ALGORITHMS FOR COMPUTING CHARACTERS FOR SYMMETRIC SPACES

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ABSTRACT. The classification theorem for semisimple Lie algebras states that up to isomorphy any finite dimensional semisimple Lie algebra $\mathfrak{g}$ defined over an algebraically closed field is uniquely determined by the root system of a maximal toral subalgebra. Moreover, if $\mathfrak{g}$ is simple, then this root system is irreducible and its Dynkin diagram must be one of A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2. This is fundamental to the study of finite-dimensional semisimple Lie algebras over algebraically closed fields. A. G. Helminck established an analogous result for local symmetric spaces where he identified twenty-four graphical structures called involution or θ-diagrams (see [Hel88]). Implicit in each of these diagrams are two root systems $\Phi(\mathfrak{a})$ and $\Phi(\mathfrak{t})$ with $\mathfrak{a}$ a maximal toral subalgebra in a local symmetric space $\mathfrak{p}$ and $\mathfrak{t} \supset \mathfrak{a}$ a maximal toral subalgebra in the corresponding semisimple Lie algebra $\mathfrak{g}$. The root system $\Phi(\mathfrak{a})$ can be described as the image of $\Phi(\mathfrak{t})$ under a projection π derived from an involution θ on $\Phi(\mathfrak{t})$. The weight lattices associated with $\Phi(\mathfrak{t})$ and $\Phi(\mathfrak{a})$ are denoted by $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{a}}$, respectively. In [GH06] it was shown that if π is extended linearly from $\Phi(\mathfrak{t})$ to $\Lambda_{\mathfrak{t}}$ then $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}}$. Explicit algorithmic formulations for the characters of each of these lattices (in terms of the other) was also given. In this paper we provide a methodology for implementing these algorithms for use in a computer algebra package.

1. INTRODUCTION

Let θ be an involution of the connected reductive algebraic group G where θ and G are defined over a (usually algebraically closed) field k, char $k \neq 2$. Denote the fixed point group of θ by $K := G^{\theta}$. The variety G/K is called a *symmetric variety*. These varieties occur in many problems in representation theory (see [BB81] and [Vog83]) and geometry (see [DCP83] and [LV83]). If $k = \mathbb{C}$, then G/K is the "complexification" of a Riemannian symmetric space and there exists a one to one correspondence between isomorphism classes of Riemannian symmetric spaces and isomorphism classes of involutions of G (see [Hel88]). Therefore the symmetric varieties G/K are also frequently called "symmetric spaces".

Essential in all studies of reductive groups defined over an algebraically closed field is their natural fine structure of a root system with its Weyl group and weight lattice coming from a maximal torus. Symmetric space have a similar fine structure, which plays an equally important role in the study of these symmetric spaces and their applications as their counterpart in the groups case. However, the fine structure of these symmetric spaces is more complicated since it involves the intricate relations of two root systems, Weyl groups, and weight lattices instead of just one. The root system of the symmetric space can be obtained as the projection of the root system of the group onto a subspace. The Weyl group of the symmetric space can be expressed in a similar manner (see [Hel88]). It was shown in [Hel88, DH04] that the fine structure of these symmetric varieties is the same as that of the Riemannian symmetric spaces. In [GH06] the authors gave algorithms for computing elements of the weight lattice of the symmetric space in terms of the weights of the group.

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A more detailed summary of our results follows. There exists a natural embedding of G/K in G via the map $\tau : G \to G$ defined by $\tau(g) := g\theta(g)^{-1}$. Denote the image of τ by $P := \{g\theta(g)^{-1} \mid g \in G\}$. Then $P \simeq G/K$. Similarly as for Lie groups we can study the fine structure of a symmetric space locally. The corresponding local symmetric space is defined as follows: Let $\mathfrak{g}$ be the Lie algebra of G and let $d\theta \in \mathrm{Aut}(\mathfrak{g})$ be the involution induced by θ on $\mathfrak{g}$. By abuse of notation we will also write θ for $d\theta$. For this case we let

$$\mathfrak{k} = \{A \in \mathfrak{g} \mid \theta(A) = A\}$$
$$\mathfrak{p} = \{A \in \mathfrak{g} \mid \theta(A) = -A\}$$

We have that $\mathfrak{k}$ and $\mathfrak{p}$ are the tangent spaces in the identity of K and P above. Furthermore as $\theta \in \mathrm{Aut}(\mathfrak{g})$ is a Lie algebra automorphism, then $\mathfrak{k}$, (the $+1$ eigenspace relative to θ), is a subalgebra of $\mathfrak{g}$. While $\mathfrak{p}$, (the -1 eigenspace), is not a subalgebra of $\mathfrak{g}$ we nevertheless have that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} \perp \mathfrak{p}$ relative to the killing form. We refer to $\mathfrak{p}$ as a *local symmetric space* of $\mathfrak{g}$ relative to θ. Let $\mathfrak{a}$ be a maximal toral subalgebra in a local symmetric space $\mathfrak{p}$ and $\mathfrak{t} \supset \mathfrak{a}$ a maximal toral subalgebra in the corresponding semisimple Lie algebra $\mathfrak{g}$. Denote the root systems of $\mathfrak{a}$ and $\mathfrak{t}$ with respect to $\mathfrak{g}$ by $\Phi(\mathfrak{a})$ and $\Phi(\mathfrak{t})$. The root system $\Phi(\mathfrak{a})$ can be described as the image of $\Phi(\mathfrak{t})$ under a projection π derived from an involution θ on $\Phi(\mathfrak{t})$. The map π maps the root lattice related to $\Phi(\mathfrak{t})$ onto the root lattice related to $\Phi(\mathfrak{a})$. Denote the weight lattices associated with $\Phi(\mathfrak{t})$ and $\Phi(\mathfrak{a})$ by $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{a}}$, respectively. Extend π linearly from $\Phi(\mathfrak{t})$ to the lattice $\Lambda_{\mathfrak{t}}$. In [Hel88] it was shown that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$, but the converse was left open as a conjecture. The truth of this conjecture was established by the authors in [GH06]. The proof involved a case by case analysis for each type of semisimple local symmetric space. This approach enabled us to give explicit algorithmic formulations for the characters of each of these lattices in terms of the other. In this paper we give a method for implementing these algorithms in a computer algebra package.

2. Local Symmetric Spaces and θ-diagrams

In this section we set the notations and recall some results from [Hel88] and [GH06]. Our basic reference for reductive groups will be the books of Humphreys [Hum75] and Springer [Spr98]. Our basic reference for Lie algebras and root systems will be [Hum72] and [Bou81]. We shall follow their notations and terminology.

2.1. **Root Space Decomposition.** Throughout this section we let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over an algebraically closed field k. Let $\mathfrak{t} \subset \mathfrak{g}$ be a toral subalgebra and let $\mathfrak{t}^*$ denote its dual. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi(\mathfrak{t})} \mathfrak{g}_\alpha \tag{1}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x,\ \forall y \in \mathfrak{t}\}$, $\Phi(\mathfrak{t}) = \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$ and $\mathfrak{g}_0$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. This decomposition of $\mathfrak{g}$ is called the *root space decomposition* of $\mathfrak{g}$ (with respect to $\mathfrak{t}$). The elements α of $\Phi(\mathfrak{t})$ are called the *roots* of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and $\{\mathfrak{g}_\alpha \mid \alpha \in \Phi(\mathfrak{t})\}$ the associated *root spaces*. If $\mathfrak{t} \subset \mathfrak{g}$ is a maximal toral subalgebra, then for each $\alpha \in \Phi(\mathfrak{t})$ we have $\dim(\mathfrak{g}_\alpha)= 1$, $\mathfrak{g}_0 = \mathfrak{t}$ and $\Phi(\mathfrak{t})$ is a root system. It is well known that every finite dimensional semi-simple Lie algebra is characterized (up to isomorphy) by its root system. Moreover, if $\mathfrak{g}$ is simple then its root system is irreducible and its Dynkin diagram must be one of A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2. See [Hum72] for a depiction of these diagrams and their corresponding Cartan matrices.

2.2. **Representations and Weights.** Let $R_{\mathfrak{t}}$ denote the integer lattice spanned by $\Phi(\mathfrak{t})$ and let $\Lambda_{\mathfrak{t}} = \{\lambda \in \mathfrak{t}^* \mid <\lambda, \alpha> \in \mathbb{Z}\ (\alpha \in \Phi(\mathfrak{t}))\}$. We call $R_{\mathfrak{t}}$ the *root lattice* of $\mathfrak{t}$ and $\Lambda_{\mathfrak{t}}$ the *weight lattice* of $\Phi(\mathfrak{t})$.

If ρ is a representation of $\mathfrak{g}$ on a vector space V, then $\rho(\mathfrak{t})$ is diagonalizable in $\mathfrak{gl}(V)$. Accordingly, we have the common eigenspace decomposition $V = \oplus V_\lambda$ where λ runs over $\mathfrak{t}^*$ and

$$V_\lambda = \{v \in V \mid \rho(y)(v) = \lambda(y)(v)\ \forall y \in \mathfrak{t}\}.$$

As before, whenever V_λ is nontrivial we call it a *weight space*, and λ a *weight* of $\mathfrak{t}$ on V. If ρ is the adjoint representation then the weights are contained in the root lattice, but in general, the weights of a representation are "outside the reach" of the root lattice, but are contained in the weight lattice. For irreducible root systems we have the following useful result.

Proposition 1 ([Hum72]). *Let $\Phi(\mathfrak{t})$ be an irreducible root system of rank n and let $\Delta(\mathfrak{t})$ be an ordered base for $\Phi(\mathfrak{t})$. Let $C(\Delta)$ be the Cartan matrix for $\Phi(\mathfrak{t})$. Define a set of vectors, $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, in $\mathfrak{t}^*$ such that the components of λ_i, with respect to $\Delta(\mathfrak{t})$, are given by the i-th row of $C(\Delta)^{-1}$. In other words,*

$$\lambda_i = \sum_{j=1}^{n} (C(\Delta)^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\} \tag{2}$$

These are called the fundamental weights relative to $\Delta(\mathfrak{t})$. Then $\Lambda_{\mathfrak{t}} = span_{\mathbb{Z}}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$.

2.3. **Local Symmetric Spaces.** We now turn our attention to a description of symmetric spaces on Lie algebras. Let $\theta \in \mathrm{Aut}(\mathfrak{g})$ be an involution. In this case we let

$$\mathfrak{k} = \{A \in \mathfrak{g} \mid \theta(A) = A\}$$
$$\mathfrak{p} = \{A \in \mathfrak{g} \mid \theta(A) = -A\}$$

Here $\mathfrak{p}$ is called a *local symmetric space* of the Lie algebra $\mathfrak{g}$. Using these symmetric spaces we obtain a decomposition of $\mathfrak{g}$ reminiscent of the root space decomposition above. Indeed, let $\mathfrak{a}$ be a toral subalgebra maximal in $\mathfrak{p}$. Then as before we have that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\omega \in \Phi(\mathfrak{a})} \mathfrak{g}_\omega$$

where $\mathfrak{g}_\omega = \{x \in \mathfrak{g} \mid [t, x] = \omega(t)x\ \forall t \in \mathfrak{a}\}$ and $\Phi(\mathfrak{a}) = \{\omega \in \mathfrak{a}^* \mid \omega \neq 0, \mathfrak{g}_\omega \neq 0\}$. In this setting the eigenspaces, $\mathfrak{g}_\omega$, are multidimensional, in general. We also have that $\Phi(\mathfrak{a})$ is a *reduced root system*, one for which twice (or half) a root may also be a root. When this occurs, then $\Phi(\mathfrak{a})$ is of type BC_n. The analysis and explicit formulation for the elements of $\Phi(\mathfrak{a})$ is carried out in [DH04].

2.4. ***θ*-diagrams.** Similar as for semisimple Lie algebras the fine structure of these local symmetric spaces can be represented by a diagram, called the θ-diagram. In this subsection we recall the construction of these diagrams from [Hel88]. Let θ be an involution on $\mathfrak{g}$ and θ^* the induced involution on the algebraic dual $\mathfrak{g}^*$. To maintain consistency with the literature, we drop the "*" on θ^*. The underlying space will be clear from the context. Let $\Lambda_0(\theta) = \{\lambda \in \Lambda_{\mathfrak{t}} \mid \theta(\lambda) = \lambda\}$ and $\Phi_0(\theta) = \Lambda_0(\theta) \cap \Phi(\mathfrak{t})$ Evidently, $\Lambda_0(\theta)$ and $\Phi_0(\theta)$ are, respectively, the characters and roots fixed by θ. Note that $\Phi_0(\theta)$ is a closed subsystem of $\Phi(\mathfrak{t})$, hence a root system itself. Let π be the canonical projection, $\pi : \Lambda_{\mathfrak{t}} \to \Lambda_{\mathfrak{t}}/\Lambda_0(\theta)$ and write $\overline{\Lambda_{\mathfrak{t}}} = \pi(\Lambda_{\mathfrak{t}})$ and $\overline{\Phi(\mathfrak{t})} = \pi(\Phi(\mathfrak{t}))$. The next proposition relates the roots of $\mathfrak{t}$ and $\mathfrak{a}$.

Proposition 2 ([Hel88]). *Let $\mathfrak{t}$ and $\mathfrak{a}$ be as above. Then for $\lambda \in \Lambda_{\mathfrak{t}}$ we have $\pi(\lambda) = \frac{1}{2}(\lambda - \theta(\lambda))$. In particular π is onto and $\Phi(\mathfrak{a}) = \overline{\Phi(\mathfrak{t})} = \{\pi(\alpha) \mid \pi(\alpha) \neq 0,\ \alpha \in \Phi(\mathfrak{t})\}$.*

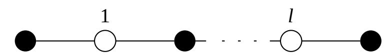
AII 

FIGURE 1. θ-diagram for the case AII

We now focus on the choice of base for $\Phi(\mathfrak{t})$ developing the notion of a "θ-base" that is, in a specific way, compatible with the action of θ.

Definition 1. An order $\succ$ on $\Phi(\mathfrak{t})$ is called a *θ-order* if whenever $\chi \in \Lambda_{\mathfrak{t}}$, with $\chi \succ 0$, $\chi \notin \Lambda_0(\theta)$, then $\theta(\chi) \prec 0$.

The base we derive from this order is called a θ-base. In [Hel88] it was shown that a θ-base can be found by choosing bases for $\Phi(\mathfrak{a})$ and $\Phi_0(\theta)$.

For the next step let $W(\mathfrak{t})$ be the Weyl group of $\Phi(\mathfrak{t})$. For a subset $S \subset \Phi(\mathfrak{t})$, denote by $W(S)$ the subgroup of $W(\mathfrak{t})$ generated by $s_\alpha,\ \alpha \in S$. Let $W_0(\theta)$ be the Weyl group of $\Phi_0(\theta)$. Then $W_0(\theta) = W(\Phi_0(\mathfrak{t}))$. Now let $\Delta(\mathfrak{t})$ be a θ-base of $\Phi(\mathfrak{t})$ and let $\Delta_0(\theta) = \Delta(\mathfrak{t}) \cap \Phi_0(\theta)$ be a base of $\Phi_0(\theta)$. Let $w_0(\theta)$ be the longest Weyl group element of $W_0(\theta)$ with respect to the base $\Delta_0(\theta)$. Then

$$w_0(\theta)[\Phi_0^+(\theta)] = \Phi_0^-(\theta)$$

This equation implies that $w_0(\theta)^2$ leaves $\Delta_0(\theta)$ fixed, hence $w_0(\theta)^2$ is the identity.

For the next step we define (another) map θ^* on $\Phi(\mathfrak{t})$ by

$$\theta^* = -\operatorname{id} \circ \theta \circ w_0(\theta) \tag{3}$$

As each involution on the right hand side of Equation (3) commutes we have that

$$\theta^* = \begin{cases} \text{id} \\ \text{Dynkin Diagram automorphism of order 2} \end{cases}$$

In light of this and Equation 3 we may recover θ. Indeed,

$$\theta = -\operatorname{id} \circ \theta^* \circ w_0(\theta) \tag{4}$$

We are now ready to define and describe the θ-diagram.

Definition 2. Let $\Delta(\mathfrak{t})$ be a θ-base for $\Phi(\mathfrak{t})$ and let D be the corresponding Dynkin Diagram. Color black each $\alpha \in \Delta_0(\theta)$. Denote the action of θ^* on D by arrows. We call these enhanced diagrams *involution diagrams* or *θ-diagrams* for short.

Similar to the Classification Theorem for semisimple Lie algebras A.G. Helminck established a correspondence between isomorphy classes of semisimple local symmetric spaces and congruence classes of θ-diagrams. In particular, up to congruence, Helminck identified nine semi-irreducible and twenty-four absolutely irreducible θ-diagrams that serve to characterize the fine structure of local symmetric spaces. Figure 2.4 gives an example of a θ-diagram. See [Hel88] for a complete list.

2.5. **Relation between weight lattices $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{a}}$.** At this point we are now able to give explicit definitions for the weight lattices of the Lie algebra and symmetric space. First, from the linearity of π and Proposition 2 we have that $\overline{R_{\mathfrak{t}}} = R_{\mathfrak{a}}$. As for the weight lattice recall from Equation (2) that

$$\lambda_i = \sum_{j=1}^{n} (C(\Delta)^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\}$$

where $C(\Delta)^{-1}$ is the inverse of the Cartan matrix of $\Phi(\mathfrak{t})$ relative to the ordered base $\Delta(\mathfrak{t})$. We define vectors γ_i with $i \in \{1, 2, \cdots, n\}$ by applying π to both sides of this equation. So that

$$\gamma_i = \pi(\lambda_i) = \sum_{j=1}^{n} (C(\Delta)^{-1})_{ij} \cdot \pi(\alpha_j) \tag{5}$$

with

$$\pi(\Lambda_{\mathfrak{t}}) = \text{span}_{\mathbb{Z}}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$$

Next let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. Let $C(\Delta(\mathfrak{a}))$ be the Cartan matrix of $\Phi(\mathfrak{a})$ relative to $\Delta(\mathfrak{a})$. Using Proposition 1 we define

$$\mu_i = \sum_{j=1}^{p} (C(\Delta(\mathfrak{a}))^{-1})_{ij} \cdot \omega_j \qquad i \in \{1, 2, \cdots, p\} \tag{6}$$

We refer to the integer span of the μ's as the *weight lattice of the symmetric space* and denote it by $\Lambda_{\mathfrak{a}}$. Evidently, $\pi(\Lambda_{\mathfrak{t}})$ is in the $\mathbb{Q}$-span of $\Delta(\mathfrak{a})$. In fact we also have that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as indicated by the following theorem.

Theorem 1 ([GH06]). *Let $\Phi(\mathfrak{t})$ and its projection $\Phi(\mathfrak{a})$ be irreducible. Then $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}}$.*

The steps in the proof serve as a basis for constructing the algorithm for computing the characters. For convenience we present them here.

Type 1. $\Phi(\mathfrak{a})$ is not of type BC_n

Step 1 Use Proposition 1 to compute the fundamental dominant weights, $\lambda_i,\ i \in \{1, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Recall that the integer span of these vectors is the lattice $\Lambda_{\mathfrak{t}}$.

Step 2 Use the implicitly defined projection map of the θ-diagram to compute $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ where $p = \mid \Delta(\mathfrak{a}) \mid$. The integer span of the γ's is the lattice $\pi(\Lambda_{\mathfrak{t}})$, (see Equation 5.)

Step 3 Again use Proposition 1 to compute the fundamental dominant weights $\mu_i,\ i \in \{1, 2, \cdots, p\}$ with respect to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$. Recall that the integer span of the μ's is the lattice $\Lambda_{\mathfrak{a}}$, (see Equation 6.)

Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ by finding integers b_i such that $\gamma_i = \sum_1^p b_i \mu_i\ \forall j = \{1, 2, \cdots, n\}$. That this is possible is guaranteed by [GH06, Theorem 2(1)].

Step 5 Use the result of Step 4 to prove the theorem for the (particular) symmetric space; i.e. show that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ explicitly by finding integers a_i such that $\mu_i = \sum_1^p a_i \gamma_i\ \forall j = \{1, 2, \cdots, p\}$.

Type 2. $\Phi(\mathfrak{a})$ is of type BC_n

Step 1 As before use Proposition 1 to compute the fundamental dominant weights, $\lambda_i,\ i \in \{1, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

Step 2 Again use θ-diagram to compute $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$. Show that the coefficients for the γ's relative to $\Phi(\mathfrak{a})$ are integers. This gives immediately an explicit formula to show that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}} = \Lambda_{\mathfrak{a}}$.

Step 3 Use the result of Step 2 to show that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}} = \Lambda_{\mathfrak{a}}$; i.e. find integers a_j such that $\omega_i = \sum_1^n a_j \gamma_j \forall i = \{1, 2, \cdots, p\}$.

2.6. **Inverses of Cartan Matrices.** With a little work one can show that the determinants of each of the Cartan matrices is nonzero. They are thus invertible. In this section we give explicit formulas for the inverses. Each case can be verified by direct computation. Table 2 gives a summary of the results.

Table 1: Inverses of Cartan Matrices

Type θ	Inverse
A_n	$(A_n^{-1})_{i,j} = \begin{cases} i(1-\frac{j}{n+1}) & i \leq j \\ j(1-\frac{i}{n+1}) & i > j \end{cases}$
B_n	$(B_n^{-1})_{i,j} = \begin{cases} i & i < j \leq n \\ j & j \leq i < n \\ \frac{j}{2} & i = n \end{cases}$
C_n	$(C_n^{-1})_{i,j} = \begin{cases} j & j < i \leq n \\ i & i \leq j < n \\ \frac{i}{2} & j = n \end{cases}$
D_n	$(D_n^{-1})_{i,j} = \begin{cases} i & i \leq j \leq n-2 \\ j & j < i \leq n-2 \\ \frac{j}{2} & i \in \{n-1, n\}, j \leq n-2 \\ \frac{i}{2} & j \in \{n-1, n\}, i \leq n-2 \\ \frac{n}{4} & i = n-1, j = n-1 \\ \frac{n-2}{4} & i = n-1, j = n \\ \frac{n-2}{4} & i = n, j = n-1 \\ \frac{n}{4} & i = n, j = n \end{cases}$
E_6	$(E_6)^{-1} = \begin{pmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 2 & 3 & 2 & 1 \\ \frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & 3 & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix}$

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Table 1: *continued*

Type θ	Inverse
E_7	$(E_7)^{-1} = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{7}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 2 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix}$
E_8	$(E_8)^{-1} = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 4 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$
F_4	$(F_4)^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}$
G_2	$(G_2)^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

2.7. **Proof for the Case *A*II.** Since the algorithm we implement involves this case we give the details of the proof in this subsection. The proofs for the other cases are constructed similarly. We require the following lemma.

Lemma 1. *Let A_n^{-1} be the inverse Cartan matrix of Table 2. Let $k \in \{1, 2, \cdots, n-1\}$ and let s be a row vector whose components are the sums of the entries of the k-th and $(k+1)$-th rows of this matrix,* *i.e.* $s_j = (A_n^{-1})_{k,j} + (A_n^{-1})_{k+1,j}$. *Then*

$$s_j = \begin{cases} j(1-\frac{k}{n+1}) + j(1-\frac{k+1}{n+1}) & 1 \le j \le k-1 \\ k(1-\frac{k}{n+1}) + k(1-\frac{k+1}{n+1}) & j = k \\ k(1-\frac{j}{n+1}) + (k+1)(1-\frac{j}{n+1}) & k+1 \le j \le n \end{cases}$$

The proof of this lemma follows immediately from the formula for $(A_n)^{-1}$ and is omitted. We are now ready to state explicitly the formulation for the containment of $\pi(\Lambda_{\mathfrak{t}})$ in $\Lambda_{\mathfrak{a}}$.

Proposition 3. *Let γ and μ be as above (relative to the case AII). We then have that*

$$\gamma_k = \begin{cases} \mu_1 & k = 1 \\ 2\mu_{\frac{k}{2}} & k = \textit{even} \\ \mu_{\frac{k-1}{2}} + \mu_{\frac{k+1}{2}} & k = \textit{odd, and } \neq 2n+1 \\ \mu_n & k = 2n+1 \end{cases}$$

Proof. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a base for $\Phi(\mathfrak{a})$. In accordance with our aforementioned strategy, we start by computing coefficients for the weights, λ_i, with respect to $\Delta(\mathfrak{t})$. For this case we have that

$$\lambda_i = \sum_{j=1}^{2n+1} (A_{2n+1}^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, 2n+1\}$$

Next, to compute the γ_i's, observe from the θ-diagram for AII that

$$\pi(\alpha_j) = \begin{cases} 0 & j = \text{odd} \\ \omega_{\frac{j}{2}} & j = \text{even} \end{cases} \qquad j \in \{1, 2, \cdots, 2n+1\}$$

Thus, with respect to $\Delta(\mathfrak{a})$, we have that

$$\gamma_i = \pi(\lambda_i) = \sum_{j=1}^{n} (A_{2n+1}^{-1})_{i,2j} \cdot \omega_j \qquad i \in \{1, 2, \cdots, 2n+1\} \tag{7}$$

For convenience, we define $[\Gamma]_\omega$ to be the $(2n+1) \times n$ matrix whose rows are the coefficients of γ_i relative to $\Delta(\mathfrak{a})$.

$$([\Gamma]_\omega)_{i,j} = ([\gamma_i]_\omega)_j$$

From Equation 7 we have that

$$\Gamma_{ij} = \begin{cases} i(1-\frac{2j}{2(n+1)}) & i \le 2j \\ 2j(1-\frac{i}{2(n+1)}) & i > 2j \end{cases} \tag{8}$$

with $1 \le i \le 2n+1$ and $1 \le j \le n$. The next step is to compute the coefficients for the weights, μ_i, with $i \in \{1, 2, \cdots, n\}$. Again we refer to the θ-diagram for AII. For this case we have that $\Phi(\mathfrak{a})$ is of type A_n. Thus,

$$\mu_i = \sum_{j=1}^{n} (A_n^{-1})_{ij} \cdot \omega_j \qquad 1 \le i, j \le n$$

As before let $[M]_\omega$ be the $n \times n$ matrix whose rows are the coefficients of the μ_i's relative to $\Delta(\mathfrak{a})$. Here we have that

$$M_{ij} = (A_n^{-1})_{i,j} = \begin{cases} i(1 - \frac{j}{n+1}) & i \leq j \\ j(1 - \frac{i}{n+1}) & i > j \end{cases} \tag{9}$$

with $1 \leq i, j \leq n$. We claim that for each $k \in \{1, 2, \cdots, 2n+1\}$, the γ_k can be written as an integer linear combination of the μ_i's. Indeed, for $k = 1$ or $2n+1$ the result is immediate. Let $k \leq 2n$ be even. From Equations 8 and 9 we have that

$$\begin{aligned} \Gamma_{kj} &= \begin{cases} k(1 - \frac{j}{n+1}) & k \leq 2j \\ 2j(1 - \frac{k}{2(n+1)} & k > 2j \end{cases} \\ &= 2M_{\frac{k}{2},j}. \end{aligned}$$

For the third case of the proposition, let $k \in \{1, 2, \cdots, n-1\}$. We make the equivalent claim that

$$\gamma_{2k+1} = \mu_k + \mu_{k+1}$$

To establish this claim, we must show that for each $k \in \{1, 2, \cdots, n-1\}$

$$\Gamma_{2k+1,j} = M_{k,j} + M_{k+1,j} \qquad 1 \leq j \leq n. \tag{10}$$

Fix $k \in \{1, 2, \cdots, n-1\}$. Obviously, we invoke Lemma 1 to show that in each case, Equation 10 is upheld. In accordance with the structure of this lemma we consider the following three cases: First, if $j \leq k-1$ then $2j < 2k+1$ so that

$$\begin{aligned} M_{k,j} + M_{k+1,j} &= j[(1 - \frac{k}{n+1}) + (1 - \frac{k+1}{n+1})] \\ &= 2j[1 - \frac{2k+1}{2(n+1)}] \\ &= \Gamma_{2k+1,j}. \end{aligned}$$

Next, if $j = k$, then (again) $2j < 2k+1$, and

$$\begin{aligned} M_{k,j} + M_{k+1,j} &= k[(1 - \frac{k}{n+1}) + (1 - \frac{k+1}{n+1})] \\ &= 2j[1 - \frac{2k+1}{2(n+1)}] \\ &= \Gamma_{2k+1,j}. \end{aligned}$$

Finally, if $j \geq k+1$ then $2j \geq 2k+1$, and

$$\begin{aligned} M_{k,j} + M_{k+1,j} &= k(1 - \frac{j}{n+1}) + (k+1)(1 - \frac{j}{n+1}) \\ &= (2k+1)(1 - \frac{j}{n+1}) \\ &= \Gamma_{2k+1,j}. \end{aligned}$$

□

The main result for case AII follows.

Theorem 2. *Let γ, μ, $\pi(\Lambda_{\mathfrak{t}})$, and $\Lambda_{\mathfrak{a}}$, be as above (relative to the case AII). We then have that*

(i)

$$\mu_k = \sum_{j=1}^{k} (-1)^{k+j} \gamma_{2j-1} \qquad k = \{1, 2, \cdots n\} \tag{11}$$

(ii) $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$

Proof. For Part 1 we use induction on k. From Proposition 3 we immediately have the result for $k = 1$. Assume that Equation 11 holds for $k \in \{1, 2, \cdots, n-1\}$. Again by Proposition 3 we have that

$$\mu_{k+1} + \mu_k = \gamma_{2k+1} \qquad k \in \{1, 2, n-1\}$$

Thus, by the inductive hypothesis, we obtain

$$\mu_{k+1} = \gamma_{2k+1} - \sum_{j=1}^{k} (-1)^{j+k} \gamma_{2j-1} \qquad k \in \{1, 2, n-1\}$$

or

$$\mu_{k+1} = \sum_{j=1}^{k+1} (-1)^{j+k+1} \gamma_{2j-1} \qquad k \in \{1, 2, \cdots, n-1\}$$

This establishes the result for any $k \in \{1, 2, \cdots, n\}$. Part 2 follows immediately. □

3. Implementation

As mentioned earlier, implementations of the algorithms in [GH06] follow naturally from the steps used for proving each case of Theorem 1. The next example gives pseudo-code for the case AII.

Example 1. Case AII.

```
/****************************************************************
*
* This is the code for computing the relationship between the
* characters for the case $A_n^{2n+1}$II. To do this we use Proposition 1 to
* compute the coefficients of the fundamental weights $\lambda_i$,
* with $i \in \{1, 2, \ldots, 2n+1\}$ of the Lie algebra relative to the
* $\theta$-base $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}\}$ of the root system $A_{2n+1}$.
* Recall that the weight lattice of the Lie algebra is given by
* $\Lambda_{\mathfrak{t}} = \mathrm{span}_{\mathbb{Z}}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}\}$.
*
****************************************************************/
read n;               // In this case $n$ is the dimension of the //
                      // root system of the symmetric space.     //
dimension = $2n + 1$;
A_inv = cartan_inverse_proc(dimension); // This routine computes the inverse of the //
                                        // Cartan matrix $A_k$, with $k$ = dimension. //
for $i = 1$ to $2n + 1$;
     for $j = 1$ to $2n + 1$;
          lambda$(i, j)$ = A_inv$(i, j)$;
     next $j$;
next $i$;

/****************************************************************
*
* Next we use Equation (5) to compute the coefficients for $\gamma_i$,
* with $i \in \{1, 2, \ldots, 2n+1\}$ relative to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \ldots, \omega_n\}$.
* Recall that the projection of the weight lattice of the Lie algebra is given by
* $\pi(\Lambda_{\mathfrak{t}}) = \mathrm{span}_{\mathbb{Z}}\{\gamma_1, \gamma_2, \ldots, \gamma_{2n+1}\}$.
*
****************************************************************/
for $i = 1$ to $2n + 1$;
     for $j = 1$ to $n$;
          gamma$(i, j)$ = A_inv$(i, 2j)$;
     next $j$;
next $i$;
```

/***
*
* We again use Proposition 1 to compute the coefficients for the
* fundamental weights μ_i, with $i \in \{1, 2, \ldots, n\}$, relative to the base
* $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \ldots, \omega_n\}$. Recall that the weight lattice of the symmetric
* space is given by $\Lambda_{\mathfrak{a}} = \text{span}_{\mathbb{Z}}\{\mu_1, \mu_2, \ldots, \mu_n\}$.
*
***/
dimension $= n$;
A_inv=cartan_inverse_proc(type, dimension);
for $i = 1$ to n;
    for $j = 1$ to n;
        $\text{mu}(i, j) = \text{A\_inv}(i, j)$;
    next j;
next i;

/***
*
* The next step is to compute the relationship between the γ's and the μ's.
* Using Proposition 4, we first show explicitly that $\gamma_i = \text{span}_{\mathbb{Z}}\{\mu_1, \mu_2, \ldots, \mu_n\}$
* for each $i \in \{1, 2, \ldots, 2n+1\}$, i.e. that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$. To this end let S be a
* $2n+1$ by n matrix used to hold the coefficients for the γ's in terms of the μ's.
*
***/
/* We start by initializing the entries of S to zero. */
for $i = 1$ to $2n+1$;
    for $j = 1$ to n;
        $S(i, j) = 0$;
    next j;
next i;
$S(1, 1) = 1$; // since by Proposition 4 $\gamma_1 = \mu_1$.//
for $i = 2$ to $2n$;
    switch;
        case($i = even$):
            $S(i, i/2) = 2$;
        case($i = odd$):
            $S(i, (i-1)/2) = 1$;
            $S(i, (i+1)/2) = 1$;
    end switch;
next i;
$S(2n+1, n) = 1$; // since By Proposition 4 $\gamma_{2n+1} = \mu_n$.//

```
/****************************************************************
*
* Using Theorem 4, we show explicitly that μ_i = span_ℤ{γ_1, γ_2, ..., γ_{2n+1}}
* for each i ∈ {1, 2, ..., n}, i.e. that Λ_𝔞 ⊆ π(Λ_𝔱). To this end let T be an
* n by 2n + 1 matrix used to hold the coefficients for the μ's in terms
* of the γ's.
*
****************************************************************/
/* We start by initializing the entries of T to zero. */
for i = 1 to n;
        for j = 1 to 2n + 1;
              T(i, j) = 0;
        next j;
next i;
for i = 1 to n;
        for j = 1 to i;
              T(i, 2j − 1) = (−1)^(i+j)
        next j;
next i;

/****************************************************************
*
* This is the code for computing the inverse of the Cartan matrix A_k
* where k = dimension. Use Table 2.
*
****************************************************************/
cartan_inverse_proc(dimension);
k = dimension
declare M[k, k] as integer; // M is a place holder for the inverse //
for i = 1 to k;
        for j = 1 to k;
              switch;
                     case(i ≤ j)
                            M(i, j) = i ∗ (1 − j/(k+1))
                     case(i > j)
                            M(i, j) = j ∗ (1 − i/(k+1))
              end switch;
        next j;
next i;
return(M);
```

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