ORBITS AND INVARIANTS ASSOCIATED WITH A PAIR OF COMMUTING INVOLUTIONS

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ABSTRACT. Let σ, θ be commuting involutions of the connected reductive algebraic group G where σ, θ and G are defined over a (usually algebraically closed) field k, $\operatorname{char} k \neq 2$. We have fixed point groups $H := G^\sigma$ and $K := G^\theta$ and an action $(H \times K) \times G \to G$, where $((h, k), g) \mapsto hgk^{-1}$, $h \in H$, $k \in K$, $g \in G$. Let $G/\!\!/(H \times K)$ denote $\operatorname{Spec} \mathcal{O}(G)^{H \times K}$ (the categorical quotient).

Let A be maximal among subtori S of G such that $\theta(s) = \sigma(s) = s^{-1}$ for all $s \in S$. There is the associated Weyl group $W := W_{H \times K}(A)$. We show:

- The inclusion $A \to G$ induces an isomorphism $A/W \xrightarrow{\sim} G/\!\!/(H \times K)$. In particular, the closed $(H \times K)$-orbits are precisely those which intersect A.
- The fibers of $G \to G/\!\!/(H \times K)$ are the same as those occurring in certain associated symmetric varieties. In particular, the fibers consist of finitely many orbits.

We investigate:

- The structure of W and its relation to other naturally occurring Weyl groups and to the action of $\sigma\theta$ on the A-weight spaces of $\mathfrak{g}$.
- The relation of the orbit type stratifications of A/W and $G/\!\!/(H \times K)$.

Along the way we simplify some of Richardson's proofs for the symmetric case $\sigma = \theta$, and at the end we quickly recover results of Berger, Flensted-Jensen, Hoogenboom and Matsuki [Ber57, FJ78, Hoo84, Mat97] for the case $k = \mathbb{R}$.

1. INTRODUCTION

1.1. Let G, θ, K, etc. be as above. We assume that k is algebraically closed and we set $\mathfrak{g} := \operatorname{Lie}(G)$, $\mathfrak{k} := \operatorname{Lie}(K)$, etc. The variety G/K is called a *symmetric variety*. These varieties occur in many problems in representation theory (see [BB81] and [Vog83]) and geometry (see [DCP83] and [LV83]). If $k = \mathbb{C}$, then G/K is the "complexification" of a Riemannian symmetric space and there exists a one to one correspondence between isomorphism classes of Riemannian symmetric spaces and isomorphism classes of involutions of G. Similarly, there is a bijection between isomorphism classes of reductive symmetric spaces and isomorphism classes of pairs of commuting involutions of G (see [Hel88]). Richardson [Ric82b] made a detailed study of the action of K (and closely related groups) on G/K, i.e., he studied $K\backslash G/K$. His results are global analogues of those of Kostant and

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Rallis [KR71] for the action of K on $T_{eK}(G/K) \simeq \mathfrak{g}/\mathfrak{k}$ ($k = \mathbb{C}$). We generalize results of Richardson to the case $H\backslash G/K$.

1.2. Results on double coset decompositions (especially over $\mathbb{R}$) are important in harmonic analysis (see, e.g., [BS97, BD92, Del98, FJ80, OS80]) and for the orbit method of Kirillov and Kostant [Kir93]. For compact groups the matrix coefficients of a finite dimensional representation of class 1 are right-H^0 and left-K^0 invariant functions (polynomials) (see [JC74, Hoo84]). Recall that an irreducible representation ϕ of G on a finite dimensional vector space V is of class 1 (relative to K) if $\dim V^K \geq 1$. Recently, T.A. Springer outlined a method to apply double coset decompositions to simplify the description of character sheaves on symmetric spaces given in [Gro92].

1.3. We now outline our main techniques and results, section by section. We rely heavily on the methods of invariant theory/transformation groups (section 2). In particular, we employ slice theorems systematically. In the case of symmetric varieties, these techniques can improve upon the traditional ones (see 6.1).

1.4. Let X be an affine G-variety. The algebra $\mathcal{O}(X)^G$ of G-invariant regular functions on X is finitely generated [Hab75]. Let $X/\!\!/G$ denote the corresponding affine variety, and let $\pi_X\colon X \to X/\!\!/G$ correspond to the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$. Then each fiber $\pi_X^{-1}(\xi)$, $\xi \in X/\!\!/G$, contains a unique closed orbit, so, in effect, $X/\!\!/G$ is the space of closed G-orbits in X.

In characteristic zero, Luna showed that there is an étale slice at every closed orbit Gx ([Lun73], see 2.4–2.7). Important for the proof are the linear reductivity of the isotropy group G_x and the surjectivity of the linear map $\mathfrak{g} = T_eG \to T_x(Gx)$. In characteristic p these properties do not always hold, but Bardsley and Richardson [BR85] showed that there is an étale slice if

- $\mathfrak{g} \to T_xGx$ is surjective.
- X is smooth at x and there is a "transversal" $R \ni x$. This means that R is a locally closed smooth G_x-stable subvariety of X such that $T_xR \oplus T_x(Gx) = T_xX$.

Given that the appropriate conditions hold, the étale slice theorem roughly says that a G-neighborhood of Gx is isomorphic to the homogeneous fiber bundle $G \times^{G_x} R$. In particular, the quotient $X/\!\!/G$ is (roughly) isomorphic to $R/\!\!/G_x$ near Gx and the fiber $\pi_X^{-1}(\pi_X(x))$ is isomorphic to $G \times^{G_x} R_x$ where R_x is the fiber $\pi_R^{-1}(\pi_R(x))$. The actual "slice" $\mathcal{S}$ is an open G_x-stable neighborhood of x in R.

1.5. In section 3 we recall facts about symmetric varieties. Since $\operatorname{char}(k) \neq 2$, we may represent G/K as a subset of G: Let $\beta\colon G \to G$, $g \mapsto g\theta(g)^{-1}$. Then β induces an isomorphism $G/K \xrightarrow{\sim} P := \beta(G)$, $gK \mapsto \beta(g)$ [Ric82b, 2.4]. The left action of G on G/K becomes the twisted action $g * x := gx\theta(g)^{-1}$, $g \in G$, $x \in P$. In particular, the $*$-action is conjugation when restricted to K. Our study has now become that of the quotient mapping $\pi_P\colon P \to P/\!\!/H$ where H acts via $*$.

1.6. In section 4 we study the appropriate tori and Weyl groups: A torus S in G is *θ-split* if $\theta(s) = s^{-1}$ for all $s \in S$ and S is *(σ, θ)-split* if it is θ-split and σ-split. Let A be a maximal (σ, θ)-split torus of G. From the twisted action $*$ we obtain a *twisted* Weyl group $W_H^*(A)$: Set $N_H^*(A) = \{h \in H \mid h * A = A\}$, $Z_H^*(A) = \{h \in H \mid h * a = a$ for every $a \in A\}$ and $W_H^*(A) = N_H^*(A)/Z_H^*(A)$. Then $W_H^*(A)$ acts on A via affine transformations (see 1.9).

In 4.5–4.6 there is a discussion/proof of the equivalence between the H-orbits on P and the $(H \times K)$-orbits on G. The latter model is used by others and is the one we adopted in the abstract.

In section 5 we establish the existence of slices:

Theorem 1.7 (see 5.11, 7.2, 3.4(9))**.** *Let $a \in A$. Then there is a reductive group $\hat{G}$ and an involution $\hat{\theta}$ with associated symmetric variety $\hat{P}$ such that*

(1) $(\hat{G})^{\hat{\theta}} = H_a^* = \{h \in H \mid h * a = a\}$.
(2) *There is an étale slice $\mathcal{S}$ at a which is isomorphic to an open H_a^*-stable subvariety of $\hat{P}$.*
(3) *There is a bijection between the (finitely many) orbits in $\pi_{\hat{P}}^{-1}(\pi_{\hat{P}}(a))$ and those in $\pi_P^{-1}(\pi_P(a))$.*

In section 6 we apply the slice theorem to establish the following variant of the Chevalley restriction theorem:

Theorem 1.8 (see 6.5)**.** *Let H act on P by $*$. Then the inclusion $A \to P$ induces an isomorphism $A/W_H^*(A) \simeq P/\!/H$. In particular:*

(1) *The closed H-orbits are exactly those which intersect A.*
(2) *If $a \in A$, then $(H * a) \cap A = W_H^*(A) * a$.*

In section 7 we study and compare natural stratifications of $P/\!/H \simeq A/W_H^*(A)$.

1.9. In sections 8 and 9 we begin a detailed analysis of $W_H^*(A)$ and its action on A: If $h \in N_H^*(A)$, then $h * e = \beta(h) \in A$, hence $\sigma(\beta(h)) = \beta(h)^{-1}$. But σ fixes h, hence fixes $\beta(h)$. Thus $\beta(h) \in A^{(2)}$, the elements of A of order 2. It follows that

Remark 1.10. $N_H^*(A) = \{h \in N_H(A) \mid \beta(h) \in A^{(2)}\}$.

We have a semidirect product $W_H(A) \ltimes A^{(2)}$ with a natural action on A: $(w, a) * a' := w(a')a$, $w \in W_H(A)$, $a \in A^{(2)}$, $a' \in A$. There is a canonical monomorphism

$$\rho\colon W_H^*(A) \to W_H(A) \ltimes A^{(2)}, \quad hZ_H^*(A) \mapsto (\mathrm{Int}(h), \beta(h)),$$

where $\mathrm{Int}(h)$ denotes conjugation by h.

There are special subgroups of $W_H^*(A)$ that play a role. First of all, there is the subgroup W_0 of "pure translations," with $W_0 = \rho^{-1}(\{e\} \times A^{(2)})$. We often identify W_0 with its image $F_0 \subset A^{(2)}$. At the other extreme is the subgroup $\rho^{-1}(W_H(A) \times \{e\})$. It is isomorphic to $W_M(A)$, where $M := K \cap H$. Finally, let F denote $\{a \in A^{(2)} \mid (w, a) \in \rho(W_H^*(A))$ for some $w \in W_H(A)\}$. To understand $W_H^*(A)$, we need to determine W_0, $W_M(A)$ and F (and more).

Using *quadratic elements* one can improve the embedding of $W_H^*(A)$ into $W_H(A) \ltimes A^{(2)}$: We say that $q \in A$ is *quadratic* if $q^2 \in Z(G)$. If q is quadratic, then one easily sees that σ, θ and $\operatorname{Int}(q)$ commute, hence $(\sigma, \tilde{\theta})$ is a pair of commuting involutions, where $\tilde{\theta} := \theta \operatorname{Int}(q)$. The analogues of K, $N_H^*(A)$, ρ, etc. for $(\sigma, \tilde{\theta})$ in place of (σ, θ) are denoted $\tilde{K}$, $\tilde{N}_H^*(A)$, $\tilde{\rho}$, etc.

Theorem 1.11 (see 9.3, 9.13)**.** *Let q be a quadratic element of A.*

(1) *$\tilde{N}_H^*(A) = N_H^*(A)$, $\tilde{Z}_H^*(A) = Z_H^*(A)$ and $\tilde{W}_H^*(A) = W_H^*(A)$ are independent of q.*
(2) *$\tilde{\rho}$ depends on q, and there is a choice of q such that $\tilde{\rho}(\tilde{W}_H^*(A)) = W_H(A) \ltimes F_0$. Equivalently, $W_{\tilde{M}}(A) = W_H(A)$, where $\tilde{M} = H \cap \tilde{K}$.*
(3) *We may choose q as in (2) such that $W_H(A) = W_{\tilde{M}^0}(A)$.*

1.12. We can characterize F_0 and F in terms of tori, as follows: Let $T \subset G$ be a (σ, θ)-stable maximal torus containing A. We denote by $T_{+-}^{\sigma,\theta}$ the maximal θ-split and σ-fixed torus in T, and similarly for $T_{--}^{\sigma,\theta} = A$, $T_{+-}^{\sigma,\theta}$ and $T_{++}^{\sigma,\theta}$. Then $T = T_{++}^{\sigma,\theta} T_{+-}^{\sigma,\theta} T_{-+}^{\sigma,\theta} T_{--}^{\sigma,\theta}$. We say that T is *standard* if

- $T_{+-}^{\sigma,\theta} A$ is a maximal θ-split torus of G.
- $T_{-+}^{\sigma,\theta} A$ is a maximal σ-split torus of G.

Standard maximal tori exist [Hel88, 5.13]. Set $\tau = \sigma\theta$ and denote $T_{+-}^{\sigma,\theta} T_{-+}^{\sigma,\theta}$ by T_-^τ.

Theorem 1.13 (see 8.8, 8.12 and 8.15)**.** *Let $T \supset A$ be a standard maximal torus. Then*

(1) *$F_0 = A \cap T_-^\tau$.*
(2) *Assume that $\beta(H) \subset H$ is connected (e.g., H is connected). Then $F = \{a \in A^{(2)} \mid a$ is contained in a θ-split torus of $Z_H(A)\}$.*

1.14. In section 9 a major role is played by the weight spaces $\mathfrak{g}_\lambda$, λ a character of A. The involutions σ and θ interchange $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda^{-1}}$ while $\tau = \sigma\theta$ leaves each $\mathfrak{g}_\lambda$ invariant. Let $m^{\pm}(\lambda, \tau)$ denote the dimension of the ± 1-eigenspace of τ on $\mathfrak{g}_\lambda$. The *signatures* $(m^+(\lambda, \tau), m^-(\lambda, \tau))$ are closely tied to the study of quadratic elements and characterizations of F and F_0.

1.15. In section 11 we consider real analogues of our results over $\mathbb{C}$. We recover much of [Ber57, FJ78, Hoo84, Mat97].

Let X be an affine variety defined over $\mathbb{R}$. Then $X(\mathbb{R})$ or $X_\mathbb{R}$ will denote the real points of X. We assume in the following that G, σ and θ are defined over $\mathbb{R}$.

Theorem 1.16 (see 11.7)**.** *Suppose that $G_\mathbb{R}$ is compact. Then $G_\mathbb{R} = H_\mathbb{R} A_\mathbb{R} K_\mathbb{R}$, $\beta(A_\mathbb{R}) = A_\mathbb{R}$ and $W_H^*(A) = W_{H(\mathbb{R})}^*(A_\mathbb{R})$. Hence $H_\mathbb{R} \backslash G_\mathbb{R} / K_\mathbb{R} \simeq A_\mathbb{R} / W_H^*(A)$.*

Theorem 1.17 (see 11.11)**.** *Suppose that θ is a Cartan involution of $G_\mathbb{R}$. Then*

(1) $G_{\mathbb{R}} = H^0_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$.
(2) $A_{\mathbb{R}}$ *is isomorphic to* $(\mathbb{R}^*)^l$ *for some* l. *Hence* $A^2_{\mathbb{R}} := \beta(A_{\mathbb{R}}) = \{a^2 \mid a \in A_{\mathbb{R}}\} \simeq (\mathbb{R}^+)^l$.
(3) $H_{\mathbb{R}} \backslash G_{\mathbb{R}} / K_{\mathbb{R}} \simeq A^2_{\mathbb{R}} / W_{M(\mathbb{R})}(A)$.

1.18. If σ and θ are (noncommuting) involutions such that σ and $\rho\theta\rho^{-1}$ commute for some choice of $\rho \in \mathrm{Aut}(G)$, then our results hold for $H\backslash G/K$. In section 12 we find conditions on σ and θ for such a ρ to exist. In these cases we recapture results of [Mat97].

At the end of this paper there is an index of notation.

1.19. The authors thank T.A. Springer for suggesting this project to us.

2. Preliminaries

2.1. Let k be an algebraically closed field. In this section only we allow $\operatorname{char} k = 2$. Using ideas of Richardson, we prove versions of some of Luna's characteristic zero results on closed orbits.

All groups, varieties, etc. will be defined over k. All finite dimensional representations of algebraic groups that we consider will be assumed rational. Let X be a smooth subvariety of an algebraic group L. We denote by X^0 the connected component of X containing the identity e (assuming $e \in X$).

Let L be a linear algebraic group, let $x \in L$ and let $\mathfrak{l}$ denote the Lie algebra of L. Then we will use $\mathrm{conj}(x)y$, $x \cdot y$ and $\mathrm{Int}(x)y$ to denote xyx^{-1}, $y \in L$ (or $\mathrm{Ad}\,(x)y$, $y \in \mathfrak{l}$). If $Z \subset L$, then $Z \cdot y$ will denote $\{x \cdot y \mid x \in Z\}$.

Most of the time we use additive notation for multiplication of characters of a torus, but on occasion we will use multiplicative notation (or even mix the two) when multiplicative notation is clearer.

We will have to deal with nonconnected linear algebraic groups L. We say that L is *reductive* if L^0 is, i.e., if $R_u(L) = \{e\}$. The symbol G will always denote a reductive algebraic group.

2.2. If X is an affine G-variety, G-reductive, then $\mathcal{O}(X)^G$, the algebra of invariant functions on X, is finitely generated [Hab75]. Let $X/\!\!/G$ denote the affine variety corresponding to $\mathcal{O}(X)^G$ and let π (or π_X) denote the morphism $X \to X/\!\!/G$ dual to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$. If $Y \subset X$, then $\mathrm{clos}(Y)$ denotes the Zariski closure of Y in X. For $x \in X$, let T_xX denote the tangent space at x, Gx the G-orbit through x and G_x the isotropy group at x. We say that the G-action on X is *stable* if there is a nonempty open subset of X consisting of closed orbits.

Theorem 2.3. *Let X, etc. be as above. Then*

(1) *The morphism* π_X *is surjective.*
(2) $\mathcal{O}(X)^G$ *separates disjoint G-stable closed subsets of X.*
(3) *If* $x \in X$, *then* $\mathrm{clos}(Gx) \setminus Gx$ *is a union of orbits of dimension less than* $\dim Gx$.

(4) *Let $\xi \in X/\!\!/G$. Then $\pi_X^{-1}(\xi)$ contains a unique closed orbit $T(\xi)$, where $T(\xi)$ is the lowest dimensional orbit in $\pi_X^{-1}(\xi)$. Thus there is a one to one correspondence between $X/\!\!/G$ and the closed orbits in X.*
(5) *If X is normal, then so is $X/\!\!/G$.*
(6) *If X irreducible and the G-action is stable, then π_X is separable.*

Let $f: Y \to X$ be a G-morphism of affine G-varieties.

(7) *There is a unique morphism $f/\!\!/G: Y/\!\!/G \to X/\!\!/G$ with the property that $\pi_X \circ f = f/\!\!/G \circ \pi_Y$.*
(8) *Suppose further that $f: Y \to X$ is inclusion of a closed (G-stable) subset. Then $f/\!\!/G: Y/\!\!/G \to X/\!\!/G$ is finite and $\pi_X(Y) = (f/\!\!/G)(Y/\!\!/G)$ is closed in $X/\!\!/G$.*

Parts (1), (2), (7) and (8) can be found in [MFK94, Ch. 1 §2 and Appendix to Ch. 1]. Parts (3) and (5) are elementary (and also in [MFK94]), and (1), (2) and (3) imply (4). Finally, (6) can be found in [Ric82b, 9.3]

If all the G-orbits in X are closed, then the quotient is called *geometric*, in which case the notation X/G is also used.

2.4. Suppose that H is a reductive subgroup of G and that Z is an affine H-variety. Define $G \times^H Z$ to be the (geometric) quotient $(G \times Z)/H$ where $h \in H$ sends $(g, z) \in G \times Z$ to (gh^{-1}, hz). Then $G \times^H Z$ is an affine G-variety with quotient $(G \times^H Z)/\!\!/G \simeq Z/\!\!/H$. The image of $(g, z) \in G \times Z$ in $G \times^H Z$ is denoted $[g, z]$.

2.5. Let X and Y be affine G-varieties. A G-morphism $\varphi : X \to Y$ is said to be *excellent* if

(1) φ is étale,
(2) the induced morphism $\varphi/\!\!/G : X/\!\!/G \to Y/\!\!/G$ is étale, and
(3) the morphism $(\varphi, \pi_X) : X \to Y \times_{Y/\!\!/G} X/\!\!/G$ is an isomorphism.

Let $x \in X$, let $\mathcal{O}_{X,x}$ denote the local ring at x, and let $\hat{\mathcal{O}}_{X,x}$ denote its completion. Then φ is étale at $x \in X$ if it induces an isomorphism of completions of local rings $\hat{\mathcal{O}}_{Y,\varphi(x)} \simeq \hat{\mathcal{O}}_{X,x}$. If $k = \mathbb{C}$, this is equivalent to φ inducing an analytic isomorphism of a neighborhood of x to a neighborhood of $\varphi(x)$. If X and Y are smooth, then φ is étale at x if and only if $d\varphi_x: T_xX \to T_{\varphi(x)}Y$ is an isomorphism. We say that φ is étale if it is étale at all points of X.

Part (3) above indicates that, although φ may fail to be an isomorphism, it induces a G-isomorphism of fibers $\pi_X^{-1}(\pi_X(x)) \simeq \pi_Y^{-1}(\pi_Y(\varphi(x)))$ for all $x \in X$.

2.6. Let X be an affine G-variety and let $x \in X$. We say that the orbit Gx is *separable* if the canonical morphism $\varphi: G \to Gx$, $g \mapsto gx$, has surjective differential $d\varphi_e: \mathfrak{g} \to T_xGx$. If Gx is separable and closed, we say that Gx is a *good orbit* and that x is a *good point*.

Let $x \in X$ be a good point, and let $\mathcal{S}$ be a G_x-stable locally closed subvariety of X containing x. We say that $\mathcal{S}$ *is an (étale) slice at* x if the canonical map $\varphi : G \times^{G_x} \mathcal{S} \to X$, $(g, s) \mapsto gs$, is excellent.

Let $x \in X$ be a smooth good point. Then the representation of G_x on $T_x(X)/T_x(Gx)$ is called the *slice representation at* x. We say that a G_x-stable connected locally closed smooth subvariety $R \ni x$ is a *transversal* to Gx at x if $T_x(X) = T_x(Gx) \oplus T_x(R)$.

The following theorem is due to Bardsley and Richardson [BR85], using techniques based on [Lun73].

Theorem 2.7. *Let X be an affine G-variety and $x \in X$ a good point. Then there is an étale slice $\mathcal{S}$ at x if*

(1) *G_x is linearly reductive, or*
(2) *$x \in X$ is a smooth point and there exists a transversal R at x.*

In each case we may arrange that $\mathcal{S}$ and $G \cdot \mathcal{S}$ are affine.

We will use the following version of a result of Richardson. Our main application will be to groups of the form $\tilde{G} = L \ltimes G$ where G is reductive and L is a 2-group (hence linearly reductive).

Theorem 2.8 ([Ric82a])**.** *Let $\tilde{G}$ be a reductive algebraic group generated by subgroups G and L, where G is reductive, L is linearly reductive and L normalizes G. Let Z denote the connected centralizer $C_G(L)^0$ and let X be an affine $\tilde{G}$-variety.*

(1) *Let $x \in X^L$, and let Gy denote the unique closed orbit in* $\operatorname{clos}(Gx)$*. Then there is a one-parameter group $\lambda\colon k^* \to Z$ such that* $\lim_{t\to 0} \lambda(t)x$ *exists and is a point of Gy. In particular, Gx is closed if and only if Zx is closed.*
(2) *Suppose that G acts transitively on X. Then Z acts transitively on each connected component of X^L. In particular, X^L consists of finitely many closed Z-orbits.*

Richardson assumes that G is connected, but this is unnecessary. The cases of most interest to us will be $\tilde{G} = L \ltimes G$ where L is a subgroup of $\{e, \theta, \sigma, \theta\sigma\}$.

We will use the following result in section 6 to prove our variant of the Chevalley restriction theorem:

Corollary 2.9. *Let $\tilde{G}$, G, etc. be as in Theorem 2.8. Then the inclusion $X^L \to X$ induces a finite morphism $\varphi\colon X^L/\!\!/C_G(L) \to X/\!\!/G$.*

Proof. We may embed X equivariantly into a $\tilde{G}$-module V. By 2.3(8), $X^L/\!\!/C_G(L) \to V^L/\!\!/C_G(L)$ is finite. Suppose that the canonical morphism $\psi\colon V^L/\!\!/C_G(L) \to V/\!\!/G$ is finite. Then the composition $\tilde{\psi}\colon X^L/\!\!/C_G(L) \to V^L/\!\!/C_G(L) \to V/\!\!/G$ is finite. But $\tilde{\psi}$ factors as $X^L/\!\!/C_G(L) \to X/\!\!/G \to V/\!\!/G$, hence $X^L/\!\!/C_G(L) \to X/\!\!/G$ is finite. Thus we may reduce to the case that $X = V$.

Let $\mathcal{N}(V, G) := \pi_V{}^{-1}(\pi_V(0))$ denote the null cone of V. Let $v \in \mathcal{N}(V, G) \cap V^L$ and let $C_G(L)w$ be the closed orbit in $\operatorname{clos}(C_G(L)v)$. Then $Gw \subset \mathcal{N}(V, G)$ is closed by 2.8, hence $w = 0$. Thus $\mathcal{N}(V, G) \cap V^L \subset \mathcal{N}(V^L, C_G(L))$. The reverse inclusion is obvious, hence $\mathcal{O}(V^L)^{C_G(L)}$ is finite over $\mathcal{O}(V)^G$. □

3. Involutions and Symmetric Varieties

3.1. We assume from now on that char $k \neq 2$ and that our reductive group G is *connected*. (Sometimes we can relax the latter condition to “almost” connected, see 3.8). We now recall results of Vust and Richardson for the symmetric case.

3.2. Let σ, θ be commuting involutions of G and denote the fixed point groups by $H := G^\sigma$ and $K := G^\theta$. By Richardson [Ric82a, 10.5.1], K and H are always reductive. Recall (1.5) the morphism $\beta\colon G \to P = P_\theta(G)$, $g \mapsto g\theta(g)^{-1}$, which induces an isomorphism $G/K \simeq P$. The left action of H on G/K transforms to the $*$ action on P, our main object of study. One can also replace H by an open subgroup; see Remark 3.6 below.

3.3. For the rest of this section we consider only the symmetric space case: $\sigma = \theta$. A torus $S \subset G$ is *split* (more precisely, *θ-split*) if $\theta(s) = s^{-1}$ for all $s \in S$. Clearly, there are split tori which are maximal among split tori; we call them *maximal split tori*.

For a torus T of G and a subgroup L of G we define the Weyl group $W_L(T)$ to be $N_L(T)/Z_L(T)$, a finite group, and we write $X^*(T)$ for the character group of T. Set $\mathfrak{l}_\lambda = \{X \in \mathfrak{l} \mid t \cdot X = \lambda(t)X \text{ for all } t \in T\}$ and $\Phi(T, L) = \{\lambda \in \mathfrak{t}^* \mid \mathfrak{l}_\lambda \neq 0 \text{ and } \lambda \neq 0\}$, the set of nonzero weights of T with respect to L. Then $\mathfrak{l} = \mathfrak{l}^T \oplus \bigoplus_{\lambda \in \Phi(T,L)} \mathfrak{l}_\lambda$. If $L = G$ we will also write $\Phi(T)$ instead of $\Phi(T, G)$. In cases important to us, $\Phi(T, L)$ will turn out to be a root system with Weyl group $W_L(T)$:

Let S be a maximal θ-split torus of G. Set $S_0 := (S \cap [G, G])^0$ and let X_0 denote the set of characters of S which are trivial on S_0. Then by Richardson [Ric82b], $\Phi(S, G)$ is a root system in $(X^*(S), X_0)$ in the sense of [BT65, §2.1]. In particular, $\Phi(S, G)$ induces a (perhaps nonreduced) root system [Bou81, Ch. 6, §1] on the real vector space it generates (see also 9.5).

Theorem 3.4 (Vust [Vus74], Richardson [Ric82b])**.** *Let G, K, $P = P_\theta(G)$, S, etc. be as above, where S is maximal θ-split and G is connected. Then*

1. $P = \{x \in G \mid \theta(x) = x^{-1}\}^0$.
2. *If θ acts nontrivially on G, then $S \neq \{e\}$. In particular, the θ-action on $Z_G(S)/S$ is trivial.*
3. *Any torus of G containing S is θ-stable.*
4. *Any two maximal θ-split tori are conjugate by an element of K^0.*
5. *$W_G(S) \simeq W_K(S)$, i.e., every element of $W_G(S)$ has a representative in $N_K(S)$.*
6. *$K = (S \cap K)K^0$, hence $W_K(S) \simeq W_{K^0}(S)$.*
7. *The Weyl group of the root system $\Phi(S)$ is $W_{K^0}(S)$.*
8. *Let $x \in P$. Then Kx is closed iff x is a semisimple element of G, and x lies in the fiber $\pi_P^{-1}(\pi_P(e))$ iff x is unipotent.*
9. *Every fiber of $\pi_P : P \to P/\!\!/K$ contains only finitely many orbits.*

(10) *The closed orbits of K in P are exactly those which intersect S.*
(11) *The inclusion $S \to P$ induces an isomorphism $S/W_K(S) \simeq P/\!/K$.*
(12) *Let $Y \subset S$ such that $k \cdot Y \subset S$ for some $k \in K$. Then there is a $k' \in N_K(S)$ such that $k \cdot y = k' \cdot y$ for all $y \in Y$.*

There is more in [Vus74] and [Ric82b] than we quote above. We generalize much of 3.4, and we give new proofs of (10)–(11).

Corollary 3.5 (Vust, Richardson)**.** *Let S be a maximal θ-split torus in G.*

(1) *Let $g \in G$. Then $g^{-1} \cdot S$ is θ-split if and only if there is a $k \in K^0$ such that $gk \in Z_G(S)$.*
(2) $N_G(S) = SN_{K^0}(S)$.
(3) $Z_G(S) = SZ_{K^0}(S)$.

Proof. Part (1) follows from 3.4(4)–(6), while (3) follows from (2). If $g \in N_G(S)$ and k is as in (1), then $k \in N_{K^0}(S)$ and we have (2). □

Remarks 3.6. (1) An open subgroup $K^\# \subset K$ differs from K^0 by $K^\# \cap S$ which acts trivially on S by conjugation. Thus the quotients $P/\!/K^\#$ and $P/\!/K^0$ are the same, although the fibers can have $K^\#$-orbits which break into finitely many K^0-orbits. Such orbits are necessarily nonclosed.

(2) If G is simply connected, then K is connected [Ste68].

There is another way to characterize when $g^{-1} \cdot S$ is θ-split:

Lemma 3.7. *Let T be a θ-stable torus in G and S a θ-split torus in G. Let $g \in G$ and set $K := G^\theta$.*

(1) *$\beta(g) \in N_G(T)$ if and only if $g^{-1} \cdot T$ is θ-stable.*
(2) *$\beta(g) \in Z_G(S)$ if and only if $g^{-1} \cdot S$ is θ-split.*

Proof. Note that $g^{-1} \cdot T$ is θ-stable if and only if $g^{-1} \cdot T = \theta(g^{-1}) \cdot T$. The latter is equivalent to $\mathrm{Int}(g) \circ \mathrm{Int}(\theta(g)^{-1})T = T$, i.e., equivalent to $\beta(g) \in N_G(T)$. The proof of (2) is similar. □

Remark 3.8. We will encounter some cases where G is not connected, but it will turn out that $G = G^0K = KG^0$. We say that (G, θ) is *almost connected*. Equivalently, $P_\theta(G) = P_\theta(G^0)$. In this case, we need to modify 3.4–3.7 as follows:

(1) Replace G by G^0 in 3.4(2).
(2) 3.4(6) is no longer valid.
(3) In 3.5, replace K^0 by K.

4. Tori and Weyl groups

We study the appropriate tori and Weyl groups corresponding to our commuting involutions.

4.1. A torus A in G is (σ,θ)-*split* if it is θ-split and σ-split. Nontrivial (σ,θ)-split tori do not always exist. For example if $G = G_1 \times G_2$, $\sigma|_{G_1} = \mathrm{id}$ and $\theta|_{G_2} = \mathrm{id}$, then G has no nontrivial (σ,θ)-split tori. We recall properties of (σ,θ)-split tori from [Hel88, §5 and §6]; note the resemblance of Theorem 4.3 below to 3.4.

Set $\tau := \sigma\theta$ and $M := H \cap K$. Let $\mathfrak{g}^\sigma_\pm$ denote the $+1$ (resp. -1)-eigenspace of σ, and define $\mathfrak{g}_{+,-} = \mathfrak{g}^\sigma_+ \cap \mathfrak{g}^\theta_-$, etc. Then $\mathfrak{g} = \mathfrak{g}_{+,+} \oplus \mathfrak{g}_{+,-} \oplus \mathfrak{g}_{-,+} \oplus \mathfrak{g}_{-,-}$.

Using 3.4(2) one easily shows:

Lemma 4.2. *The following are equivalent:*

(1) $A = \{e\}$.
(2) $(G^\tau)^0$ *contains no non-trivial* σ*-split tori.*
(3) σ *acts trivially on* $(G^\tau)^0$.
(4) $\mathfrak{g}^\tau = \mathfrak{g}_{+,+}$.
(5) $\mathfrak{g}_{-,-} = (0)$.
(6) *Every maximal* σ*-split torus of* G *lies in* G^θ.

Theorem 4.3. *Let* G, K, H, *etc. be as above and let* A *be a maximal* (σ,θ)*-split torus. Then*

(1) *If* G *is simple and both* θ *and* σ *act nontrivially, then* $A \neq \{e\}$.
(2) *Maximal* θ*-split (or maximal* σ*-split) tori of* G *containing* A *are* θ*-stable and* σ*-stable.*
(3) *Any two maximal* (σ,θ)*-split tori are conjugate by an element of* M^0.
(4) $\Phi(A)$ *is a root system with Weyl group* $W_{K^0}(A)$.

Proof. Let S be a maximal τ-split torus in G. Then the Weyl group of $\Phi(S)$ has representatives in $(G^\tau)^0$. If $A = \{e\}$, then $\mathrm{Ad}(G^\tau)^0 = \mathrm{Ad}\, M^0$ preserves $\mathfrak{g}_{+,-}$ and $\mathfrak{g}_{-,+}$. By ([Hel88, 7.17]), $\Phi(S)$ cannot be the union of two mutually orthogonal nontrivial σ-stable and θ-stable subsets, and (1) is immediate. The equivalence of 4.2(1) and 4.2(6) shows that any maximal τ-split torus S of $Z_G(A)/A$ splits as $(S^\sigma)^0(S^\theta)^0$, and (2) follows. Since maximal (σ,θ)-split tori are maximal σ-split tori of $(G^\tau)^0$, we obtain (3) from 3.4(4). For (4), see [Hel88, §6]. □

Analogously to Corollary 3.5 one establishes

Corollary 4.4. (1) $M = M^0 N_M(A)$.
(2) $W_{M^0}(A) = W(A, (G^\tau)^0) = W_{K\cap H^0}(A)$.
(3) (H,θ) *is almost connected (see 3.8) iff* $H = H^0 N_M(A)$.

4.5. **Another viewpoint.** Let A be a maximal (σ,θ)-split torus in G. In the following it will be useful to use a second copy $\tilde{A}$ of A and to consider the "squaring" homomorphism $\tilde{A} \to A$, $a \mapsto a^2$. Recall the definition of $N^*_H(A)$, $Z^*_H(A)$ and $W^*_H(A)$ from 1.6 and 1.9

As in the abstract and in [Hoo84] and [Mat97], we can construct a Weyl group as follows: We have the action $(H \times K) \times G \to G$, $((h,k),g) \mapsto hgk^{-1}$. Let $\tilde{Z}_{H\times K}(\tilde{A})$

(resp. $\tilde{N}_{H\times K}(\tilde{A})$) denote the centralizer (resp. normalizer) of $\tilde{A}$ for this action and set $\tilde{W}_{H\times K}(\tilde{A}) := \tilde{N}_{H\times K}(\tilde{A})/\tilde{Z}_{H\times K}(\tilde{A})$. If $(h, k) \in \tilde{N}_{H\times K}(\tilde{A})$, then we denote its class in $\tilde{W}_{H\times K}(\tilde{A})$ by $[h, k]$. Now $\beta\colon \tilde{A} \to A$ has kernel $\tilde{A}^{(2)}$, so we obtain a (faithful) action of $W_{H\times K}(A) := \tilde{W}_{H\times K}(\tilde{A})/\tilde{A}^{(2)}$ on $A = \tilde{A}/\tilde{A}^{(2)}$. Let $\tilde{W}_{H\times K,0}$ denote $\{[h, k] \in \tilde{W}_{H\times K}(\tilde{A}) \mid h \in Z_H(\tilde{A})\}$, and set $W_{H\times K,0} = \tilde{W}_{H\times K,0}/\tilde{A}^{(2)}$.

Theorem 4.6. *There is an isomorphism of groups acting on A, $\eta\colon W_{H\times K}(A) \xrightarrow{\sim} W^*_H(A)$, $[h, k] \mapsto [h]$. Moreover, $\eta(W_{H\times K,0}) = W_0$.*

Proof. Let $(h, k) \in \tilde{W}_{H\times K}(\tilde{A})$. Then $\tilde{b} := hk^{-1} \in \tilde{A}$, and $\tilde{b}^{-1} = kh^{-1} = \theta(\tilde{b}) = \theta(h)k^{-1}$. If $\tilde{a} \in \tilde{A}$, then $h\tilde{a}k^{-1} = h\tilde{a}\theta(h)^{-1}\theta(h)k^{-1} = (h * \tilde{a})\tilde{b}^{-1} \in \tilde{A}$, so $h \in N^*_H(\tilde{A}) = N^*_H(A)$. If $h \in Z_H(A)$, then $[h] \in W_0$. If $h \in Z^*_H(A)$, then $\tilde{b} = hk^{-1} = \theta(h)k^{-1} = \tilde{b}^{-1}$, so that $\tilde{b} \in \tilde{A}^{(2)}$, and $[h, k]$ has trivial image in $\tilde{W}_{H\times K}(\tilde{A})/\tilde{A}^{(2)}$. Conversely, given $[h] \in W^*_H(A)$, choose $s \in A$ such that $s^2 = \beta(h)$. Then $k := s^{-1}h \in K$ and $[h, k] \in \tilde{W}_{H\times K}(\tilde{A})$ with $\eta([h, k]) = [h]$. □

Remark 4.7. If we compute $G/\!\!/(H \times K)$ by first quotienting by H, then we are considering the closed K-orbits on $P_\sigma(G)$. The Weyl group of A with respect to K is isomorphic to $W_{H\times K}(A) \simeq W^*_H(A)$.

5. Slices, slices everywhere

5.1. We use the slice theorem to show that the action of H on $P = P_\theta(G)$ locally resembles an action arising from a symmetric variety.

Lemma 5.2. *Let $x \in P$ such that $\sigma(x) = x^{-1}$. Then the following are equivalent*

(1) *x is semisimple.*
(2) *$K \cdot x$ is closed.*
(3) *$M \cdot x$ is closed.*
(4) *$H * x$ is closed.*

Proof. The equivalences of (r) and $(r + 1)$, $1 \le r \le 3$ follow from:

(1) Theorem 3.4(8).
(2) Theorem 2.8 applied to $\{e, \tau\} \ltimes K$.
(3) Theorem 2.8 applied to $\{e, \tau\} \ltimes H$.

□

Remark 5.3. The equivalences above are false if $\sigma(x) \neq x^{-1}$: Let $T \subset H$ (resp. $U \subset H$) be a θ-stable maximal torus (resp. (T, θ)-stable maximal unipotent subgroup). If θ acts nontrivially on the commutator subgroup (H^0, H^0), then it acts nontrivially on U, hence there is a $u' \in U$ such that $u := \beta(u') \neq e$. It follows that $u * e = u^2$ is unipotent and nontrivial (since char $k \neq 2$). Thus $H * e$ contains nontrivial unipotent elements. Of course, $\sigma(u^2) = u^{-2} \neq u^2$.

In Example 7.4 below, with $n = 2$, one has $H = \mathrm{SL}_2$, $A = \{\left(\begin{smallmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{smallmatrix}\right)\}$, $T = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\}$ and θ acts on $U = \{u(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\}$ by sending t to $-t$. Then $H * e$ contains U.

5.4. Let $x \in P$ be σ-split (equivalently, τ-fixed). We show that there is a slice at x if $H * x$ is closed (in which case x is semisimple). We translate our problem to the identity via multiplication by x^{-1} and compute $T_e(Px^{-1})$ and $T_e((H * x)x^{-1})$. We show that a complement to $T_e((H * x)x^{-1}) \subset T_e(Px^{-1})$ is $T_e\tilde{P}$, where $\tilde{P}$ is a symmetric space of a group $\tilde{G}$. This gives us the transversal required to apply the Bardsley-Richardson version of Luna's slice theorem.

We have a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the ± 1-eigenspaces of θ and $\mathfrak{k} = \mathrm{Lie}(K)$. Similarly, we have a σ-eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. We have the translated "orbit mapping" $\psi : G \to Px^{-1}$ sending $g \in G$ to $(g * x)x^{-1}$. We denote its derivative at e by φ. Then $\varphi(X) = X - x \cdot \theta(X)$, $X \in \mathfrak{g}$. We will denote the isotropy group of H at x by H^*_x.

Lemma 5.5. *Let x etc. be as above. Then*

(1) $Px^{-1} = \{y \in G \mid \theta(y) = x^{-1} \cdot y^{-1}\}^0$.
(2) $T_e(Px^{-1}) = \{X \in \mathfrak{g} \mid X = -x \cdot \theta(X)\}$.
(3) *If $h \in H^*_x$ and $y \in Px^{-1}$, then $h * (yx) = (h \cdot y)x$.*
(4) *H^*_x acts on $T_e(Px^{-1})$ by: $h, X \mapsto h \cdot X$.*
(5) $(H * x)x^{-1} = \{hx\theta(h)^{-1}x^{-1} \mid h \in H\}$.
(6) $T_e((H * x)x^{-1}) = \varphi(\mathfrak{h}) = \{X - x \cdot \theta(X) \mid X \in \mathfrak{h}\}$.
(7) *x is a good point.*

Proof. If $y \in Px^{-1}$, then $\theta(yx) = (yx)^{-1}$, and (1) and (2) follow easily. If $h \in H^*_x$, then $h * (yx) = hyx\theta(h)^{-1} = hyh^{-1}x$, giving (3) and (4). Parts (5) and (6) are trivial, and (6) implies that $H * x$ is separable, hence x is good. □

5.6. Write $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$, where $\mathrm{Ad}\,x$ acts with eigenvalue λ on $\mathfrak{g}_\lambda$. Since θ and σ send x to x^{-1}, they interchange the eigenspaces $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda^{-1}}$. We have a direct sum decomposition

$$\mathfrak{g} = (\mathfrak{g}_{\pm 1} := \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}) \oplus (\mathfrak{g}' := \bigoplus_{\lambda \neq \pm 1} (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{\lambda^{-1}}))$$

which induces a direct sum decomposition $\mathfrak{h} = \mathfrak{h}_{\pm 1} \oplus \mathfrak{h}'$, and similarly for $T_e(Px^{-1})$ and $\mathfrak{k}$.

Let λ be an eigenvalue of $\mathrm{Ad}(x)$. Since τ commutes with $\mathrm{Ad}\,x$, we may choose a basis of τ-eigenvectors $x_1, \dots, x_r$ for $\mathfrak{g}_\lambda$, where $\tau(x_i) = \epsilon(i)x_i$, $\epsilon(i) \in \{\pm 1\}$, $i = 1, \dots, r$. Note that $\{\sigma(x_i)\}$ is a basis of $\mathfrak{g}_{\lambda^{-1}}$.

We now show that a complement to $T_e((H * x)x^{-1}) \subset T_e(Px^{-1})$ is $(\mathfrak{p} \cap \mathfrak{q})_1 \oplus (\mathfrak{k} \cap \mathfrak{q})_{-1}$.

Proposition 5.7. *Let φ and λ be as above. Then*

(1) *φ is multiplication by* 2 *on* $\operatorname{Im}\varphi = T_e(Px^{-1})$.
(2) $\varphi(\sigma(x_i)) = -\epsilon(i)\lambda\varphi(x_i)$, $i = 1, \ldots, r$.
(3) *If* $\lambda \neq \pm 1$, *then* $\varphi(x_i + \sigma(x_i))$ *is a nonzero multiple of* $\varphi(x_i)$.
(4) $\varphi(\mathfrak{h}') = T_e(Px^{-1})'$.
(5) $\operatorname{Lie}(H_x^*) = \operatorname{Ker}(\varphi|_{\mathfrak{h}}) = (\mathfrak{h} \cap \mathfrak{k})_1 \oplus (\mathfrak{h} \cap \mathfrak{p})_{-1}$.
(6) $T_e(Px^{-1}) = \varphi(\mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})_1 \oplus (\mathfrak{k} \cap \mathfrak{q})_{-1}$.

Proof. Parts (1), (2) and (3) follow by direct calculation, and they clearly imply (4). For (5) and (6), one computes that φ kills $(\mathfrak{h} \cap \mathfrak{k})_1 \oplus (\mathfrak{h} \cap \mathfrak{p})_{-1}$ and is multiplication by 2 on $(\mathfrak{h} \cap \mathfrak{k})_{-1} \oplus (\mathfrak{h} \cap \mathfrak{p})_1$. That leaves only $(\mathfrak{p} \cap \mathfrak{q})_1 \oplus (\mathfrak{k} \cap \mathfrak{q})_{-1}$ for the complement of $\varphi(\mathfrak{h}) \subset T_e(Px^{-1})$. □

5.8. Set $\tilde{\sigma} = \sigma$, $\tilde{\theta} = \theta \operatorname{Int}(x)$, $\tilde{\tau} = \tilde{\sigma}\tilde{\theta} = \tau \operatorname{Int}(x) = \operatorname{Int}(x)\tau$, $\tilde{G} = G^{\tilde{\tau}}$, $\tilde{H} := \tilde{G}^{\tilde{\sigma}}$ and $\tilde{P} := P_{\tilde{\theta}}(\tilde{G})$. Of course, $\tilde{h} \in \tilde{H}$ acts on $\tilde{P}$ by $\tilde{*}$, where $\tilde{h}\tilde{*}\tilde{y} = \tilde{h}\tilde{y}\tilde{\theta}(\tilde{h})^{-1}$, $\tilde{y} \in \tilde{P}$. The symmetric variety $\tilde{P}$ is (after translating and throwing away some components) the transversal we are looking for.

Lemma 5.9. *Let $\tilde{G}$, $\tilde{\tau}$, etc. be as above. Then*
(1) $x^2 \in Z(\tilde{G})$.
(2) $\tilde{\theta} = \tilde{\sigma}$ *on* $\tilde{G}$.
(3) $\tilde{H} = H_x^*$
(4) *The $\tilde{*}$-action of $\tilde{H}$ on $\tilde{P}$ is ordinary conjugation.*
(5) *$T_x(\tilde{P}x)$ is complementary to $T_x(H * x)$ in $T_x(P)$.*

Proof. Since $\tau(g) = x^{-1} \cdot g$ for $g \in \tilde{G}$, $g = x^{-1} \cdot x^{-1} \cdot g$, and (1) follows. Part (2) is immediate. Now $\tilde{H} = \tilde{G}^{\tilde{\sigma}} = (G^{\sigma})^{\tilde{\theta}}$, and for $h \in H$, $h \in \tilde{H}$ iff $\theta(h) = x^{-1} \cdot h$, which is equivalent to $h \in H_a^*$, and we have (3). Part (4) holds since $\tilde{\theta}$ is the identity on $\tilde{H}$.

Since $x^2 \in Z(\tilde{G})$, $T_e(\tilde{P}) \subset \mathfrak{g}_{\pm 1}$ and $T_e(\tilde{P}) = \{X \in \mathfrak{g}_{\pm 1} \mid \sigma(X) = -X \text{ and } \theta(X) = -x \cdot X\}$. Thus $T_e(\tilde{P}) = (\mathfrak{q} \cap \mathfrak{p})_1 \oplus (\mathfrak{q} \cap \mathfrak{k})_{-1}$. By 5.7, $T_x(\tilde{P}x)$ is complementary to $T_x(H * x)$. □

5.10. Let $\hat{P}(x)$ (or just $\hat{P}$) denote the component of $P_{\tilde{\theta}}(\tilde{G})$ containing e and set $\hat{G}(x)$ (or just $\hat{G}$) $:= \{g \in \tilde{G} \mid g\tilde{\theta}(g)^{-1} \in \hat{P}(x)\}$. Then $\hat{G}$ is an open subgroup of $\tilde{G}$ and $(\hat{G}, \tilde{\theta})$ is almost connected (3.8).

Theorem 5.11. *Let $\hat{G}$, etc. be as above. Then $(\hat{G})^{\tilde{\theta}} = \tilde{H}$, and we have an isomorphism $(P_{\tilde{\theta}}(\hat{G}), \tilde{H}) = (\hat{P}, \tilde{H}) \xrightarrow[\sim]{\nu} (\hat{P}x, \tilde{H})$, where $\nu(y) = yx$ and $\nu(\tilde{h} \cdot y) = \tilde{h} * (yx)$, $y \in \hat{P}$, $\tilde{h} \in \tilde{H} = H_x^*$. Thus $R_x := \hat{P}x$ is a transversal at $x \in P$, and the H-action has a slice at x.*

Proof. Since $\tilde{H}$ maps to $\{e\} \in \tilde{P}$, it must lie in $\hat{G}$, and we get that $\tilde{H} = \hat{G}^{\tilde{\theta}}$. The rest follows from 5.9. □

Corollary 5.12. *Let $x \in P$ with $\tau(x) = x$ and $H * x$ closed (equivalently, $M \cdot x$ closed). Then x is good and*

(1) *The maximal $(\tilde\sigma = \tilde\theta)$-split tori in $\hat G \subset \tilde G$ have dimension* $\dim A$.
(2) *If $\sigma = \theta$, then the transversal at x arises from an open subgroup of the pair (G^x, θ), and G^x contains a maximal θ-split subtorus of G.*

Proof. The dimension of $\hat P/\!/(\hat G)^{\tilde\theta}$ is the dimension of a maximal $\tilde\theta$-split torus of $\hat G$ (and $\tilde G$), and by the slice theorem this is the same as the dimension of $P/\!/H$. If we choose $x = e$, then a maximal $(\tilde\theta = \theta)$-split torus is A. We have (1), and (2) is a special case of (1). □

6. Closed orbits

We begin by giving a short proof of Theorem 3.4(10), simplifying the one in [Ric82b]. Recall that we have not made use of 3.4(10) so far.

Theorem 6.1. *Let (G, θ) be almost connected (see 3.8), and let S be a maximal θ-split subtorus of G. Then every closed K-orbit in P intersects S.*

Proof. Let $x \in P$ and write $x = \beta(y)$ for some $y \in G^0$. Assume that Kx is closed. From Corollary 5.12 we know that G^x contains a θ-split subtorus S' of dimension $\dim S$. We can find $k \in K^0$ such that $k^{-1} \cdot S = S'$. Then $k \cdot x = \beta(k \cdot y) \in Z_G(S)$, so we may assume that $x \in P \cap Z_G(S)$.

By 3.5, 3.7 and 3.8, we can find $k \in K$ such that $yk \in Z_G(S)^0 = SZ_K(S)^0$. Hence $x = \beta(y) \in S$. □

6.2. We establish the analogue of the Chevalley restriction theorem 3.4(11).

Definition 6.3. We say that $a \in A$ is *generic* or *principal* if $H^*_a = Z^*_H(A) = Z_M(A)$. We denote the principal points of A by A_{pr}.

Proposition 6.4. (1) $A_{\mathrm{pr}} \neq \emptyset$.

(2) *If $a \in A_{\mathrm{pr}}$, then A is the unique maximal σ-split torus of $G^{\tilde\tau} = \{g \in G \mid \tau(g) = a^{-1} \cdot g\}$.*
(3) *The slice at any $a \in A_{\mathrm{pr}}$ is an open subset of A with trivial group action. In particular, the slice representation is trivial.*
(4) *A_{pr} is open (and dense) in A.*
(5) *The canonical morphism $H \times^{Z^*_H(A)} A \to P$ is étale at points of the form $[h, a]$, $h \in H$, $a \in A_{\mathrm{pr}}$.*
(6) *If $a \in A_{\mathrm{pr}}$, $h \in H$ and $h * a \in A$, then $h \in N^*_H(A)$.*

Proof. Let $a \in A$ such that $Z_G(a^2) = Z_G(A)$ and let $h \in H^*_a$. By 5.9, h commutes with a^2, so $h \in Z_H(A)$, and $h * a = a\beta(h) = a$, so $\beta(h) = e$. In other words, $h \in Z^*_H(A) = Z_M(A)$, and we have (1).

Let $a \in A_{\mathrm{pr}}$. Then $(G^{\tilde{\tau}})^\sigma = H_a^* = Z_H^*(A)$ acts trivially on A. By 3.4, the root system $\Phi(A, (G^{\tilde{\tau}})^0)$ is empty, hence $(G^{\tilde{\tau}})^0 = A \cdot G_1$ where G_1 has trivial σ-action, hence trivial $\tilde{\theta}$-action. Thus $P_{\tilde{\theta}}((G^{\tilde{\tau}})^0) = A$. This establishes (2) and (3), and (4) and (5) then follow from the slice theorem. Finally, if a, and h are as in (6), let b denote $h * a$ and let $G^{\tilde{\tau}_b}$ denote $\{g \in G \mid \tau(g) = b^{-1} \cdot g\}$. Since $h \cdot H_a^* = H_b^*$ and $H_a^* = Z_H^*(A) \subset H_b^*$, we must have $b \in A_{\mathrm{pr}}$. Moreover, $h \cdot G^{\tilde{\tau}} = G^{\tilde{\tau}_b}$, and then (2) shows that $h \in N_H(A)$, hence $h \in N_H^*(A)$. □

Theorem 6.5. *Let H act on P by $*$. Then the inclusion $A \to P$ induces an isomorphism $A/W_H^*(A) \xrightarrow[\sim]{\iota} P/\!/H$. In particular:*

1. *The closed H-orbits are exactly those which intersect A.*
2. *If $a \in A$, then $(H * a) \cap A = W_H^*(A) * a$.*

Proof. Consider the action of $\{e, \tau\} \ltimes H$ on P. Then, by 2.9, $P^\tau/\!/H^\tau \to P/\!/H$ is finite. Since $H^\tau = M$ and $(P^\tau)^0$ is the component of P^τ through e, $(P^\tau)^0/\!/M \to P/\!/H$ is finite. By 3.4(11), $A \to (P^\tau)^0/\!/M$ is finite, hence $A \to P/\!/H$ and ι are finite. By 6.4 and the slice theorem, ι is étale and injective on $A_{\mathrm{pr}}/W_H^*(A)$, hence ι is birational and surjective. Since $P/\!/H$ is normal, Zariski's Main Theorem shows that $A/W_H^*(A) \simeq P/\!/H$. □

Remark 6.6. The theorem above establishes the analogue of 3.4(11). The analogue of 3.4(9) follows from 3.4(9) and 7.2 below. For 3.4(12) we have:

Proposition 6.7. *Let H act on P by $*$. Let $Y \subset A$, and suppose that $h * Y \subset A$ for some $h \in H$. Then there is an $h' \in N_H^*(A)$ such that $h' * y = h * y$ for all $y \in Y$.*

Proof. Let $\tilde{G}^Y$ denote $\{g \in G \mid \tau(g) = y^{-1} \cdot g$ for all $y \in Y\}$, and define $\tilde{G}^{Y'}$ similarly, where $Y' = h * Y$. Then $h \cdot \tilde{G}^Y = \tilde{G}^{Y'}$. By 5.12, A is a maximal σ-split torus in $\tilde{G}^Y$ and $\tilde{G}^{Y'}$, and $h^{-1} \cdot A$ is maximal σ-split in $\tilde{G}^Y$. We can find $g^{-1} \in (\tilde{G}^Y)^\sigma$ such that $(g^{-1}h^{-1}) \cdot A = A$. By 5.9, $(\tilde{G}^Y)^\sigma$ fixes Y pointwise, hence $hg \in N_H(A)$ (forcing $hg \in N_H^*(A)$), and hg acts identically to h on Y. □

7. The Slice at e and Stratifications

7.1. Let X be an affine L-variety, where L is reductive. For $x \in X$, we let $\mathcal{F}(X)_x$ (or just $\mathcal{F}_x$) denote the fiber $(\pi_X)^{-1}\pi_X(x)$. If $R(X)_x$ is a transversal at the good point x, then $\mathcal{F}(X)_x = L \times^{L_x} \mathcal{F}(R(X)_x)_x = L \times^{L_x} \mathcal{F}(\mathcal{S}(X)_x)_x$ by the slice theorem, where $\mathcal{S}(X)_x \subset R(X)_x$ is a slice at x.

Let $a \in A$, and let $\hat{P} = \hat{P}(a)$ and $\hat{G} = \hat{G}(a)$ be as in 5.10. Recall that $\hat{P}a$ is a transversal at a for the action of H_a^*.

Proposition 7.2. (1) $\mathcal{F}(P)_a \simeq H \times^{H_a^*} \mathcal{F}(R_a)_a$.

(2) $\mathcal{F}(R_a)_a = \{ua \mid u \in P_{\tilde{\theta}}(\hat{G})$ *is unipotent*$\}$.

(3) $H_a^* \cap Z_G(a) = M_a = (H \cap K)_a$.

(4) *Suppose that* $[H_a^* : M_a] < \infty$. *Then* $\mathcal{F}(P)_a = H \times^{H_a^*} \mathcal{F}(\hat{P})_a$.

Proof. Parts (1) and (2) follow from 3.4(8) and our discussion above, and part (3) follows from the definitions. In (4), the transversal $R(P)_a$ to the orbit $H * a$ has actions of $H_a^* \supset M_a$. Since the two groups have the same dimension, the fiber $\pi_{\hat{P}}^{-1}(\pi_{\hat{P}}(a))$ is the same for the actions of both groups. ☐

Corollary 7.3. *Let* $\hat{P} = \hat{P}(e)$ *be the transversal for* $H_e^* = M$, *and let* $\varphi : H \times^M \hat{P} \to P$ *be the canonical map. Then the following are equivalent:*

(1) φ *is étale at* $[e, a] \in H \times^M \hat{P}$.
(2) φ *has finite fibers at* $[e, a]$
(3) $[H_a^* : M_a] < \infty$

If these conditions hold at every $a \in A$ *we have*

(4) *The canonical morphism*

$$\hat{\varphi} : H \times^M \hat{P} \to P \times_{P/\!/H} (\hat{P}/\!/M)$$

is an isomorphism iff $A/W_M(A) \to A/W_H^*(A)$ *is a covering iff* $(W_H^*(A))_a = (W_M(A))_a$ *for all* $a \in A$.

(5) *The canonical morphism* $\varphi : H \times^M \hat{P} \to P$ *is an isomorphism iff* $W_H^*(A) = W_M(A)$.

Proof. For (1), (2) and (3) we can reduce to transversals. So, we have $\varphi : H_a^* \times^{M_a} R(\hat{P})_a \to R(P)_a$. Since φ collapses the orbit $H_a^*[e, a]$ to $a \in R(P)_a$, it is clear that both (1) and (2) imply (3). Conversely, if $[H_a^* : M_a] < \infty$, then our arguments above show that we may assume that $R(P)_a = R(\hat{P})_a$, and (1) and (2) follow.

It is clear that (4) implies (5) and that $A/W_M(A) \to A/W_H^*(A)$ is a covering iff $(W_H^*(A))_a = (W_M(A))_a$ for all $a \in A$. Suppose that $\hat{P}/\!/M \simeq A/W_M(A) \to A/W_H^*(A) \simeq P/\!/H$ is a cover. Then $P \times_{P/\!/H} (\hat{P}/\!/M)$ is smooth. For every $a \in A$, $d\varphi_{[e,a]}$ is an isomorphism, hence so is $d\hat{\varphi}_{[e,a]}$, and $\hat{\varphi}$ is étale at $[e, a]$. By the fundamental lemma [BR85, 6.2], $\hat{\varphi}$ is an excellent morphism in an H-neighborhood of $[e, a]$ for every $a \in A$. Since $\hat{\varphi}$ induces an isomorphism on the quotient spaces by H, it is an isomorphism.

Conversely, suppose that $\hat{\varphi}$ is an isomorphism. Then φ is excellent, hence $\varphi/\!/H : \hat{P}/\!/M \to P/\!/H$ is étale. It follows that $A/W_M(A) \to A/W_H^*(A)$ is étale (and it is finite), hence it is a cover. ☐

We now give an example where 7.3 applies and another where it does not, as well as an example showing that one cannot always look at connected groups. In the examples, i denotes a square root of -1.

Example 7.4. Let $G = \mathrm{GL}_{2n}(k)$, $\theta(g) = L({}^t g^{-1})L^{-1}$, $\sigma(g) = J({}^t g^{-1})J^{-1}$ and $\tau(g) = \sigma\theta(g) = EgE^{-1}$, where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \ L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{ and } E = LJ = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then $K \simeq O_{2n}$, $H \simeq SP_{2n}$ and $\tilde{G} = G^\tau = \{\mathrm{diag}(X, Y) \mid X, Y \in GL_n\}$. Moreover, $M := H \cap K = \tilde{G}^\theta = \tilde{G}^\sigma = \{\mathrm{diag}(X, {}^tX^{-1}) \mid X \in GL_n\} \simeq GL_n$ and we may choose $A = \{\mathrm{diag}(D, D) \mid D$ diagonal, nonsingular$\}$.

Now $\mathfrak{h} = \{y \in \mathfrak{gl}_{2n} \mid JyJ^{-1} = -y^t\}$. Define $\varphi(x) = xE$, $x \in P$. Then a computation shows that $\operatorname{Im}\varphi$ is the open subset $U \subset \mathfrak{h}$ consisting of matrices which are nonsingular (as elements of GL_{2n}). Moreover, $\varphi(h * x) = h\varphi(x)h^{-1}$, $x \in P$, $h \in H$. The image of A is $\{\mathrm{diag}(D, -D) \mid D$ diagonal, nonsingular$\}$.

Let $a := \mathrm{diag}(D, -D) \in \varphi(A)$. Let the eigenvalues of D be $s_1, \ldots, s_r$ where s_j has multiplicity m_j, $j = 1, \ldots, r$. Then a matrix computation shows that H_a^* is a product $\prod_{j=1}^r GL_{m_j} \subset M$. Thus $H_a^* = M_a$ for every $a \in A$, $N_H^*(A) = N_M(A)$ and $P \simeq H \times^M \hat{P}$.

Example 7.5. Let $G = SL_2(k)$, and let θ and σ be conjugation by the matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, respectively. Then $A := \{\mathrm{diag}(a, a^{-1}) \mid a \in k^*\}$ is a maximal torus in G which is (σ, θ)-split, and $H = \{\left(\begin{smallmatrix} c & d \\ d & c \end{smallmatrix}\right) \mid c^2 - d^2 = 1\}$. Calculation shows that $\theta(h) = h^{-1}$ for $h \in H$ and that $\{h \in H \mid \beta(h) = h^2 \in A\} = \{h_0, h_0^2, h_0^3, h_0^4 = e\}$, where $h_0 := \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Since h_0 normalizes A, $W_H^*(A)$ is generated by its image w, which sends $\mathrm{diag}(a, a^{-1})$ to $\mathrm{diag}(-a^{-1}, -a)$. Note that w has order 2 and does not lie in $W_H(A)$.

The transversal at $e \in P$ is $P_\theta(G^\tau) = A$ with the (trivial) action of $M = H \cap K = \{\pm I\}$ by conjugation. Now w has fixed points $a_\pm := \pm i$, and $H_{a_\pm}^* = H$, so the fibers of P and $H \times^M \hat{P}$ have different dimensions at these points. At all other points we have $H_a^* = M_a = M$, and 7.2 applies.

Example 7.6. This is a case where $W_M(A) \neq W_{M^0}(A)$. Let $G = SO_3(k)$, where G preserves the usual quadratic form $\sum_{j=1}^3 x_j^2$. Let θ and σ be conjugation by the matrices $\mathrm{diag}(-1, 1, -1)$ and $\mathrm{diag}(-1, -1, 1)$, respectively. Then H and K are isomorphic to copies of O_2. For example,

$$H^0 = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$$

and the other component contains $\mathrm{diag}(-1, 1 - 1)$. The (σ, θ)-split torus A is the same as H^0, except that the a's and b's occur in rows and columns 2 and 3. Here $W_G(A) = W_H(A) = W_K(A)$ have order 2 (since A has rank 1, we cannot get more!), and there are generators in both components of H and in $H \cap K^0$, but not in $H^0 \cap K^0$ nor in $M^0 = (H \cap K)^0 = \{e\}$. There is no "twisting" in this example, since $N_H^*(A) = N_M(A) = \{\mathrm{diag}(\pm 1, \pm 1, \pm 1)\} \cap SO_3$.

7.7. We now consider stratifications of the quotients $P/\!\!/H$ and $A/W_H^*(A)$: Let X be an affine G-variety. We have the *isotropy type* stratification $X/\!\!/G = \bigcup_L (X/\!\!/G)^{(L)}$, where $(X/\!\!/G)^{(L)}$ denotes the closed orbits with isotropy group conjugate to the subgroup L of G. Suppose that X is smooth and that all closed orbits are separable and have slices. Then there is a finer stratification by *slice type* , where we consider the isomorphism classes of the slice representations $(T_x(X)/T_x(Gx), G_x)$ for Gx closed.

Proposition 7.8. *The slice type and isotropy type stratifications of $P/\!/H$ coincide, and the same is true of $A/W_H^*(A)$.*

Proof. Let $a \in A$. Then $(T_a(A), W_H^*(A)_a)$ is the slice representation of $W_H^*(A)_a$ at a, so it clearly only depends upon $W_H^*(A)_a$. Suppose that $\bar{a} \in A$ and $H_a^* = H_{\bar{a}}^*$. Then one computes that $\mathrm{Int}(\bar{a}a^{-1})$ gives an isomorphism of $T_a(\mathcal{S}_a)$ with $T_{\bar{a}}(\mathcal{S}_{\bar{a}})$ commuting with the $(H_a^* = H_{\bar{a}}^*)$-actions, where $\mathcal{S}_a$ and $\mathcal{S}_{\bar{a}}$ are slices at a and $\bar{a}$, respectively. □

Theorem 7.9. *Let $a \in A$. Identify $A/W_H^*(A)$ and $P/\!/H$ via the inclusion $A \to P$. Then the connected components of $(A/W_H^*(A))^{(W_H^*(A)_a)}$ and $(P/\!/H)^{(H_a^*)}$ through $\pi_A(a) = \pi_P(a)$ coincide.*

Proof. Using a slice, we can reduce to the case where $a = e$ and $\sigma = \theta$. Then the transversal at $e \in P$ is just P itself. We need to establish that $B^0 = C^0$, where $B := A^{W_K(A)}$ and $C := P^K$. If $c \in C$, then $K \cdot c = \{c\}$ is a closed orbit, hence it passes through A, and we get that $C = A^K \subset B$.

Write $\mathfrak{g} = \mathfrak{a} \oplus \sum_{\alpha \in \Phi_A} \mathfrak{g}_\alpha$ where Φ_A is the root system of $(\mathfrak{g}, A)$. Let Δ be a basis of Φ_A. By [Ric82b], for every $\alpha \in \Phi_A$ there is a $k \in N_{K^0}(A)$ which sends α to α^{-1}. Thus α and α^{-1} have the same restriction to B, and α is trivial on B^0. It follows that $\mathfrak{g}^{B^0} = \mathfrak{g}$, so $B^0 \subset A^{K^0}$. By 3.4(12), the fixed points of K and $W_K(A)$ in A^{K^0} coincide, hence $B^0 \subset C$. □

Example 7.10. It is possible that a stratum of $A/W_H^*(A)$ corresponds to a union of several different $P/\!/H$-strata, even in the case that $\sigma = \theta$: Let $G = \mathrm{SO}_3(k)$. Set $\alpha = \mathrm{diag}(-1, -1, 1)$ and $\theta = \mathrm{conj}(\alpha)$. Then K is generated by $\{\left(\begin{smallmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid a^2 + b^2 = 1\}$ and $\delta := \mathrm{diag}(i, i, -1)$, and $S = \{Y(c, d) := \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & c & d \\ 0 & -d & c \end{smallmatrix}\right) \mid c^2 + d^2 = 1\}$ is a maximal θ-split torus whose Weyl group W is generated by α. Identifying S with k^* by sending $Y(c, d)$ to $c + \mathrm{id}$, we see that $\mathrm{Int}\,\alpha$ sends $c + \mathrm{id}$ to $c - \mathrm{id}$, so that the W-fixed points are $Y(\pm 1, 0)$. However, K fixes $Y(1, 0)$ (the identity!), while $Y(-1, 0)$ is only fixed by $\{e, \delta\} \subset K$. Thus the W-fixed points in A correspond to two different strata for $P/\!/K$.

8. Elementary properties of $W_H^*(A)$

8.1. Since $W_H(A)$ acts on A, we have a semidirect product $W_H(A) \ltimes A$, with multiplication $(w, a) * (w', a') = (ww', aw(a'))$, $w, w' \in W_H(A)$, $a, a' \in A$. We identify A with $\{(e, a) \mid a \in A\}$. Recall that $N_H^*(A) = \{h \in H \mid h * A \subset A\}$, $Z_H^*(A) = \{h \in H \mid h * a = a$ for every $a \in A\}$ and $W_H^*(A) = N_H^*(A)/Z_H^*(A)$. The action of $W_H^*(A)$ on A factors through the homomorphism

$$\rho\colon W_H^*(A) \to W_H(A) \ltimes A^{(2)}, \quad [h] \mapsto (\mathrm{Int}(h), \beta(h)),$$

where $[h]$ is the class of $h \in N_H^*(A)$ in $W_H^*(A)$. We have

Lemma 8.2. *The homomorphism $\rho : W_H^*(A) \to W_H(A) \ltimes A^{(2)}$ is injective.*

8.3. The composition $\phi\colon W_H^*(A) \xrightarrow{\rho} W_H(A) \ltimes A^{(2)} \to W_H(A)$ is a group homomorphism with kernel $W_0 \simeq (Z_H(A) \cap N_H^*(A))/Z_H^*(A)$. The composition $\psi\colon W_H^*(A) \xrightarrow{\rho} W_H(A) \ltimes A^{(2)} \to A^{(2)}$ is a homomorphism (and equals β) when restricted to W_0, and it induces an isomorphism $W_0 \simeq F_0 := \psi(W_0)$. The image of ψ is denoted by F. We identify $W_M(A)$ with the corresponding subgroups of $W_H^*(A)$ and $W_H(A)$. The set F_0 is $W_H(A)$-stable, but F is not necessarily $W_H(A)$-stable; see 10.11.

We leave the proof of the following trivialities to the reader.

Proposition 8.4. *Let $[h] \in W_H^*(A)$, where $\rho([h]) = (w, a) \in W_H(A) \ltimes A^{(2)}$.*
1. *$[h] \in W_M(A) \subset W_H^*(A)$ iff $a = e$.*
2. *$w \in W_M(A) \subset W_H(A)$ iff $a \in F_0$.*
3. *$F_0 \subset A^{(2)}$ is $W_H(A)$-stable.*
4. *F is $W_M(A)$-stable.*

Corollary 8.5. *The following are equivalent:*
1. *$W_M(A) = W_H(A)$.*
2. *$F = F_0$.*
3. *$\rho(W_H^*(A)) = W_H(A) \ltimes F_0$.*

If the conditions above hold, we say that (σ, θ) is *split*.

8.6. We establish characterizations of F and F_0. Recall that we assume that G is connected. Adding components to G can obviously change F and F_0. A good example to keep in mind is 7.5, where one changes G from $\mathrm{SL}_2(k)$ to $\{g \in \mathrm{GL}_2(k) \mid \det(g) = \pm 1\}$. Then F_0 changes from the trivial group to $A^{(2)}$.

8.7. **Standard Tori.** Let $T \subset G$ be a torus. If T is invariant under an involution η, then we use T_+^η to denote $(T^\eta)^0$ and T_-^η to denote the (unique) maximal η-split subtorus of T. Then $T = T_+^\eta T_-^\eta$.

If T is stable under our commuting involutions σ and θ, then we define $T_{++}^{\sigma,\theta}$ to be $(T_+^\sigma \cap T_+^\theta)^0$, and similarly for $T_{--}^{\sigma,\theta}$, $T_{+-}^{\sigma,\theta}$ and $T_{-+}^{\sigma,\theta}$. Then

$$T = T_{++}^{\sigma,\theta} T_{+-}^{\sigma,\theta} T_{-+}^{\sigma,\theta} T_{--}^{\sigma,\theta}.$$

Proposition 8.8 (see [Hel88, 5.13] or proof of 4.3(2))**.** *There is a (σ, θ)-stable maximal torus T of G such that*
1. *$A = T_{--}^{\sigma,\theta}$.*
2. *$AT_{+-}^{\sigma,\theta}$ is a maximal θ-split torus.*
3. *$AT_{-+}^{\sigma,\theta}$ is a maximal σ-split torus.*

We call the maximal tori in 8.8 *standard*. All standard maximal tori are conjugate under $H \cap K$. Hence the class of standard maximal tori is uniquely determined by the pair of commuting involutions (σ, θ).

8.9. For the remainder of this section we fix a standard maximal torus $T \supset A$. Note that $T^\tau_- := T^{\sigma,\theta}_{+-} T^{\sigma,\theta}_{-+}$ is not necessarily maximally τ-split in G (see Example 7.5).

From 4.2 and 4.3 applied to $Z_G(A)/A$ we obtain

Proposition 8.10. (1) *T^τ_- is a maximal τ-split torus of $Z_G(A)$.*
(2) $Z_G(A)^\tau = A Z_M(A)$.
(3) *All maximal τ-split tori in $Z_G(A)$ are $Z_M(A)^0$-conjugate.*

Proposition 8.11. (1) $H = H^0(H \cap T^\sigma_-)$.
(2) $N_H(A) = (H \cap T^\sigma_-) N_{H^0}(A)$.
(3) $W_H(A) = W_{H^0}(A)$.
(4) $H \cap T^\sigma_- \subset N^*_H(A) \cap Z_H(A)$
(5) $N^*_H(A) = (H \cap T^\sigma_-) N^*_{H^0}(A)$.
(6) $W^*_H(A) = W^*_{H^0}(A) W_0$.
(7) $F = P_\theta(H) \cap A$.
(8) $F_0 = P_\theta(AT^\tau_- \cap H) \cap A$.

Proof. Part (1) is 3.4(6), and (1) implies (2)-(3). Let $t = t_{-+}t_{--} \in H \cap (T^{\sigma,\theta}_{-+} T^{\sigma,\theta}_{--} = T^\sigma_-)$. Then $\beta(t) = t^2_{--} \in A$, giving (4), which in turn implies (5)–(6).

If $a = \beta(h)$ for $h \in N^*_H(A)$, then clearly $a \in P_\theta(H) \cap A$. Conversely, if $A \ni a = \beta(h)$ for some $h \in H$, then by 3.7(1), $h^{-1} \cdot A$ is (σ, θ)-split, hence there is an $m \in M^0$ such that $hm \in N_H(A)$. Since $hm \in N^*_H(A)$ and $\beta(hm) = a$, we have (7).

For (8), the only nontrivial part is to show that any $h \in N^*_H(A) \cap Z_H(A)$ has a representative in $(AT^\tau_-)^\sigma$. Since $\tau(h) = \beta(h)$ centralizes T^τ_-, 3.7(1) shows that $h^{-1} \cdot T^\tau_-$ is τ-split. By 8.10, there is an $m \in Z_M(A)^0$ such that $hm \in H \cap Z_{G'}(T^\tau_-)$, where $G' := Z_G(A)$. But, by 3.5(3) and 8.10(2), we may write $hm = h'm'$ where $m' \in Z_M(A)$ and $h' \in AT^\tau_-$. □

Theorem 8.12. (1) $F_0 = A \cap T^\tau_- = (A \cap T^{\sigma,\theta}_{+-})(A \cap T^{\sigma,\theta}_{-+})$.
(2) $T^\sigma_- \cap T^\theta_- = (T^{\sigma,\theta}_{+-} \cap T^{\sigma,\theta}_{-+})A$.
(3) *F_0 is isomorphic to*

$$\frac{(T^{\sigma,\theta}_{+-} \cap A) \times (T^{\sigma,\theta}_{-+} \cap A)}{T^{\sigma,\theta}_{+-} \cap T^{\sigma,\theta}_{-+} \cap A},$$

where the denominator sits diagonally inside the product.

Proof. If $t \in AT^\tau_-$, we write $t = t_{--}t_{+-}t_{-+}$ where $t_{+-} \in T^{\sigma,\theta}_{+-}$, etc. If $a \in F_0$, then we know that $a = \beta(t) = t^2_{--}t^2_{+-}$ for some $t \in H \cap AT^\tau_-$. Since $t \in H$, $\beta(t) = t\tau(t)^{-1} = t^2_{+-}t^2_{-+}$. Thus $t^2_{-+} = t^2_{--} \in A$ and $t^2_{+-} \in A$. Hence $a = s_{+-}s_{-+}$ where $s_{+-} \in T^{\sigma,\theta}_{+-} \cap A$, $s_{-+} \in T^{\sigma,\theta}_{-+} \cap A$.

Now suppose that $a = s_{+-}s_{-+} \in T_{-}^{\tau} \cap A$. Then s_{+-} is fixed and split by σ and θ, and similarly for s_{-+}. If $t_{+-} \in T_{+-}^{\sigma,\theta}$ with $t_{+-}^2 = s_{+-}$, then $\beta(t_{+-}) = s_{+-} \in A$, so we may reduce to the case that $a = s_{-+}$. Choose $t = t_{--}t_{-+} \in AT_{-+}^{\sigma,\theta}$ such that $t_{--}^2 = t_{-+}^2 = a$. Then t_{--} and t_{-+} have order 4, $\sigma(t) = t^3 = t \in H$ and $\beta(t) = t_{--}^2 = a \in F_0$. We have (1).

Let $s_{+-}s_{--} = s_{-+}s'_{--} \in T_{-}^{\theta} \cap T_{-}^{\sigma}$. Then $a := s_{+-}s_{-+}^{-1} \in A \cap T_{-}^{\tau} = F_0$, hence there are $t_{+-} \in T_{+-}^{\sigma,\theta} \cap A$ and $t_{-+} \in T_{-+}^{\sigma,\theta} \cap A$ such that $a = t_{+-}t_{-+}$. Thus modulo $F_0 \subset A$ we may assume that $s_{+-}s_{-+}^{-1} = e$, giving (2). Part (3) follows from (1). □

Corollary 8.13. *If A is maximally θ-split and maximally σ-split, then $F_0 = \{e\}$.*

Remark 8.14. The converse is false; see Example 10.8.

Theorem 8.15. *Assume that $P_\theta(H)$ is connected (e.g., H is almost connected 3.8 or G is simply connected 3.6(2)). Let $a \in A^{(2)}$. Then the following are equivalent:*

(1) *$a \in F$.*
(2) *$a \in P_\theta(Z_H(a)^0)$.*
(3) *$a \in S$ where S is a θ-split torus in $Z_H(a)$.*

If any of these conditions holds, then there is a $w \in W_H(A)$ such that $(w, a) \in \rho(W_H^*(A))$ and $w(a) = a$.

Proof. We may assume that $a \neq e$. If (1) holds, then a is a semisimple element of $P_\theta(H^0)$, hence it lies in a maximal θ-split torus $S \subset H$. Consequently, $a \in S \subset Z_H(a)^0$, or, equivalently, $a \in P_\theta(Z_H(a)^0)$.

Conversely, suppose that $a \in S \subset Z_H(a)^0$ where S is θ-split. Choose $s \in S$ such that $s^2 = \beta(s) = a$. Since $\beta(s)$ centralizes A, $s^{-1} \cdot A$ is (σ, θ)-split, and we can find an $m \in Z_H(a)^0 \cap M$ such that $h := sm \in N_H(A)$. Then $\beta(h) = a \in A$, establishing (1). Note that $h \cdot a = a$, so that $\rho([h]) = (w, a)$ where $w(a) = a$. □

Remark 8.16. Essentially we reduce the problem of computing whether or not $a \in A$ is in F to the case $a \in Z(G)$.

9. Quadratic Elements

9.1. An element $q \in A$ is called *quadratic* if $q^2 \in Z(G)$. We let $Q(A)$ denote the set of quadratic elements in A. Given $q \in Q(A)$, we produce a new automorphism $\tilde{\theta} := \theta\,\mathrm{Int}(q)$. Since $\mathrm{Int}(q) = \mathrm{Int}(q^{-1})$, $\tilde{\theta}$ is an involution of G which commutes with σ. Moreover, A is a maximal $(\sigma, \tilde{\theta})$-split torus. We show that there are q such that $(\sigma, \tilde{\theta})$ is split (see 8.5).

9.2. Let $\tilde{Z}_H^*(A)$, $\tilde{N}_H^*(A)$, etc. denote the groups corresponding to $Z_H^*(A)$, $N_H^*(A)$, etc. when we replace θ by $\tilde{\theta}$. We denote by $\tilde{\rho}$ the canonical injection of $\tilde{W}_H^*(A)$ into $W_H(A) \ltimes A^{(2)}$ as in 8.2, and $\tilde{\beta}\colon G \to P_{\tilde{\theta}}(G)$, $g \mapsto g\tilde{\theta}(g)^{-1}$ is the analogue of β.

For $w \in W_H(A)$, let $a_{w,q}$ denote $w(q^{-1})q$. Then $a_{w,q}^2 = w(q^{-2})q^2 = e$ since $q^2 \in Z(G)$. Thus $a_{w,q} \in A^{(2)}$, and $a_{w,q}$ only depends upon the class of q modulo $Z(G)$. Let η_q denote the automorphism of $W_H(A) \ltimes A^{(2)}$ sending (w, a) to $(w, a_{w,q}a)$, $w \in W_H(A)$, $a \in A^{(2)}$. One computes that:

Theorem 9.3. *Let q, etc. be as above. Let $h \in N_H(A)$ and let w denote $\operatorname{Int}(h) \in W_H(A)$. Then*

1. *$\tilde{\beta}(h) = a_{w,q}\beta(h)$. In particular, η_q is the identity on F_0 and $\tilde{\beta}(h) \in A$ if and only if $\beta(h) \in A$.*
2. *We have equalities $\tilde{N}_H^*(A) = N_H^*(A)$, $\tilde{Z}_H^*(A) = Z_H^*(A)$, $\tilde{W}_H^*(A) = W_H^*(A)$, $\tilde{W}_0 = W_0$ and $\tilde{F}_0 = F_0$.*
3. *The mapping η_q is a group isomorphism which sends $\rho(W_H^*(A))$ isomorphically onto $\tilde{\rho}(\tilde{W}_H^*(A))$.*

9.4. The group of multiplicative one parameter subgroups of A is denoted by $X_*(A)$, and it is naturally $W_G(A)$-equivariantly dually paired with the character group $X^*(A)$: $\langle \alpha, \lambda \rangle = m \in \mathbb{Z}$ where $\alpha \circ \lambda \colon t \mapsto t^m$, $t \in k^*$, $\lambda \in X_*(A)$, $\alpha \in X^*(A)$. Since A is maximally (σ, θ)-split in both G and G^τ we have root systems (see below) $\Phi(A) = \Phi(A, G) \supset \Phi(A, (G^\tau)^0)$. This is in general a proper inclusion (see example 7.5).

9.5. **Nonreduced root systems.** (See [Bou81].) Let Φ be a nonreduced root system. We say that a root α is *indivisible* if it is not of the form $c\gamma$ where $\gamma \in \Phi$ and $c > 1$. In fact, the only possibility for c is 2. Let $\Phi' \subset \Phi$ consist of the indivisible elements. Then Φ' is reduced, and Φ and Φ' have the same Weyl group. Any base Δ of Φ lies in Φ' and is also a base of Φ'. If Φ is irreducible, then there is only one possibility for Φ, and that is the root system of type BC_n [Bou81].

Remark 9.6. We can reduce many of our questions to the case that $\Phi(A)$ is irreducible, as follows. Let $T \supset A$ be a standard maximal torus of G. Following [Hel88, 7.17], we say that (σ, θ) is *irreducible* if $\Phi(T)$ is not the union of two mutually orthogonal σ-stable and θ-stable subsets Φ_1 and Φ_2 (e.g., G is simple). If (σ, θ) is irreducible, then $\Phi(A)$ is irreducible [Hel88, 7.17].

9.7. Let $T \supset A$ be standard. Let $\mathfrak{g}(A, \lambda)$ denote the root space corresponding to $\lambda \in \Phi(A)$. Since $\sigma(\lambda) = \theta(\lambda) = -\lambda$, $\tau = \sigma\theta$ stabilizes $\mathfrak{g}(A, \lambda)$. Set

$$\mathfrak{g}(A, \lambda)^\tau_\pm = \{X \in \mathfrak{g}(A, \lambda) \mid \tau(X) = \pm X\}$$
$$m^\pm(\lambda, \tau) = \dim \mathfrak{g}(A, \lambda)^\tau_\pm$$

For $\lambda \in \Phi(A)$ call $(m^+(\lambda, \tau), m^-(\lambda, \tau))$ the *signature* of λ, and let s_λ denote the reflection through λ. Note that $s_\lambda = s_{2\lambda}$ if 2λ is also a root.

Lemma 9.8. *Let A and Δ be as above.*

1. *If $\lambda \in \Phi(A)$, then $\lambda \in \Phi(A, (G^\tau)^0)$ iff $m^+(\lambda, \tau) > 0$.*

(2) *If* $\lambda \in \Delta$*, then* $s_\lambda \in W_{(G^\tau)^0}(A) = W_{M^0}(A)$ *iff* $m^+(\lambda, \tau) > 0$ *or* $m^+(2\lambda, \tau) > 0$ (see 3.4).

(3) *If* $q \in Q(A)$*,* $\lambda \in \Phi(A)$ *and* $\lambda(q) = -1$*, then* $m^\pm(\lambda, \tau) = m^\mp(\lambda, \tau\mathrm{Int}(q))$.

(4) *If* $q \in Q(A)$*,* $2\lambda \in \Phi(A)$ *and* $\lambda(q) = -1$*, then* $m^\pm(\lambda^2, \tau) = m^\pm(\lambda^2, \tau\mathrm{Int}(q))$.

Definitions 9.9. Following [Hel88, 6.11] we say that (σ, θ) is a *standard pair* (resp. *weakly-standard pair*) if $m^+(\lambda, \tau) \geq m^-(\lambda, \tau)$ (resp. $m^+(\lambda, \tau) \neq 0$ or $m^+(2\lambda, \tau) \neq 0$) for any $\lambda \in \Delta$. We say that $q \in Q(A)$ is *standard* (resp. *weakly-standard*) for (σ, θ) if $(\sigma, \tilde{\theta})$ is standard (resp. weakly-standard).

Remarks 9.10. (1) If (σ, θ) is standard or weakly standard, then the inequalities on the signatures hold for all $\lambda \in \Phi(A)^+$.

(2) $W_{M^0}(A) = W_H(A)$ if and only if (σ, θ) is weakly-standard.

(3) From Examples 7.6 and 10.5 one can see that the conditions split (see 8.5), weakly-standard and standard are distinct.

(4) If G is connected and simply connected, then H is connected 3.6(2), and 3.4(5) and 3.4(6) show that $W_M(A) = W_{M^0}(A)$. Hence split is the same as weakly-standard.

(5) If $\Phi(A)$ contains no factor of type BC_n and the root subspaces of $\mathfrak{g}$ are all of dimension 1, then standard and weakly-standard are equivalent.

9.11. The quadratic elements can be constructed as follows. Set $\mathrm{Ker}_A := \bigcap_{\lambda\in\Delta} \mathrm{Ker}\,\lambda$. Then $\mathrm{Ker}_A = A \cap Z(G)$ and Δ is a basis of the character group of A/Ker_A. We can find $\gamma_\lambda \in X_*(A)$, $\lambda \in \Delta$, such that $\langle \lambda', \gamma_\lambda\rangle = 0$ if $\lambda \neq \lambda'$ and $\lambda \circ \gamma_\lambda \colon k^* \to k^*$ has kernel $\mathrm{Ker}_\lambda = \gamma_\lambda^{-1}(\mathrm{Ker}_A)$. Set $q_\lambda := \gamma_\lambda(t_\lambda)$ where $t_\lambda \notin \mathrm{Ker}_\lambda$, $t_\lambda^2 \in \mathrm{Ker}_\lambda$. Then $q_\lambda^2 \in \mathrm{Ker}_A$, so that $q_\lambda \in Q(A)$. The q_λ, $\lambda \in \Delta$, are called the *basic quadratic elements*.

9.12. Results of Borel and de Siebenthal [BdS49] (see also [Hel88, Theorem 8.13]) classify the $W_H(A)$-orbits in $Q(A)/(Q(A) \cap Z(G))$: Let $q \in Q(A)$, and write $\Phi(A) = \Phi_1 \cup \dots \cup \Phi_r$ with each Φ_i irreducible. Then

(1) There are $w_j \in W(\Phi_j)$ and $\lambda_j \in \Delta'_j = \Delta(\Phi_j) \cup \{0\}$ such that $q \prod_j w_j(q_j) \in Z(G)$, where we interpret $q_j = q_{\lambda_j}$ as e when $\lambda_j = 0$.

(2) We can choose a basis $\Delta = \Delta_1 \cup \dots \cup \Delta_r$ of $\Phi(A)$ and $q_j = q_{\lambda_j}$ such that $q \prod_j q_j \in Z(G)$.

Theorem 9.13. *Let* (σ, θ) *and* Δ *be as above. Then there is a standard* $q \in Q(A)$*. If* q *is split (e.g., standard), then*

(1) $\rho^{-1}\eta_{q^{-1}} : W_H(A) \ltimes F_0 \to W_H^*(A)$ *is an isomorphism.*

(2) $W_H(A) = W_{\tilde{M}}(A)$.

(3) *There are isomorphisms*

$$(W_H(A) \ltimes F_0)/W_M(A) \xrightarrow{\sim} F, \quad (w, a) \mapsto a_{w,q}\, a$$

and

$$W_H(A)/W_M(A) \xrightarrow{\sim} F/F_0, \quad w \mapsto a_{w,q}\, F_0.$$

Proof. A standard q is the product of those q_λ, $\lambda \in \Delta$, for which $m^+(\lambda, \tau) < m^-(\lambda, \tau)$. The rest follows from Lemma 9.8. □

10. F, F_0 and Standardness

We find criteria for pairs (σ, θ) to be weakly-standard or split, and we determine the $W_H(A)$-orbits in F_0. The complete picture of the structure of the groups $W_H^*(A)$ is still a bit beyond our grasp. We feel, however, that the following conjecture is true, which would allow a classification.

Conjecture 10.1. *Assume that G is adjoint and that $q \in Q(A)$ is standard. Define $F^q = \{a_{w,q} \mid w \in W_H(A)\}$. Then $F^q \cap F_0 = \{e\}$ or $F^q \subset F_0$.*

From 9.8, 9.10 and 8.4 we get

Proposition 10.2. *Let (σ, θ), Δ be as in 9.12, and let $q := q_\lambda \in Q(A)$ be weakly-standard. Then the following are equivalent:*

(1) *(σ, θ) is weakly-standard.*
(2) $W_{M^0}(A) = W_H(A)$.
(3) $W_M(A) = W_{\tilde{M}}(A)$.
(4) $m^-(\lambda, \tau) m^+(\lambda, \tau) > 0$ *or* $m^+(2\lambda, \tau) > 0$.

10.3. We find a criterion for (σ, θ) to be split. Let $q \in Q(A)$ and set $W_q := \{w \in W_H(A) \mid w(q) = q\}$ and $M_q = \{m \in M \mid m \cdot q = q\}$.

Proposition 10.4. *Let $(\sigma, \tilde{\theta} = \theta \operatorname{Int}(q))$ be split. Then*

(1) $W_q = W_{M^q}(A)$.
(2) $W_M(A) = \{w \in W_H(A) \mid a_{w,q}\ (= w(q^{-1})q) \in F_0\}$.

Suppose that $q = q_\lambda$, $\lambda \in \Delta$. Then

(3) *$W_H(A)$ is generated by W_q and s_λ, and the following are equivalent*
   (a) *(σ, θ) is split.*
   (b) $a_{s_\lambda, q_\lambda}\ (= s_\lambda(q_\lambda^{-1}) q_\lambda) \in F_0$.

Proof. Everything follows from the following two observations:

- $W_M(A) \ltimes F_0 = \eta_{q^{-1}}(W_H(A) \ltimes F_0)$ and $\eta_q = \eta_q^{-1} = \eta_{q^{-1}}$.
- By construction of the basic quadratic elements, the only simple reflection which does not fix q_λ modulo $Z(G)$ is s_λ itself.

□

Example 10.5. Here is an example where (σ, θ) is split but not weakly-standard: Let $I_{2,2}$ denote $\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ where I_2 is the 2×2 identity matrix. Let $G = \mathrm{PGL}_4 = \mathrm{PGL}_4(k)$, $\theta(g) = ({}^t g^{-1})$ and $\sigma(g) = I_{2,2} g I_{2,2}$. Then $\mathfrak{k} \simeq \mathfrak{o}_4$, $\mathfrak{h} \simeq \mathfrak{gl}_2 \times \mathfrak{gl}_2$. If A is a maximal (σ, θ)-split torus, then $\Phi(A)$ is of type C_2. Let q_1 and q_2 be the basic quadratic elements and set $\tilde{\theta} = \theta \operatorname{Int}(q_2)$. Since $m^-(\lambda_2, \tau) = 0$ the pair $(\sigma, \tilde{\theta})$ is not weakly-standard. However $s_2(q_2)q_2^{-1} = q_1 \in F_0$, and 10.4 gives that $F = F_0$.

10.6. **$W_H(A)$-orbits in F_0.** Let $\Phi = \cup_j \Phi_j$, etc. be as in 9.12. If we can determine the elements $\lambda_{ij} \in \Delta_j$ such that $q_{ij} \in F_0$, then 9.12 gives us (another) characterization of F_0, together with its $W_H(A)$-orbit structure. It suffices to consider the case that $\Phi(A)$ is irreducible. Then the classification of signatures in [Hel88] does the trick:

Theorem 10.7. *Let Δ be a basis of $\Phi(A)$, $\lambda \in \Delta$ where $\Phi(A)$ is irreducible. Then*

(1) $q_\lambda \in F_0$ *if and only if* $m^+(\lambda, \tau) = m^-(\lambda, \tau)$.
(2) $F_0 = \{e\}$ *iff* $m^+(\lambda, \tau) \neq m^-(\lambda, \tau)$ *for all* $\lambda \in \Delta$.
(3) $F_0 = A^{(2)}$ *iff* $m^+(\lambda, \tau) = m^-(\lambda, \tau)$ *for all* $\lambda \in \Delta$.

Proof. By Theorem 9.3 we may assume that (σ, θ) is a standard pair. Let $T \supset A$ be a standard maximal torus. Then $m^+(\lambda, \tau) = m^-(\lambda, \tau)$ iff there exists $t \in T$, $\sigma(t) = t$, such that $a = t\theta(t)^{-1}$ [Hel88, Cor. 8.7 and Theorem 8.14], which is equivalent to $a \in F_0$ by Theorem 8.12. This gives (1), and (2) and (3) are immediate from 9.12 and (1). □

Example 10.8. Here is an example where $F_0 = \{e\}$, but not by an application of 8.13: Let $G = \mathrm{SO}_{2n}$. Let I_p denote the unit matrix of order p and put $I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$. Let $\sigma(g) = I_{2n-p,p}\, g\, I_{2n-p,p}$, $(p \le n)$ and $\theta(g) = I_{2n-1,1}\, g\, I_{2n-1,1}$. Then $H = \mathrm{SO}_{2n-p} \times \mathrm{SO}_p$, $K = \mathrm{SO}_{2n-1} \times \mathrm{SO}_1$. Now $\Phi(A)$ is of type A_1, $m^+(\lambda, \tau) = 2n - p - 1 > 0$, $m^-(\lambda, \tau) = p - 1 > 0$ and $F_0 = \{e\}$. Thus one can have $F_0 = \{e\}$ even without $A = T_-^\sigma$ or $A = T_-^\theta$ (compare 8.13).

While F_0 is a subgroup of $A^{(2)}$, the same does not necessarily hold for F. We leave it as an exercise to show:

Lemma 10.9. *Let w, $w' \in W_H(A)$. Then*

(1) $a_{w,q} = a_{w,q^{-1}}$.
(2) $a_{w,q}\, a_{w',q} = w(a_{w^{-1}w',q})$.

Corollary 10.10. *F is $W_H(A)$-stable iff F is a subgroup of $A^{(2)}$.*

Example 10.11. Take $G = \mathrm{SL}_5$, $\theta(g) = {}^t g^{-1}$, $\sigma = \theta \operatorname{Int}(q)$ and $A = T = \operatorname{diag}(\mathrm{SL}_5)$, a maximal torus. Then $W_M(A) = W_q$ and $F_0 = \{e\}$. The basic quadratic elements are:

$q_1 = \operatorname{diag}(1, -1, -1, -1, -1)$, $q_2 = \operatorname{diag}(-1, -1, 1, 1, 1)$,
$q_3 = \operatorname{diag}(1, 1, 1, -1, -1)$ and $q_4 = \operatorname{diag}(-1, -1, -1, -1, 1)$.

Up to conjugacy under the Weyl group we may assume that $q = e$, q_1 or q_2.

$q = e$: Here $\sigma = \theta$ and $W_H(A) = W_M(A) = W_H^*(A) \simeq S_5$.
$q = q_1$: There is a unique W_q-orbit in F^q, generated by $a_{s_1,q_1} = q_2$, and $F = \{\mathrm{Id}, q_2, q_1q_2q_3, q_1q_3q_4, q_1q_4\}$.
$q = q_2$: The W_q-orbits in F^q are represented by $a_{s_2,q_2} = q_1q_3$ and $s_2s_3s_1a_{s_2,q_2}$, and $F = \{\mathrm{Id}, q_1q_3, q_1q_2q_3q_4, q_1q_2q_4, q_1q_2q_3, q_1q_3q_4, q_1q_4, q_4, q_3q_4, q_2q_3\}$.

Note that F is not a subgroup of $A^{(2)}$ when q is q_1 or q_2.

11. Real reductive symmetric spaces

We consider applications to the classification of double coset spaces $H'\backslash G'/K'$ arising from commuting involutions (σ, θ) for G' a real group. If K' is compact we may assume that θ is a Cartan involution of G'. We can then reduce to computing over $\mathbb{C}$ (cf. [HW93]). Throughout this section we assume that G is defined over $\mathbb{R}$.

11.1. If X is a complex variety defined over $\mathbb{R}$, then we denote by $X(\mathbb{R})$ or $X_{\mathbb{R}}$ the real algebraic set of $\mathbb{R}$-rational points of X.

From [Bir71] we have:

Lemma 11.2. *Let X be an affine G-variety with X and the G-action defined over $\mathbb{R}$. Let $x \in X_{\mathbb{R}}$. Then Gx is closed if and only if $G_{\mathbb{R}}x$ is closed.*

11.3. The set of $\mathbb{R}$-automorphisms of G is denoted by $\mathrm{Aut}_{\mathbb{R}}(G)$, and σ and θ will always denote a pair of commuting involutions in $\mathrm{Aut}_{\mathbb{R}}(G)$. By definition, $\beta\colon G_{\mathbb{R}}/K_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}}) := \beta(G_{\mathbb{R}})$ is surjective. However, $P_\theta(G_{\mathbb{R}}) \neq P_\theta(G)_{\mathbb{R}}$ in general. One has to deal with real semialgebraic sets.

Given a maximal $\mathbb{R}$-split and θ-split torus $A_0 \subset G^\tau$, there is always a maximal (σ, θ)-split torus A, defined over $\mathbb{R}$, which contains A_0 ([HW93]). We fix such an A and A_0. Note that $Z^*_H(A)$ is defined over $\mathbb{R}$, and that $(Z^*_H(A))_{\mathbb{R}} = Z^*_{H(\mathbb{R})}(A_{\mathbb{R}}) = Z^*_{H(\mathbb{R})}(A)$. We have analogous results for $N^*_{H(\mathbb{R})}(A)$. Since $Z^*_H(A) \cap N^*_{H(\mathbb{R})}(A) \subset Z^*_{H(\mathbb{R})}(A)$, we may identify $W^*_{H(\mathbb{R})}(A) := N^*_{H(\mathbb{R})}(A)/Z^*_{H(\mathbb{R})}(A)$ with its image in $W^*_H(A)$.

Lemma 11.4. (1) *Let L be a reductive $\mathbb{R}$-subgroup of G such that $L_{\mathbb{R}}$ is compact and let S be an $\mathbb{R}$-torus in G. Then the natural map $W_{L(\mathbb{R})}(S) \xrightarrow{\sim} W_L(S)$ is an isomorphism.*

(2) *Let A' be a maximal (σ, θ)-split $\mathbb{R}$-torus in G whose intersection with G^τ is maximally $\mathbb{R}$-split, and suppose that $M_{\mathbb{R}}$ is compact. Then A' is $M^0_{\mathbb{R}}$-conjugate to A.*

Proof. Since $L_{\mathbb{R}}$ is compact, L is the complexification of $L_{\mathbb{R}}$, and we may write any $g \in L$ in the form $g = g_0 y$ where $g_0 \in L_{\mathbb{R}}$ and $y = \exp(iY)$, $Y \in \mathfrak{l}_{\mathbb{R}}$. If $g \in N_L(S)$, then $\bar{g}^{-1} g = y g_0^{-1} g_0 y = y^2 \in N_L(S)$. Since $W_L(S)$ is finite, some power y^{2m} is in $Z_L(S)^0$, which implies that $Y \in Z_{\mathfrak{l}}(\mathfrak{b})$. Thus y acts trivially on S, and we have (1).

For (2), first suppose that $A_0 = \{e\}$. Then $A_{\mathbb{R}}$ is a maximal compact subgroup of A. There is an $m \in M^0$ such that $m \cdot A = A'$, and $m \cdot A_{\mathbb{R}}$ is the maximal compact subgroup $A'_{\mathbb{R}}$ of A'. Write $m = m_0 y$ where $m_0 \in M^0_{\mathbb{R}}$ and $y \in \exp(i\mathfrak{m}_{\mathbb{R}})$. Replacing A' by $m_0^{-1} \cdot A'$ we may suppose that $m = y$. If $a \in A_{\mathbb{R}}$, then $y \cdot a = \bar{y} \cdot \bar{a} = y^{-1} \cdot a$. It follows that y^2 and y centralize $A_{\mathbb{R}}$, and we have $m_0 \cdot A = A'$.

In general, there is an $m \in M^0_{\mathbb{R}}$ which conjugates A_0 into the maximal $\mathbb{R}$-split subtorus A'_0 of A' [HW93], hence we may suppose that $A_0 = A'_0$. Now A/A_0 and A'/A_0 are maximal (σ, θ)-split $\mathbb{R}$-tori of $G_1 := Z_G(A_0)/A_0$, hence they are conjugate under $\tilde{M} := (G_1^\sigma \cap G_1^\theta)^0$. Since $(A/A_0)_{\mathbb{R}}$ and $(A'/A_0)_{\mathbb{R}}$ are compact, they are conjugate by an $m \in \tilde{M}^0_{\mathbb{R}}$, which is in turn the image of an element of $M^0_{\mathbb{R}}$. ☐

Corollary 11.5. *Suppose that either*

(1) *$G_{\mathbb{R}}$ is compact, or*
(2) *$M_{\mathbb{R}}$ is compact and every maximal (σ, θ)-split $\mathbb{R}$-torus is $\mathbb{R}$-split.*

Then any two maximal (σ, θ)-split $\mathbb{R}$-tori of G are $M^0_{\mathbb{R}}$ conjugate.

In the following we characterize the double cosets $H_{\mathbb{R}}\backslash G_{\mathbb{R}}/K_{\mathbb{R}}$ and Weyl groups $W^*_{H(\mathbb{R})}(A)$ in the two extreme cases that $A_{\mathbb{R}}$ is compact (resp. $\mathbb{R}$-split). We get the obvious extremes $W_{M(\mathbb{R})}(A) \subset W^*_H(A)$ for the Weyl groups as well. In other cases the geometry of the double cosets and the Weyl group is much more complicated. We intend to study this in a future paper.

11.6. **Compact groups with a pair of commuting involutions.** We first look at the case that $G_{\mathbb{R}}$ is compact. Note that $A_{\mathbb{R}}$ is then a compact torus and $\beta(A_{\mathbb{R}}) = A_{\mathbb{R}}$.

Theorem 11.7. *Let A be a maximal (σ, θ)-split $\mathbb{R}$-torus in G where $G_{\mathbb{R}}$ is compact. Then*

(1) $W_H(A) = W_{H(\mathbb{R})}(A)$.
(2) $W^*_H(A) = W^*_{H(\mathbb{R})}(A)$.
(3) *All $H_{\mathbb{R}}$-orbits in $P_\theta(G_{\mathbb{R}})$ and H-orbits intersecting $P_\theta(G_{\mathbb{R}})$ are closed.*
(4) *The $*$-action $H_{\mathbb{R}} \times A_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}})$ is surjective.*
(5) $G_{\mathbb{R}} = H_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$.
(6) *The inclusion $A_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}})$ induces an isomorphism $A_{\mathbb{R}}/W^*_{H(\mathbb{R})}(A) \to P_\theta(G_{\mathbb{R}})/H_{\mathbb{R}}$.*

Proof. Parts (1) and (2) follow from 11.4(1) and the observation that $A^{(2)} \subset A_{\mathbb{R}}$, and (3) follows from 11.2 and compactness of $H_{\mathbb{R}}$.

Note that (4), (5) and (6) are equivalent. First we consider the symmetric space version of (4)–(6). Let S be a maximal θ-split $\mathbb{R}$-torus of G and let $x \in P_\theta(G_{\mathbb{R}})$. Then the orbit $K \cdot x$ is closed, hence, by 5.12, G^x contains a maximal θ-split torus S' of G. Since G^x is defined over $\mathbb{R}$, we may assume that S' is also. By 11.5, we may assume that $S' = S$. Then the proof of Theorem 6.1 shows that $x \in S$, hence $x \in S_{\mathbb{R}}$. Thus $G_{\mathbb{R}} = K_{\mathbb{R}} S_{\mathbb{R}} K_{\mathbb{R}}$.

Now let $a \in A_{\mathbb{R}}$ where A is a maximal (σ, θ)-split $\mathbb{R}$-torus in $P_\theta(G_{\mathbb{R}})$. The differentiable slice theorem (see [Slo89]) and our calculations in Section 5 show that there is an $(H^*_a)_{\mathbb{R}}$-invariant open subset $\mathcal{S}$ of $\hat{P} := P_{\tilde{\theta}}((G^{\tilde{\tau}}_{\mathbb{R}})^0)$ such that $H_{\mathbb{R}} \times^{(H^*_a)_{\mathbb{R}}} \mathcal{S}a \to P_\theta(G_{\mathbb{R}})$ covers a $G_{\mathbb{R}}$-neighborhood of $H_{\mathbb{R}} * a$ (recall that $\tilde{\theta} = \theta \operatorname{Int}(a)$). By construction, $\hat{P} \supset A$, and our argument above shows that every point of $\mathcal{S}a$ is $(H^*_a)_{\mathbb{R}}$-conjugate to a point of A. Thus the image $H_{\mathbb{R}} * A_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}})$ is open. Since the image is closed and $P_\theta(G_{\mathbb{R}})$ is connected, $H_{\mathbb{R}} * A_{\mathbb{R}} = P_\theta(G_{\mathbb{R}})$. □

Remarks 11.8. It is well-known [RS90, 8.3.1] that if a compact real algebraic group C acts algebraically on a real algebraic set X, then any orbit Cx, $x \in X$, is algebraic and isomorphic to C/C_x. Thus we have

1. $P_\theta(G_{\mathbb{R}}) = (P_\theta(G))_{\mathbb{R}}$ is real algebraic.
2. If $a \in A_{\mathbb{R}}$, then $(H * a)(\mathbb{R}) = H_{\mathbb{R}} * a$.

11.9. **Noncompact groups with a Cartan involution.** Assume that $G_{\mathbb{R}}$ is noncompact. The involution θ is called a *Cartan involution* (of G or $G_{\mathbb{R}}$) if and only if $K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. In this case we can change any involution σ by an element of $\mathrm{Aut}_{\mathbb{R}}(G)$ such that σ and θ commute (see [Ber57] or [HW93]).

Conversely, given commuting involutions $\sigma, \theta \in \mathrm{Aut}(G)$ there is a σ and θ-stable conjugation δ of G whose fixed points are a compact real form [Hel88]. Set $\bar{\theta} = \theta\delta$. Then $\bar{\theta}$ determines a real form $G_{\mathbb{R}}$ of G, σ and θ are defined over $\mathbb{R}$ and θ is a Cartan involution.

We assume for the rest of this section that $G_{\mathbb{R}}$ is noncompact and that θ is a Cartan involution of $G_{\mathbb{R}}$.

11.10. We have $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$, and $G_{\mathbb{R}} = K_{\mathbb{R}} \exp(\mathfrak{p}_{\mathbb{R}})$ is the Cartan decomposition of $G_{\mathbb{R}}$. It follows that $P_\theta(G_{\mathbb{R}}) \simeq \exp(\mathfrak{p}_{\mathbb{R}})$ can be represented as real diagonalizable matrices with positive eigenvalues. Moreover, $P_\theta(G_{\mathbb{R}})$ contains no compact tori, hence $A_{\mathbb{R}} \simeq (\mathbb{R}^*)^s$ for some s. If $s \neq 0$, then $A_{\mathbb{R}}$ is not contained in $P_\theta(G_{\mathbb{R}})$, but the subgroup $A^2_{\mathbb{R}} := \beta(A_{\mathbb{R}})$ does lie in $P_\theta(G_{\mathbb{R}})$. Note that θ is a Cartan involution of $H_{\mathbb{R}}$ and that $M_{\mathbb{R}}$ is compact.

Analogously to 11.7 we have:

Theorem 11.11. *Let A be a maximal (σ, θ)-split $\mathbb{R}$-torus in P_θ. Then*

1. $W_M(A) = W_{M(\mathbb{R})}(A)$.
2. $N^*_{H(\mathbb{R})}(A) = N_{M(\mathbb{R})}(A)$, *hence* $W^*_{H(\mathbb{R})}(A) = W_{M(\mathbb{R})}(A)$.
3. *All $H_{\mathbb{R}}$-orbits in $P_\theta(G_{\mathbb{R}})$ and H-orbits intersecting $P_\theta(G_{\mathbb{R}})$ are closed.*
4. *The $*$-action $H^0_{\mathbb{R}} \times A^2_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}})$ is surjective.*
5. $G_{\mathbb{R}} = H^0_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$.
6. *The inclusion $A^2_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}})$ induces isomorphisms $A^2_{\mathbb{R}}/W_{H^0(\mathbb{R})}(A) \to P_\theta(G_{\mathbb{R}})/H^0_{\mathbb{R}}$ and*
   $A^2_{\mathbb{R}}/W_{H(\mathbb{R})}(A) \to P_\theta(G_{\mathbb{R}})/H_{\mathbb{R}}$.

Proof. Part (1) follows from 11.4(1). Let $h \in N^*_{H(\mathbb{R})}(A)$. Then $\beta(h) = h * e \in A \cap P_\theta(G_{\mathbb{R}})$ has order 2 and positive real eigenvalues (see 11.10). Hence $\beta(h) = e$ and $h \in N_{M(\mathbb{R})}(A)$ giving (2).

Let $y \in G_{\mathbb{R}}$. Then $H_{\mathbb{R}} * \beta(y)$ is the image of $H_{\mathbb{R}} y$ under the proper mapping $G_{\mathbb{R}} \to P_\theta(G_{\mathbb{R}})$. Hence $H_{\mathbb{R}} * \beta(y)$ and $H * \beta(y)$ are closed, giving (3).

As in 11.7, (4)–(6) are equivalent. Since $K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$, it intersects every component of $G_{\mathbb{R}}$, so that (5) is equivalent to the version with $H^0_{\mathbb{R}}$ replaced by $H_{\mathbb{R}}$. For notational convenience, we switch the roles of K and H and assume that σ is a Cartan involution of $G_{\mathbb{R}}$. Then $H_{\mathbb{R}}$ is compact and acts on the manifold $P_\theta(G_{\mathbb{R}})$ via $*$. The argument is now the same as in 11.7(5). □

Remarks 11.12. (1). The real Weyl groups $W^*_{H(\mathbb{R})}(A) = W_{M(\mathbb{R})}(A)$ of pairs (σ, θ) and $(\sigma, \tilde{\theta})$ may not be isomorphic. For example, if (σ, θ) is a standard pair, then $W_{M(\mathbb{R})^0}(A) = W_H(A)$, but if $(\sigma, \tilde{\theta})$ is not weakly-standard then $\tilde{W}_{M(\mathbb{R})^0}(A) \subsetneq W_H(A)$.

(2). In general it is difficult to determine $W_{M(\mathbb{R})}(A)$. On the other hand, the Weyl group $W_{M(\mathbb{R})^0}(A) = W(A, (G^\tau)^0)$ can be determined from the classification of pairs of commuting involutions in [Hel88]. So it is natural to ask when $W_{M(\mathbb{R})}(A) = W_{M(\mathbb{R})^0}(A)$.

Proposition 11.13. *Let A be a maximal (σ, θ)-split $\mathbb{R}$-torus in $P_\theta(G)$. Then the following are equivalent.*

(1) $W_{M(\mathbb{R})^0}(A) = W_{M(\mathbb{R})}(A)$.
(2) $W^*_{H(\mathbb{R})^0}(A) = W^*_{H(\mathbb{R})}(A)$.

Proof. Clearly (1) and (2) are equivalent to

(3) $N_{M(\mathbb{R})}(A) = N_{M(\mathbb{R})^0}(A) Z_{M(\mathbb{R})}(A)$ and
(4) $N^*_{H(\mathbb{R})}(A) = N^*_{H(\mathbb{R})^0}(A) Z^*_{H(\mathbb{R})}(A)$, respectively.

Now apply 11.11(2). □

12. Noncommuting involutions

12.1. We say that (σ, θ) *commute up to* $\operatorname{Aut}(G)$ *(resp.* $\operatorname{Int}(G)$*)* if there is a $\rho \in \operatorname{Aut}(G)$ (resp. $\operatorname{Int}(G)$) such that σ and $\theta' := \rho\theta\rho^{-1}$ commute. If (σ, θ) commute up to $\operatorname{Aut}(G)$, then $H\backslash G/K \simeq H\backslash G/K'$, and we may apply our theory above.

In [Hel88, 4.8 and Table II] the notions of *type* and *restricted rank* of involutions are defined, and it is shown [Hel88, 3.8] that two involutions are conjugate under the action of $\operatorname{Aut}(G)$ if and only if they have the same type and restricted rank. Then we get

Theorem 12.2. *The pair (σ, θ) commute up to* $\operatorname{Aut}(G)$ *if and only if there exists a pair of commuting involutions of the same type and restricted rank.*

Proof. Assume that (σ_1, θ_1) are a commuting pair of the same type and rank as (σ, θ). By [Hel88, 3.8] there is a $\rho \in \operatorname{Aut}(G)$ such that $\sigma_1 = \rho\sigma\rho^{-1}$, so conjugating the pair (σ, θ) by ρ, we can reduce to the case that $\sigma = \sigma_1$. Then $\theta_1 = \eta\theta\eta^{-1}$ for some $\eta \in \operatorname{Aut}(G)$. □

Remark 12.3. If (σ, θ) commute up to $\operatorname{Aut}(G)$, then they also commute up to $\operatorname{Int}(G)$, except when both the involutions are of type $D(III_a)$. One can show this result by combining [Hel88, 3.7] and [Hel91, 7.19].

For G not simple one can have many noncommuting pairs of involutions which are not isomorphic to a commuting pair by exchanging a number of the simple components of G. Here is an example:

Example 12.4. Let G_1 be a simple group with involutions σ_1 and θ_1. Let $G = G_1 \times G_1 \times G_1$, and for $(x_1, x_2, x_3) \in G$, define $\sigma(x_1, x_2, x_3) = (x_2, x_1, \sigma_1(x_3))$ and $\theta(x_1, x_2, x_3) = (\theta_1(x_1), x_3, x_2)$. Then σ and θ are not isomorphic to a commuting pair. For our discussion in 12.8 below, we note that $\sigma\theta$ is semisimple if and only if $\sigma_1\theta_1$ is semisimple.

To determine which cases can lead to a commuting pair of involutions we need to characterize the possibilities for a commuting pair. We can reduce to the case that (σ, θ) is irreducible. In the following, G_1 denotes a simple group and σ_1, θ_1 are commuting involutions of G_1.

Proposition 12.5 ([Hel88])**.** *Let (σ, θ) be irreducible with $\sigma\theta = \theta\sigma$. Up to inner isomorphisms of G there are the following possibilities:*

(1) *G is simple.*
(2) *$G = G_1 \times G_1$, and θ stabilizes both simple factors. Then $\sigma(x, y) = (\sigma_1(y), \sigma_1(x))$ and $\theta(x, y) = (\theta_1(x), \theta_1(y))$, $(x, y) \in G$.*
(3) *$G = G_1 \times G_1$, and neither σ nor θ stabilizes the simple factors. Then $\sigma(x, y) = (\sigma_1(y), \sigma_1(x))$ and $\theta(x, y) = (\theta_1(y), \theta_1(x))$, $(x, y) \in G$.*
(4) *$G = G_1 \times G_1 \times G_1 \times G_1$, $\sigma(x_1, x_2, x_3, x_4) = (\sigma_1(x_2), \sigma_1(x_1), \sigma_1(x_4), \sigma_1(x_3))$ and $\theta(x_1, x_2, x_3, x_4) = (\theta_1(x_4), \theta_1(x_3), \theta_1(x_2), \theta_1(x_1))$, $(x_1, x_2, x_3, x_4) \in G$.*

Remarks 12.6. (1). Suppose that (σ, θ) are as in 12.5(2), 12.5(3) or 12.5(4), but we drop the assumption that σ_1 and θ_1 commute. Then one easily shows that (σ, θ) commute up to $\mathrm{Aut}(G)$ if and only if (σ_1, θ_1) commute up to $\mathrm{Aut}(G_1)$. Thus, it suffices to analyze the case G simple.

(2). From the classification of pairs of commuting involutions in [Hel88, Table II,IV] it follows that if G is simple, two involutions of G commute up to $\mathrm{Aut}(G)$ except in the following 2 cases:

(i) θ is of type $A(II)$ and σ of type $A_m^p(III_a)$, with p odd.
(ii) θ is of type $D(III)$ and σ of type $D_m^p(I_a)$, with p odd.

These cases are discussed in detail in the following examples. In each case, up to an element of $\mathrm{Aut}(G)$, we have that $\tau = \sigma\theta$ has order 4.

Example 12.7. (1) Let $G = \mathrm{SL}_{2n}(k)$, $\theta(g) = J({}^t g^{-1})J^{-1}$ and $\sigma(g) = I_{p,q}({}^t g^{-1})I_{p,q}^{-1}$, where $I_{p,q}$ is as in example 10.8 with $p + q = 2m$ and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Then σ is of type $A_m^p(III_a)$ and θ is of type $A(II)$. From [Hel88, Table IV] it follows that σ and θ are isomorphic to a commuting pair of involutions iff p is even. If p is odd, $\tau = \sigma\theta$ is a semisimple automorphism of order 4.

(2). (See [Mat97, Remark 1]) $G = \mathrm{O}_{2m}$, $\sigma(g) = I_{p,q} g I_{p,q}$, $\theta(g) = J_m g J_m^{-1}$ where $I_{p,q}$ is as in example 10.8 with $p + q = 2m$, and J_m is the block diagonal product of m copies of J_1, where $J_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then σ is of type $D(I_a)$ and θ is of type $D(III)$. In this case, $\tau = \sigma\theta$ is a semisimple automorphism of order 4 whose square has the same form as σ, but with a different p and q, and τ^2 is inner. From [Hel88, Table IV] it follows that σ and

θ are isomorphic to a commuting pair of involutions iff p is even. Note that σ is an outer automorphism iff $p \neq q$ and p is odd.

12.8. **τ semisimple.** In [Mat97] Matsuki studies double cosets $H\backslash G/K$ for a real Lie group G. He does not assume that σ and θ commute, but imposes conditions which, in fact, imply that $\tau = \sigma\theta$ is semisimple. As we saw above, for G simple, one can always reduce to the case that τ has order 4 (hence is semisimple!). For general G the semisimplicity condition has more bite. The element $x := \tau^2$ plays an essential role, and to get anywhere, we need to assume that it is inner.

Remark 12.9. Using the classification in [Hel88] one can show that if G is simple, then $x \in \mathrm{Int}(G)$, except when G is of type D_4 and both σ and θ are of type $D_4^1(I_a)$. But then by 12.6, they commute up to $\mathrm{Aut}(G)$!

Proposition 12.10. *Suppose that $\tau := \sigma\theta$ is semisimple and that $x := \tau^2 \in \mathrm{Int}(G)$. Then:*

(1) *If $\tau \in \mathrm{Int}(G)$, then $x \in P_\theta((\mathrm{Int}\, G)^\tau)$.*
(2) *If $x \in P_\theta((\mathrm{Int}\, G)^\tau)^0$, then there exists $t \in P_\theta((\mathrm{Int}\, G)^\tau)^0$ such that σ and $t\theta t^{-1}$ commute.*
(3) *If (G^τ, θ) is almost connected (e.g., G is simply connected) and $\tau \in \mathrm{Int}(G)$, then (σ, θ) commute up to* $\mathrm{Aut}(G)$.

Proof. Part (1) follows from the fact that $x = \tau\theta(\tau)^{-1} = \tau\theta\tau^{-1}\theta$. For (2), we first show that θ preserves G^τ and $\mathrm{Int}(G)^\tau$. If $\tau(g) = g$, then $\tau\theta(g) = \sigma(g) = \sigma\tau(g) = \theta(g)$, so θ preserves G^τ. The same argument applies to $\mathrm{Int}(G)$, where τ, etc. act by conjugation. Since τ is semisimple, its fixed groups G^τ and $\mathrm{Int}(G)^\tau$ are reductive. If $x \in P_\theta((\mathrm{Int}\, G)^\tau)^0$, then 5.12 shows that there is a maximal θ-split torus S of $\mathrm{Int}(G)^\tau$ containing x. Let $t \in S$ with $x = (\sigma\theta)^2 = t^4$. Then the element $t\theta t^{-1}$ has the desired property. Finally (3) follows from (1) and (2). □

Remark 12.11. Assume that G^τ is connected.

(1). We can put our involutions in a normal form: Choose a θ-stable maximal torus T of $(\mathrm{Int}\, G)^\tau$ such that T_θ^- is a maximal θ-split torus of $(\mathrm{Int}\, G)^\tau$. Since $x \in Z((\mathrm{Int}\, G)^\tau)$ it follows that $x \in T$. Write $x = x_1x_2$ with $x_1 \in T_\theta^+$ and $x_2 \in T_\theta^-$. Let $t \in T_\theta^-$ such that $t^2 = x_2$, and set $\theta' = t\theta t^{-1}$ and $\tau' = \sigma\theta'$. Then $x' := (\tau')^2$ satisfies $x' \in T_\theta^+ \subset (\mathrm{Int}\, G)^\sigma \cap (\mathrm{Int}\, G)^\theta$, $(x')^2 = \mathrm{id}$. (Note that if $\tau \in \mathrm{Int}(G)$, we get $x' = \mathrm{id}$.) So we can reduce to the case that $x \in T_\theta^+$, where T is a torus as above, and $x \in (\mathrm{Int}\, G)^\sigma \cap (\mathrm{Int}\, G)^\theta$ has order 2.

(2). Using similar arguments as in 12.10, a detailed case analysis as in [Hel88] and 12.3 one can show that involutions (σ, θ) in normal form commute iff they commute up to $\mathrm{Aut}(G)$ iff $x = \mathrm{id}$.

Index of Notation

$\beta\colon G \to G$ the map $g \mapsto g\theta(g)^{-1}$	1.5
$P = P_\theta(G) = \beta(G)$	1.5
$*$ twisted action: $g * x := gx\theta(g)^{-1}$, $g \in G$, $x \in P$	1.5
A maximal (σ, θ)-split torus of G	1.6
$N_H^*(A) = \{h \in H \mid h * A = A\}$	1.6
$Z_H^*(A) = \{h \in H \mid h * a = a$ for every $a \in A\}$	1.6
$W_H^*(A) = N_H^*(A)/Z_H^*(A)$	1.6
$A^{(2)} = \{a \in A \mid a^2 = e\}$	1.9
k, algebraically closed base field, char $k \neq 2$ except in section 2	2.1
G reductive algebraic group, assumed connected unless otherwise specified	2.1
$\mathfrak{g}$ Lie algebra of G	2.1
X an affine G-variety	2.2
T_xX the tangent space in $x \in X$	2.2
Gx the G-orbit though $x \in X$	2.2
G_x the isotropy group of $x \in X$	2.2
$\mathcal{O}(X)^G$ the algebra of invariant functions on X	2.2
$X/\!\!/G$ the affine variety corresponding to $\mathcal{O}(X)^G$	2.2
π the morphism $X \to X/\!\!/G$ dual to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$	2.2
$\operatorname{clos}(Y)$ the Zariski closure of Y in X	2.2
X/G geometric quotient	2.3
$\mathcal{S} = \mathcal{S}(X)_x$ a slice at x	2.6
R a transversal to Gx at $x \in X$	2.6
σ and θ commuting involutions of G except in section 12	3.2
$H = G^\sigma$ fixed point group of σ	3.2
$K = G^\theta$ fixed point group of θ	3.2
S maximal θ-split torus of G	3.3
$\tau = \sigma\theta$	4.1
$M = K \cap H$	4.1
H_x^* isotropy group of H at x	5.4
$\tilde{\sigma} = \sigma$	5.8
$\tilde{\theta} = \theta \operatorname{Int}(x)$	5.8
$\tilde{\tau} = \tilde{\sigma}\tilde{\theta} = \tau \operatorname{Int}(x) = \operatorname{Int}(x)\tau$	5.8
$\tilde{G} = G^{\tilde{\tau}}$	5.8
$\tilde{H} = \tilde{G}^{\tilde{\sigma}}$	5.8
$\tilde{P} = P_{\tilde{\theta}}(\tilde{G})$	5.8
$\hat{P}(x) = \hat{P}$ the component of $P_{\tilde{\theta}}(\tilde{G})$ containing e	5.10
$\hat{G}(x) = \hat{G} = \{g \in \tilde{G} \mid g\tilde{\theta}(g)^{-1} \in \hat{P}(x)\}$	5.10
A_{pr} the principal points A	6.3

$\mathcal{F}(X)_x = \mathcal{F}_x$ the fiber $(\pi_X)^{-1}\pi_X(x)$	7.1
$[h]$ the class of $h \in N^*_H(A)$ in $W^*_H(A)$	8.1
$\rho\colon W^*_H(A) \to W_H(A) \ltimes A^{(2)}$ the map $[h] \mapsto (\mathrm{Int}(h), \beta(h))$	8.1
$W_0 \simeq (Z_H(A) \cap N^*_H(A))/Z^*_H(A)$ kernel of $\phi\colon W^*_H(A) \xrightarrow{\rho} W_H(A) \ltimes A^{(2)} \to W_H(A)$	8.3
$\psi\colon W^*_H(A) \xrightarrow{\rho} W_H(A) \ltimes A^{(2)} \to A^{(2)}$	8.3
F image of ψ	8.3
$F_0 = \psi(W_0)$	8.3
$Q(A)$ the set of quadratic elements in A	9.1
$X_*(A)$ multiplicative one parameter subgroups of A	9.4
$X^*(A)$ character group of A	9.4
$\Phi(A)$ root system of A	9.4
Δ a base of $\Phi(A)$	9.5
$\mathfrak{g}(A, \lambda)$ the root space corresponding to $\lambda \in \Phi(A)$	9.7
$(m^+(\lambda, \tau), m^-(\lambda, \tau))$ the signature of $\lambda \in \Phi(A)$	9.7
s_λ denotes the reflection through $\lambda \in \Phi(A)$	9.7
$q_\lambda, \lambda \in \Delta$ the basic quadratic elements	9.11

References

[BB81] A. Beilinson and J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C.R. Acad. Sci. Paris **292** (1981), no. I, 15–18.

[BD92] J.-L. Brylinski and P. Delorme, *Vecteurs distributions H-invariants pour les séries principales géneralisées d'espaces symétriques réductifs et prolongement méromorphe d'integrales d'Eisenstein*, Invent. Math. **109** (1992), 619–664.

[BdS49] A. Borel and J. de Siebenthal, *Les sous-groupes fermés de rang maximumdes groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221.

[Ber57] M. Berger, *Les espaces symétriques non-compacts*, Ann. Sci. École Norm. Sup. **4** (1957), 85–177.

[Bir71] D. Birkes, *Orbits of linear algebraic groups*, Annals of Math. **93** (1971), 459–475.

[Bou81] N. Bourbaki, Groupes et algébres de Lie, Chapitres 4, 5 et 6, Éléments de Mathématique, Masson, Paris, 1981.

[BR85] P. Bardsley and R. W. Richardson, *Étale slices for algebraic transformation groups in characteristic p*, Proc. London Math. Soc. **(3) 51** (1985), no. 2, 295–317.

[BS97] E. van den Ban and H. Schlichtkrull, *The most continuous part of the Plancherel decomposition for a reductive symmetric space I*, Ann. Math. **145** (1997), 267–364.

[BT65] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–152.

[DCP83] C. De Concini and C. Procesi, *Complete symmetric varieties*, Invariant theory (Montecatini, 1982), Springer, Berlin, 1983, pp. 1–44.

[Del98] P. Delorme, *Formule de Plancherel pour les espaces symétrique réductifs*, Annals of Math. **147** (1998), no. 2, 417–452.

[FJ78] M. Flensted-Jensen, *Spherical functions on a real semisimple Lie group. A method of reduction to the complex case*, J. Funct. Anal. **30** (1978), 106–146.

[FJ80] M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Annals of Math. **111** (1980), 253–311.

[Gro92] I. Grojnowski, *Character sheaves on symmetric spaces*, Ph.D. thesis, Massachusetts Institute of Technology, June 1992.

[Hab75] W. J. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. **102** (1975), 67–84.

[Hel88] A. G. Helminck, *Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces*, Adv. in Math. **71** (1988), 21–91.

[Hel91] A. G. Helminck, *Tori invariant under an involutorial automorphism I*, Adv. in Math. **85** (1991), 1–38.

[Hoo84] B. Hoogenboom, *Intertwining functions on compact lie groups*, CWI Tract, vol. 5, CWI, Amsterdam, The Netherlands, 1984.

[HW93] A. G. Helminck and S. P. Wang, *On rationality properties of involutions of reductive groups*, Adv. in Math. **99** (1993), 26–96.

[JC74] A. T. James and A. G. Constantine, *Generalized jacobi polynomials as spherical functions of the grassmann manifold*, Proc. London Math. Soc. **29** (1974), 174–192.

[Kir93] A. Kirillov, *The orbit method. I. Geometric quantization*, Representation theory of groups and algebras (Providence, RI), Contemp. Math., vol. 145, Amer. Math. Soc., 1993, pp. 1–32.

[KR71] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[Lun73] D. Luna, *Slices étales*, Bull. Soc. Math. France **Mémorie 33** (1973), 81–105.

[LV83] G. Lusztig and D.A. Vogan, *Singularities of closures of K-orbits on flag manifolds*, Invent. Math. **71** (1983), 365–379.

[Mat97] T. Matsuki, *Double coset decompositions of reductive Lie groups arising from two commuting involutions*, J. of Alg. **197** (1997), 49–91.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Springer Verlag, Berlin–New York, 1994.

[OS80] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators in an affine symmetric space*, Invent. Math. **57** (1980), 1–81.

[Ric82a] R.W. Richardson, *On orbits of algebraic groups and lie groups*, Bull. Austral. Math. Soc. **25** (1982), 1–28.

[Ric82b] R.W. Richardson, *Orbits, invariants and representations associated to involutions of reductive groups*, Invent. Math. **66** (1982), 287–312.

[RS90] R.W. Richardson and P.J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. **42** (1990), 409–429.

[Slo89] P. J. Slodowy, *Der scheibensatz für algebraische transformationsgruppen*, Algebraic Tranformation Groups and Invariant Theory (Basel-Boston), DMV Seminar, vol. 13, Birkhäuser Verlag, Basel-Boston, 1989, pp. 89–113.

[Ste68] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., vol. 80, Amer. Math. Soc., Providence, RI, 1968.

[Vog83] D. A. Vogan, *Irreducible characters of semi-simple Lie groups III. Proof of the Kazhdan-Lusztig conjectures in the integral case*, Invent. Math. **71** (1983), 381–417.

[Vus74] T. Vust, *Opération de groupes reductifs dans un type de cônes presque homogènes*, Bull. Soc. Math. France **102** (1974), 317–334.

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