatives of the isomorphy classes of involutions..

(5). **$\mathfrak{p}$-adic numbers $\mathbb{Q}_p$.** If $p \neq 2$, there are four involutions and $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. This can be seen as follows. Recall that $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ relative to the $\mathfrak{p}$-adic norm, and can be represented as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=-n}^{\infty} a_i p^i \;\middle|\; a_i = 0, 1, \ldots, p-1, \text{ with } a_n \neq 0, n \in \mathbb{Z} \right\}.$$

The norm of $x = \sum_{i=-n}^{\infty} a_i p^i$ with $a_n \neq 0,\ a_i = 0, 1, \ldots, p-1$ is defined as p^{-n}, i.e. $|x|_p = p^{-n}$. With this norm, the distance of any two points is one of the countable choices $p^{-n}, n \in \mathbb{Z}$. Now let's turn to the squares of $\mathfrak{p}$-adic numbers. First of all, p can't be a square. Otherwise if we have $x \in \mathbb{Q}_p$ such that $x^2 = p$, then x has to begin with positive power of p, which makes the square of x begin with p^2 at least and disagree with p. Note that the squares of the $\mathfrak{p}$-adic numbers must have its leading coefficient in $\mathbb{F}_p^{*2}$, so N_p (same notation as (4)) can't be a square. $N_p p^{-1}$ is not a square either by the same argument. That means p and N_p are in different cosets and $pN_p = p^{-1} N_p p^2$ isn't a square either. It's easy to check 1, p, N_p are pN_p are in different cosets. And actually they are representatives for $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, i.e. the matrices corresponding to different involutions for $\mathbb{Q}_p (p \neq 2)$ are

$$\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, a = 1, p, N_p \; or \; pN_p. \tag{1}$$

Interestingly $\mathbb{Q}_2$ has eight involutions. The corresponding matrices are

$$\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, a \in \{1, -1, 2, -2, -3, 3, 6, -6\} \tag{2}$$

The details about cosets for $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ can also be found in Mahler's book [Mah81].

3. FIXED POINT GROUPS AND SYMMETRIC k-VARIETIES

The fixed-point group $H = G^\delta$ for an involution δ over G is defined by

$$H = G^\delta = \{x \in G \mid \delta(x) = x\}.$$

The fixed point group determines a lot of the structure of the corresponding symmetric k-variety $X := \{g\delta(g)^{-1} \mid g \in G\}$. It is easy to see that $X \simeq G/G^\delta$. Moreover if G^δ is compact then from [HW93] it follows that X consists of semisimple elements:

Proposition 1 ([HW93, Proposition 10.8])**.** *Let G be a connected reductive algebraic k-group with* $\mathrm{char}(k) = 0$ *and $X = \{g\theta(g)^{-1} | g \in G\}$. Suppose that $H \cap [G,G]$ is anisotropic over k. Then X_k consists of semi-simple elements.*

In view of this result it is important to determine which involutions have an k-anisotropic fixed point group. For $k = \mathbb{R}$ or $\mathbb{Q}_p$ all k-anisotropic subgroups are compact. In this section we determine the fixed point groups of the involutions in the previous section. Our main focus will be given to the $\mathfrak{p}$-adic fields $\mathbb{Q}_p$. From [HW93, Proposition 1.2] we know that two involutions are conjugate if and only if their corresponding fixed-point groups are conjugate. Combined with Corollary 1, it follows that it suffices to determine the fixed-point group of the involutions $\delta = I_A$ with $A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$ and a representative of k^*/k^{*2}. We will also write H^a for G^δ. We note that

$$H^a = \left\{ \{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, xw - yz = 1 \right\}.$$

That is

$$H^a = \left\{ \begin{pmatrix} x & y \\ ay & x \end{pmatrix} \;\middle|\; x^2 - ay^2 = 1 \right\}.$$

3.1. $k = \mathbb{C}$ or $\mathbb{R}$.

For k the complex numbers there is only one involution and for k the real numbers there are two. The fixed point groups corresponding to the involutions θ and τ as section 2 are:

$$G^\theta = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2 + b^2 = 1 \right\}.$$

and

$$G^\tau = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \;\middle|\; a^2 - b^2 = 1 \right\}.$$

They are conjugate over the complex numbers but not over the real numbers, since over the real numbers G^θ is compact while G^τ is not compact.

3.2. $\boldsymbol{k = \mathbb{Q}_p}$. For $\mathbb{Q}_p$, choose a from (1) or (2), then the fixed-point groups corresponding to these involutions are

$$\left\{\begin{pmatrix} x & y \\ ay & x \end{pmatrix} \middle| \, x^2 - ay^2 = 1\right\}$$

Consider the curves

$$\mathbf{O}_a = \{(x, y) \mid x^2 - ay^2 = 1, x, y \in \mathbb{Q}_p\}$$

Where a is chosen from (1) or (2). Before we go any further, we recall some useful Lemmas:

Lemma 6. *Let* $x \in \mathbb{Q}_p (p \neq 2)$. *If* $|x|_p \leq p^{-1}$, *then* $1 + x \in Q_p^{*2}$. *If* $x \in \mathbb{Q}_2$ *and* $|x|_p \leq \frac{1}{8}$, *then* $1 + x \in \mathbb{Q}_2^{*2}$.

Lemma 7. *For* $p \neq 2$, $x^2 - ay^2 \leq 1$ *implies* $\max(|x|_p, |y|_p) \leq 1$. *Where* $a =$ p, N_p *or* pN_p *and* $x, y \in \mathbb{Q}_p$.

Lemma 8. *Suppose* $x, y \in \mathbb{Q}_2$, $a = -1, -2, -6, 2, 3$ *or* 6, *then* $x^2 - ay^2 \leq 1$ *implies* $\max(|x|_p, |y|_p) \leq 1$; $x^2 + 3y^2 \leq 1$ *implies* $\max(|x|_p, |y|_p) \leq 2$.

Similar results can be found in [Mah81], we omit the proofs here.

Lemma 9. *A bounded sequence in* $\mathbf{O}_p$ *has a limit.*

Proof. Without loss of generality, suppose $\{x_n \in \mathbf{O}_p \mid |x_n|_p \leq 1, n = 1, 2, \ldots\}$ is a bounded sequence. Name the sequence Φ^0 and write every $x_n^0 = x_n$ as the standard form

$$x_n^0 = \sum_{i=0}^{\infty} a_{ni}^0 p^i, a_{ni} = 0, 1, \ldots, p-1, \text{ and } n = 0, 1, 2, \ldots$$

Since a_{n0}^0 has only p choices, there must be a infinite subsequence $\{x_{n_k}^0\}$ with the same value for its first coefficients, i.e. all $a_{n_k 0}^0$'s are all the same, say b_0. Name the subsequence Φ^1 and write every x_n^1 as

$$x_n^1 = b_0 + \sum_{i=1}^{\infty} a_{ni}^1 p^i, a_{ni} = 0, 1, \ldots, p-1, \text{ and } n = 0, 1, 2, \ldots$$

To choose b_1 and b_i, perform the same choice on Φ^1 and $a_{n_1}^1$ and so on. By construction, the sequence has limit

$$x = \sum_{i=0}^{\infty} b_i p^i$$

□

Theorem 2. (i) *The orbits of the curves* $\mathbf{O}_a$, $a \neq 1$ *are all closed, bounded and (sequentially) compact.*

(ii) *The orbit of the curve* $\mathbf{O}_1$ *is closed, but neither compact nor bounded.*

Proof. (i). Take the norm of $\mathbb{Q}_p \times \mathbb{Q}_p$ to be the maximum of that of its components, i.e. for $z = (x, y) \in \mathbb{Q}_p \times \mathbb{Q}_p$, the norm of z is defined by $\|z\|_p = \max(|x|_p, |y|_p)$. Consider first the functional $f(x, y)$ defined by:

$$(x, y) \in \mathbb{Q}_p \times \mathbb{Q}_p \to f(x, y) = x^2 - ay^2 \in \mathbb{Q}_p.$$

Since $|x^2 - ay^2|_p \leq |x|_p^2 + |a|_p |y|_p^2$, $f(x, y)$ is continuous. The orbit, which is the preimage of the closed set $\{1\}$ is also closed. Boundedness is immediate from Lemma 7 and 8. For $\mathbf{Q}_p$, compactness and sequential compactness are equivalent. For a infinite sequence in one of the orbits, it has a limit by Lemma 9. Since it's closed, the limit lies on the orbit. This proves the compactness and thus the first part of the theorem as well.

(ii). The same argument proves that $\mathbf{O}_1$ is closed. Now assume $x = p^{-n}\overline{x}$ and $y = p^{-n}$, we prove for some choices of $\overline{x}$, (x, y) lies on the orbit. Actually

$$x^2 - y^2 = p^{-2n}\overline{x}^2 - p^{-2n} = 1.$$

But then $\overline{x}^2 = 1 + p^{2n}$. By Lemma 6, there is a solution for $\overline{x}$ for $n \geq 2$. And $|y|_p = p^n$ converges to infinity as n goes to infinity. So $\mathbf{O}_1$ is unbounded and thus noncompact. ☐

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