INFINITE DIMENSIONAL FLAG MANIFOLDS IN INTEGRABLE SYSTEMS

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Abstract. In this paper we present several instances where infinite dimensional flag varieties and their holomorphic line bundles play a role in integrable systems. As such, we give the correspondance between flag varieties and Darboux transformations for the KP-hierarchy and the n-th KdV-hierarchy. We construct solutions of the n-th $MKdV$-hierarchy from the space of periodic flags and we treat the geometric interpretation of the Miura transform. Finally we show how the group extension connected with these line bundles shows up at integrable deformations of linear systems on $\mathbb{P}^1(\mathbb{C})$.

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Contents

1. Introduction

Let H be a finite dimensional complex Hilbert space. Then a flag $V = (V(i))$ in H is a chain of subspaces in H,

$$\{0\} = V(0) \subsetneq V(1) \subsetneq \cdots \subsetneq V(m) = H.$$

If we put $\underline{d} = (d_1, \dots, d_m)$, with $d_i = \dim(V(i))$ and we write $\mathcal{F}_{\underline{d}}$ for the collection of all flags $V = (V(i))$ in H such that $\dim V(i) = d_i$, then $\mathcal{F}_{\underline{d}}$ is a compact Kähler manifold. The flag variety of type $\underline{d}$. The flag varieties $\mathcal{F}_{\underline{d}}$ are the main ingredients in field theory for the definition of the fundamental complex manifolds of twistor geometry, see [22]. If $m = 2$, then $\mathcal{F}_{\underline{d}}$ is simply the collection of all subspaces of H of dimension d_1 and is, as such, better known as the Grassmann manifold $Gr(d_1, n - d_1)$. These manifolds are the natural scene for Ricatti type equations and play, as such, an important role in control theory, see [9].

The manifolds $\mathcal{F}_{\underline{d}}$ are homogeneous spaces for both $Gl(H)$ and $U(H)$. By adding additional requirements to the flags under consideration, one obtains flag varieties corresponding to other groups.

Holomorphic line bundles over the varieties $\mathcal{F}_{\underline{d}}$ lead to a geometric realization of the irreducible unitary representations of $U(H)$ as the natural action of $U(H)$ on the holomorphic sections of these line bundles. An important example from mathematical physics, where these bundles are crucial is the Penrose transform. For details and generalizations of this theme, we refer to [2]

In the second section of this paper we give the geometric structure of the flag varieties and their holomorphic line bundles as they are needed in the context of integrable systems.

The third section describes the systems of equations and transformations that will be fit into the geometric framework.

In the fourth section, we recall first some results for the KP-hierarchy. Next we show how Darboux transformations can be seen geometrically and we conclude with the construction of solutions for the n-th MKdV-hierarchy and the interpretation of the Miura transformation.

The final section describes the role the group extension related to the line bundles plays at integrable deformations of a linear system on $\mathbb{P}^1(\mathbb{C})$.

2. Flag varieties

2.1. **The type of flags.** Like with infinite dimensional vector spaces, there is a wide range of manifold structures that can be considered for infinite dimensional flag varieties. Since we like to discuss analytic properties of our varieties and apply them in analytic situations, we will not take the algebraic set-up from [13], e.g. We will also not try to generalize the Sato Grassmanian, see [12] and [5] e.g., or the one from [26]. Our choice will be a natural generalization of the Grassmann manifolds in [20].

Let H be a separable complex Hilbert space with inner product $< \cdot | \cdot >$. Since we work in a topological context, it is reasonable to consider merely chains $F = (F(i))$ in H,

$$\{0\} = F(0) \subsetneq F(1) \subsetneq \cdots \subsetneq F(m) = H,$$

where all the $F(i)$ are closed subspaces of H. If we put for all i, $1 \leq i \leq m$,

$$F_i := F(i-1)^{\perp} \cap F(i),$$

then we see that each flag $F = (F(i))$ corresponds precisely to an orthogonal decomposition of H

$$H = F_1 \oplus \cdots \oplus F_m.$$

We will use both notations $F = (F(i))$ and $F = (F_i)$ to describe a flag. Our starting point is a given orthogonal decomposition

$$H = H_1 \oplus \cdots \oplus H_m \quad \text{, with } H_i \perp H_j \text{ for } i \neq j. \tag{2.2.1}$$

The flag corresponding to the (H_i), we denote by $F^{(0)}$ and we call it the *basic flag*. We will take together all flags "similar" to the basic flag and make this into some kind of manifold. First we present two examples of decompositions that occur naturally at modelling certain equations from mathematical physics. We start with what will be the leading example during the rest of this paper

Example 2.1. Let H be the Hilbert space

$$L^2(S^1, \mathbb{C}) = \{\sum_{n \in \mathbb{Z}} a_n z^n, a_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}} | a_n |^2 < \infty\}$$

with the inner product

$$< \sum_{n \in \mathbb{Z}} a_n z^n \mid \sum_{n \in \mathbb{Z}} b_n z^n > = \sum_{n \in \mathbb{Z}} a_n \overline{b}_n.$$

Let $\underline{s} = (s_1, \cdots, s_{m-1})$, where $s_i \in \mathbb{Z}$ and $s_{i+1} < s_i$, then we take the basic flag defined by

$$H(i) = \{\sum_{n \geq s_i} a_n z^n \in H\} \quad \text{for } i, 1 \leq i \leq m-1 \quad \text{and } H(m) = H.$$

We will see later on that the flag manifold corresponding to this flag is the "basis" of the modified equations.

Example 2.2. In quantum field theory the states of the system are vectors in a Fock space. In the fermionic case, this space is built up from the splitting of a basic Hilbert space into positive and negative energy states. E.g., at the Dirac theory for a one dimensional particle of mass $m \geq 0$, one considers, see [3], the Dirac Hamiltonian

$$D = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

It acts on the Hilbert space $H = L^2(\mathbb{R})^2$ and the relevant decomposition of H for the Fock space representation is $H = H_+ \oplus H_-$, where H_+ is the subspace of H corresponding to the positive spectrum of D and H_- the one corresponding to the negative spectrum of D. In general, see [16], Dirac operators are associated to a number of geometric data, like a spin bundle over an oriented Riemannian manifold, another vector bundle over this manifold and a connection. A similar splitting of the square-integrable extended Dirac spinors is the starting point for the construction of a representation of an extension of the group of gauge transformations, see [16].

In the finite dimensional case it was sufficient to fix the dimension of each F_i and then to take all flags of the same size. This idea is too vague in this context and requires adaptation. For each $i, i \leq i \leq m$, let p_i denote the orthogonal projection from H onto H_i. Then we introduce

Definition 2.3. The *flag variety* $\mathcal{F}$ corresponding to the decomposition (2.2.1) is the collection of flags $F = \{F_1, \cdots, F_m\}$ in H, satisfying $\dim(F_i) = \dim(H_i)$ and for all i and j with $j \neq i$, the orthogonal projection $p_j : F_i \to H_j$ is a Hilbert-Schmidt operator.

If H and $\underline{s}$ are as in example 2.1, then we write $\mathcal{F} = \mathcal{F}(\underline{s})$. The flag variety $\mathcal{F}$ is a homogeneous space for an analogue adapted to this situation of the general linear group. The Banach structure of this group follows directly from that of its Lie algebra. This requires a

Notation 2.4. If g belongs to $\mathcal{B}(H)$, the space of bounded linear operators from H to H, then $g = (g_{ij}), 1 \leq i \leq m$ and $1 \leq j \leq m$, denotes the decomposition of g w.r.t. the $\{H_i \mid 1 \leq i \leq m\}$. That is to say $g_{ij} = p_i \circ g \mid H_j$.

Now we come to the Lie algebra of the analogue of the general linear group.

Definition 2.5. A *restricted endomorphism* of H is a $u = (u_{ij})$ in $\mathcal{B}(H)$ such that u_{ij} is a Hilbert-Schmidt operator for all $i \neq j$. We denote the space of all restricted endomorphisms of H by $\mathcal{B}_{\text{res}}(H)$.

The algebra $\mathcal{B}_{\text{res}}(H)$ becomes a Banach algebra if we equip it with the norm $\| \cdot \|_2$ defined by

$$\| u \|_2 = \| u \| + \sum_{i \neq j} \| u_{ij} \|_{\mathcal{HS}} .$$

where $\| \cdot \|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt norm. The Lie group corresponding to $\mathcal{B}_{\text{res}}(H)$ is.

Definition 2.6. The *restricted linear group*, $Gl_{\text{res}}(H)$, is the group of invertible elements in $\mathcal{B}_{\text{res}}(H)$. As such, it is a natural Banach Lie group.

One easily shows that $GL_{\text{res}}(H)$ can be described as

$$GL_{\text{res}}(H) = \left\{ g = (g_{ij}) \in Gl(H) \;\middle|\; \begin{array}{l} g_{ii} \text{ is a Fredholm operatorfor all } i \\ g_{ij} \in \mathcal{HS}(H_j, H_i), \quad \text{for } i \neq j. \end{array} \right\}$$

Moreover $Gl_{\text{res}}(H)$ acts on $\mathcal{F}$ through its natural action on subspaces of H and this action is transitive, see [8]. Hence, we have $\mathcal{F} = Gl_{\text{res}}(H)/P$, where P is the parabolic subgroup stabilizing the basic flag.
Let $\tau : Gl_{\text{res}}(H) \to \mathcal{F}$ be the projection $\tau(g) = g \cdot F^{(0)}$. From the definition of the parabolic group P one sees directly that the Lie algebra of P is given by

$$L(P) = \{g \mid g = (g_{ij}) \in \mathcal{B}_{\text{res}}(H), g_{ij} = 0 \text{ for all } i > j\}$$

and that a complement of $L(P)$ in $\mathcal{B}_{\text{res}}(H)$ is the Hilbert space $(E, \| \cdot \|_{\mathcal{HS}})$ with

$$E = \bigoplus_{\substack{1 \le j \le m-1 \\ i > j}} \mathcal{HS}(H_j, H_i).$$

From [8] we know then that the homogeneous space $\mathcal{F} = GL_{\text{res}}(H)/P$ carries an analytic E-manifold structure for which τ is a submersion and for which the natural action of $GL_{\text{res}}(H)$ on $\mathcal{F}$ is analytic.
It is shown in [8] that the group of unitary operators in $Gl_{\text{res}}(H)$ acts transitively on $\mathcal{F}$. Hence, to make $\mathcal{F}$ into a hermitian manifold it suffices to give on the tangent space E of the basic flag $F^{(0)}$ a hermitian form, which is invariant under the stabilizer of $F^{(0)}$. Consider on $E \times E$ the form $H(\cdot|\cdot)$ given by

$$H(X \mid Y) = H((X_{ij})|(Y_{ij})) = \text{Trace}\left(\sum_{\substack{i=2 \\ j<i}}^{m} X_{ij} Y_{ij}^* \right)$$

Let $\omega(X, Y)$ be the imaginary part of $H(X \mid Y)$. One verifies directly that ω is a nondegenerate bilinear form. By direct computation or by using the Lie algebra 2-cocycles for $\mathcal{B}_{\text{res}}(H)$ from [8], one shows that ω is closed: $d\omega = 0$. Hence $\mathcal{F}$ is a symplectic manifold and even a Kähler manifold, another property that carries over from the finite dimensional $F_{\underline{d}}$.

Remark 2.7. For simplicity we restrict ourselves in this paper, to finite chains of subspaces. However, one could just as well consider infinite chains of closed subspaces. They also arise naturally in the context of integrable systems, see [6] and in an implicit form [7].

2.2. **Properties of $\mathcal{F}$.** As in the Grassmanian case, one shows that the connected component of $\mathcal{F}$ containing the basic flag equals the orbit of the connected component $Gl_{\text{res}}^{(0)}(H)$, i.e.

$$\mathcal{F}^{(0)} = \{g \cdot \mathcal{F}^{(0)} \mid g \in Gl_{\text{res}}^{(0)}(H)\} = \{g \cdot F^{(0)} \mid g = (g_{ij}),\ \text{index}\ \ (g_{ii}) = 0\ \text{ for all }\ i\}$$

It may look that all these flag varieties are homogeneous spaces for different groups. However, from the basic properties of Fredholm operators one deduces

Proposition 2.8. *Let G_2 be the subgroup of $Gl_{\text{res}}(H)$ with the induced topology, defined by*

$$G_2 = \{g \mid g \in Gl_{\text{res}}(H), g - Id \in \mathcal{HS}(H)\}.$$

Then G_2 acts transitively on all the connected components of the $\mathcal{F}$.

In the rest of this subsection, we will take H and the basic flag as in example 2.1. For that example, we also need a description of the other connected components. Also here there holds

Proposition 2.9. *The connected components of $\mathcal{F}(\underline{s})$ are given by*

$$\mathcal{F}^{(k)}(\underline{s}) = \{g \cdot F^{(0)} \mid \ \ \textit{index}\ \ (g_{11}) = k\}, k \in \mathbb{Z}.$$

For a proof we refer to [8]. An important operator in $Gl_{\text{res}}(H)$ is the shift operator Λ defined by

$$\Lambda(\sum_{n\in\mathbb{Z}} a_n z^n) = \sum_{n\in\mathbb{Z}} a_n z^{n+1}.$$

As Λ_{11} has index -1, we see that for each l and k in $\mathbb{Z}$, Λ^l maps $\mathcal{F}^{(k)}$ onto $\mathcal{F}^{(k-l)}(\underline{s})$. Taking this into account, we get the following result:

> For each strictly decreasing sequence $\underline{s} = (s_1, \cdots, s_{m-1})$ and each $l \in \mathbb{Z}$,
> we have $\mathcal{F}(\underline{s}) = \mathcal{F}(\underline{t})$, where $\underline{t} = (t_1, \cdots, t_{m-1})$ is defined by (2.2.2)
> $t_i = s_i + l$ for all i.

Finally, we introduce a group of commuting flows on $\mathcal{F}(\underline{s})$ that plays a central role in the sequel. Let U be a connected neighbourhood of S^1 in $\mathbb{C}$ and let $\Gamma(U)$ be the collection of all non-zero holomorphic maps $\gamma : U \to \mathbb{C}^*$. In a natural way $\Gamma(U)$ is a group. If $U_1 \supset U_2$, then we get by restriction an embedding of $\Gamma(U_1)$ into $\Gamma(U_2)$. We write Γ for the inverse limit of the $\{\Gamma(U)\}$. Each $\gamma \in \Gamma$ has a Fourier series

$$\sum_{i\in\mathbb{Z}} \gamma_i z^i, \gamma_i \in \mathbb{C},$$

that converges uniformly on some neighbourhood of S^1. Clearly, the multiplication with γ, gives the element $\sum_{i\in\mathbb{Z}} \gamma_i \Lambda^i$ of $B_{\text{res}}(H)$ and this defines a continuous injective group homomorphism from Γ to $Gl_{\text{res}}(H)$. The closure of the image of Γ in $Gl_{\text{res}}(H)$ is a maximal abelian subgroup of $Gl_{\text{res}}(H)$. In Γ we consider the following subgroups

$$\begin{array}{l}\Gamma_{+}=\{\gamma \mid \gamma\in\Gamma, \gamma=\exp(\sum_{i=1}^{\infty}t_i z^i)\}\\ \Gamma_{-}=\{\gamma \mid \gamma=\sum_{j\leq 0}\gamma_j z^j\in\Gamma\}\ \text{ and}\\ \Delta=\{z^k \mid k\in\mathbb{Z}\}.\end{array}$$

Then we have, see [10].

Proposition 2.10. *The group* Γ *is the direct product of these 3 groups, i.e.* $\Gamma=\Gamma_{+}\Delta\Gamma_{-}$.

This decomposition will play a role at the geometric description of the KP-hierarchy.

2.3. **Holomorphic line bundles over $\mathcal{F}^{(0)}$.** If one wants to construct an analogue of the holomorphic line bundles over finite dimensional flag varieties, one needs a description of $\mathcal{F}^{(0)}$ as a homogeneous space of a smaller group, for which certain "minors" exist. Consider

$$\mathfrak{G}=\left\{g|g=(g_{ij})\in Gl(H)\ \begin{array}{l} g_{ii}-Id\ \text{ is trace-class}\\ g_{ij}\in\mathcal{HS}(H_j,H_i)\ \text{ for }\ i\neq j.\end{array}\right\}$$

Then one verifies directly that $\mathfrak{G}$ is a group. We give it the topology based on

$$\oplus_{i\neq j}\mathcal{HS}(H_j,H_i)\oplus\oplus_{i=1}^{m}\mathcal{N}(H_i),$$

where $\mathcal{N}(H_i)$ is the space of trace class operators on H_i, equiped with the trace norm. The group $\mathfrak{G}$ is chosen such that for each i, $1\leq i\leq m$, and for each $g=(g_{ij})$ in $\mathfrak{G}$ the minor

$$det\begin{pmatrix} g_{11} & \cdots & g_{1i}\\ \ldots & \vdots & \ldots\\ g_{i1} & \cdots & g_{ii}\end{pmatrix}$$

exists. The natural embedding of $\mathfrak{G}$ into $Gl_{\text{res}}(H)$ is continuous. Moreover, it can be shown, see [8], that $\mathfrak{G}$ acts transitively on $\mathcal{F}^{(0)}$. Hence, we have that $\mathcal{F}^{(0)}$ is isomorphic to $\mathfrak{G}/\mathfrak{T}$, with $\mathfrak{T}$ the stabilizer of $F^{(0)}$ in $\mathfrak{G}$, i.e.

$$\mathfrak{T}=\left\{t=\begin{pmatrix} t_{ii} & & t_{1m}\\ 0 & \ldots & \\ \vdots & \ldots & \\ 0 & \cdots 0 & t_{mm}\end{pmatrix}, t\in\mathfrak{G}\right\}$$

For each $\underline{k}=(k_1,\cdots,k_m)\in\mathbb{Z}^m$, we define the analytic characters $\psi_{\underline{k}}:\mathfrak{T}\to\mathbb{C}^*$ by

$$\psi_{\underline{k}}(t)=\det(t_{11})^{k_1}\cdots\det(t_{mm})^{k_m}.$$

To each $\psi_{\underline{k}}$ there is associated a holomorphic line bundle $L(\underline{k})$ over $\mathcal{F}^{(0)}$. It can be defined as follows: consider on the space $\mathfrak{G}\times\mathbb{C}$ the equivalence relation

$$(g_1,\lambda_1)\sim(g_2,\lambda_2)\Leftrightarrow g_1=g_2\circ t,\quad \text{with }\ t\in\mathfrak{T}\quad \text{and }\ \lambda_2=\lambda_1\psi_{\underline{k}}(t).$$

The space $\mathfrak{G} \times \mathbb{C}$ modulo this equivalence relation is $L(\underline{k})$. We denote the class of the pair (g, λ), $g \in \mathfrak{G}$, $\lambda \in \mathbb{C}$, by $[g, \lambda]$.

Remark 2.11. If we take $m = 2$, $\dim(H_1) = \dim(H_2) = \infty$, $k = (-1, 0)$ resp. $k = (1, 0)$ then the line bundle $L((-1, 0))$ resp. $L((1, 0))$ are the determinant bundle and its dual introduced in [13].

The group $\mathfrak{G}$ acts naturally on $L(\underline{k})$ by left translations. However, e.g. the group of flows Γ_+ from subsection 2.2 is not contained in $\mathfrak{G}$ and one would like to let it act nevertheless. In general one cannot lift the natural action of $Gl_{\text{res}}^{(0)}(H)$ on $\mathcal{F}^{(0)}$ to an action of the whole group on $L(\underline{k})$ and one is forced to pass to an extension G of $Gl_{\text{res}}^{(0)}(H)$. If D is the subgroup of $Gl_{\text{res}}^{(0)}(H)$, given by

$$D = \{g = (g_{ij}) \in Gl_{\text{res}}^{(0)}(H), g_{ij} = 0 \quad \text{if } \ i \neq j\}$$

then G is defined by

$$G = \{(g, d) \mid g \in Gl_{\text{res}}^{(0)}(H), d \in D, gd^{-1} \in \mathfrak{G}\}$$

The parabolic subgroup P of $Gl_{\text{res}}^{(0)}(H)$ embeds as follows into G:

$$p = (p_{ij}) \mapsto (p, \begin{pmatrix} p_{11} & & 0 \\ & \cdots & \\ 0 & & p_{mm} \end{pmatrix})$$

The group G acts on $L(\underline{k})$ by

$$(g, d) \cdot [g_1, \lambda_1] = [gg_1d^{-1}, \lambda_1].$$

In particular, there is thus a lifting of the action of Γ^+ on $\mathcal{F}^{(0)}$ to one on $L(\underline{k})$. This action of G leads to a natural representation of G on the holomorphic sections of $L(\underline{k})$. These representations are the analogue of the finite dimensional irreducible representations of $Gl_n(\mathbb{C})$.

3. The equations

3.1. **The KP-hierarchy.** We first recall the Lax form of this system of equations. One starts out with some commutative ring of functions depending of the variables $\{x_1, t_1, t_2, t_3, \cdots\}$. Next one assumes that $\partial := \frac{\partial}{\partial x}$ and all the $\partial_k := \frac{\partial}{\partial t_k}$ form derivations of R. To R and ∂ is associated the ring $R[\xi, \xi^{-1}]$ of pseudodifferential operators, whose elements are expressions

$$\sum_{i=-\infty}^{N} a_i \xi^i \quad , \text{with } \ a_i \in R.$$

Addition and multiplication in $R[\xi, \xi^{-1}]$ are defined by

$$\sum_i a_i\xi^i + \sum_i b_i\xi^i = \sum_i (a_i + b_i)\xi^1$$
$$(\sum_i a_i\xi^i) \cdot (\sum_j b_j\xi^j) = \sum_{\substack{i,\, j \\ m \geq 0}} \begin{pmatrix} i \\ m \end{pmatrix} a_i\partial^m(b_j)\xi^{i+j-m}$$

If $P = \sum_j p_j\xi^j$ belongs to $R[\xi, \xi^{-1}]$, then we denote its differential operator part $\sum_{j\geq 0} p_j\xi^j$ by P_+ and we write P_- for $\sum_{j<0} p_j\xi^j$.
In $R[\xi, \xi^{-1}]$ one considers so-called *Lax operators*, i.e. operators L of the form

$$L = \xi + \sum_{j>0} u_{j+1}\xi^{-j}. \tag{3.3.1}$$

Examples of Lax operators can be obtained by the *dressing procedure*: take a $K = \xi^n + \sum_{j<n} k_j\xi^j$, then K is invertible in $R[\xi, \xi^{-1}]$ and $L = K\xi K^{-1}$ is a Lax operator. All Lax operators can be obtained in this way if and only if ∂ is surjective. An important class of examples is the set of roots of differential operators. For, if $\mathcal{L} \in R[\xi]$ has the form

$$\mathcal{L} = \xi^n + \sum_{j=2}^{n} l_j\xi^{n-j}. \tag{3.3.2}$$

Then there exist a unique Lax operator L with $L^n = \mathcal{L}$ and this L is also denoted as $\mathcal{L}^{\frac{1}{n}}$. We extend the derivations ∂ and $\{\partial_k \mid k \geq 1\}$ from R to derivations of $R[\xi, \xi^{-1}]$ by letting them act coefficientwise. Now we can introduce the KP-hierarchy in Lax form

Definition 3.1. The *K(adomtsev) P(etviashvili)-hierarchy* is the following system of equations for a Lax operator L:

$$\partial_k(L) = [(L^k)_+, L] \quad \text{for all } k \geq 1. \tag{3.3.3}$$

A Lax operator that satisfies the equations (3.3.3) is called a *solution* of the KP-hierarchy. For $k = 1$, we see that this equation says that $\partial = \partial_1$ on the coefficients of L. Therefore we may just as well assume $x = t_1$ and this will be done from now on. If $L = \mathcal{L}^{\frac{1}{n}}$, with $\mathcal{L}$ as in (3.3.2), it is not difficult to show that the equations (3.3.3) are equivalent to

$$\partial_k(\mathcal{L}) = [(\mathcal{L}^{\frac{k}{n}})_+, \mathcal{L}] \quad \text{, for all } k \geq 1. \tag{3.3.4}$$

For $n = 2$, $\mathcal{L} = \xi^2 + 2u$ and the equation (3.3.4) for $k = 3$ is then equivalent to the KdV-equation for u. Therefore one calls the equations (3.3.4) for $n = 2$, the KdV-hierarchy and for general n, the n-th KdV-hierarchy. The next subsection describes some linear equations related to (3.3.3).

3.2. **The linearization.** Let L be a Lax operator. The equations (3.3.3) can be seen as the compatibility of the equations

$$L \cdot \psi = z\psi \quad \text{and} \quad \partial_k(\psi) = (L^k)_+ \cdot \psi, \qquad \text{for all } \ k \geq 1 \tag{3.3.5}$$

and the equations (3.3.5) are called the *linearization of the KP-hierarchy*. In order to be able to obtain (3.3.3) from (3.3.5), one must first give a context in which operators like L and the $(L^k)_+$ act on "functions" ψ. Further we should be able to speak of $\partial_k(\psi)$ and, finally the manipulations with the equations should hold. This context is an appropriate $R[\xi, \xi^{-1}]$-module. To justify its form, we consider the trivial solution of the KP-hierarchy $L = \xi$. In that case

$$g(z) = \exp(\sum_{k=1}^{\infty} t_k z^k)$$

is a solution of (3.3.5). In viewing the general solutions of the KP-hierarchy as "perturbations" of the trivial solution, one takes the ψ as "perturbations" of $g(z)$. Thus one comes to consider

$$M(g(z)) = \{ f(z) \cdot g(z) \mid f(z) = \sum_{i=-\infty}^{N} a_i z^i \quad \text{, with } \ a_i \in R\}.$$

The product $f(z) \cdot g(z)$ is a formal one, for in general the product of these 2 series in z makes no sense. $M(g(z))$ becomes an $R[\xi, \xi^{-1}]$-module by linear extension of

$$(\textstyle\sum_{i \leq N} a_i z^i) \cdot g(z) + (\sum_{i \leq N} b_i z^i) \cdot g(z) = (\sum_{i \leq N} (a_i + b_i) z^i) \cdot g(z)$$
$$p\xi^j \cdot \{(\textstyle\sum a_i z^i) \cdot g(z)\} = (\displaystyle\sum_{\substack{i \\ m \geq 0}} \begin{pmatrix} j \\ m \end{pmatrix} p\partial^m(a_i) z^{i+j-m}) \cdot g(z).$$

In particular, one sees that ξ acts as differentiating the formal product w.r.t. the variable x. We let ∂_k act in the same fashion on $M(g(z))$:

$$\partial_k((\sum a_i z^i) g(z)) = (\sum_i \partial_k(a_i) z^i + \sum_i a_i z^{i+k}) g(z).$$

Note that it makes sense now to consider the equations (3.3.5) inside $M(g(z))$. Another important observation is that $M(g(z))$ is a free $R[\xi, \xi^{-1}]$-module with generator $g(z)$, for we have

$$(\sum_i a_i \xi^i) \cdot g(z) = (\sum_i a_i z^i) \cdot g(z).$$

Thus one can translate relations in $M(g(z))$ to equations in $R[\xi, \xi^{-1}]$. In $M(g(z))$ we are interested in elements of a special form

Definition 3.2. The element ψ in $M(g(z))$ is called of *type* z^n if we have $\psi = (z^n + \sum_{j<n} a_j z^j) g(z)$.

So, for each element ψ of type z^n there is a unique operator $K_\psi = \xi^n +$ "lower order in ξ" such that $\psi = K_\psi \cdot g(z)$. The crucial notion is now

Definition 3.3. An element ψ in $M(g(z))$ of type z^n is called a *Baker-function* of type z^n for the Lax operator L if the equations (3.3.5) hold.

Remark 3.4. In the literature one can also find the names wavefunction and Baker-Achieser function.

The main result is now

Theorem 3.5. *If L is a Lax operator and ψ is a Baker function of type z^n for L, then L is a solution of the KP-hierarchy and $L = K_\psi \xi K_\psi^{-1}$.*

3.3. **Darboux transformations.** If the Lax operator L is a solution of the KP-hierarchy, one might wonder of simple transformations that yield out of L new solutions to the KP-hierarchy. A typical example of such a transformation could be conjugation with an element

$$K = a_n \xi^n + \ ''\text{lower order in } \ \xi''$$

with a_n invertible in R. One directly computes that $\widetilde{L} = KLK^{-1}$ is a Lax operator without constant term iff $\partial_1(a_n) = 0$, but then we can just as well assume $a_n = 1$. In particular we are interested in differential operators

$$P = \sum_{i=0}^{n} p_i \xi^i = \xi^n + \sum_{i<n} p_i \xi^i$$

such that $\widetilde{L} = PLP^{-1}$ is a solution of the KP-hierarchy. If ψ is a Baker function of type z^m, then $\widetilde{\psi} = P \cdot \psi$ is the candidate Baker function for $\widetilde{L}$ and it will be a solution of the KP-hierarchy, if we can show for all $k \geq 1$

$$\begin{aligned} \partial_k(\widetilde{\psi}) &= \{\partial_k(P)P^{-1} + PB_kP^{-1}\}\widetilde{\psi} \\ &= \{\partial_k(P)P^{-1} + PB_kP^{-1}\}_+\widetilde{\psi} \end{aligned}$$

In fact, it even suffices to show for each $k \geq 1$, that there is a C_k in $R[\xi]$ such that

$$\partial_k(\widetilde{\psi}) = C_k \cdot \widetilde{\psi}.$$

This equation implies namely directly that

$$C_k = (\widetilde{L})^k_+ = (\partial_k(P)P^{-1} + PB_kP^{-1})_+.$$

Likewise one can put for solutions L of the KP-hierarchy that are of the form $L = \mathcal{L}^{\frac{1}{m}}$, with

$$\mathcal{L} = \sum_{j=0}^{m} l_j \xi^j = \xi^m + \sum_{j=0}^{m-2} l_j \xi^j,$$

the question which differential operators $P = \xi^n + \sum_{i<n} P_i \xi^i$ yield by the transformation $\mathcal{L} \to P\mathcal{L}P^{-1} = \widetilde{\mathcal{L}}$, again an operator $\widetilde{\mathcal{L}}$ in $R[\xi]$. In that case $\widetilde{\mathcal{L}}$ is clearly of the same form as $\mathcal{L}$. We call the transformation $\mathcal{L} \to \tilde{\mathcal{L}}$ a *Darboux transformation of order n*. Consider namely the special case of the Schrödinger operator $\mathcal{L} = \xi^2 - u$. In 1882, G. Darboux associated with $\mathcal{L}$ another Schrödinger operator

$$\widetilde{\mathcal{L}} = \xi^2 - (u - 2\frac{\partial(\psi)}{\psi}) \quad \text{, with } \ \psi \neq 0, \mathcal{L}\psi = c\psi.$$

It was shown in [4] that this Darboux-Bäcklund transformation $\mathcal{L} \to \widetilde{\mathcal{L}}$ is simply the conjugation of $\mathcal{L}$ with $A = \psi\xi\psi^{-1} = \xi - \frac{\partial(\psi)}{\psi}$.

3.4. **Miura transformations.** In this subsection we follow [25]. To explain the name of this subsection we consider again the Schrödinger operator $\mathcal{L} = \xi^2 - u$. Consider a splitting of $\mathcal{L}$ into linear factors,

$$\mathcal{L} = (\xi - v)(\xi + v). \tag{3.3.6}$$

One computes directly that u is related to from v by the so-called Miura transformation

$$u = \partial(v) - v^2. \tag{3.3.7}$$

Miura showed that it had the following remarkable property: if v satisfies the modified KdV-equation

$$\frac{\partial v}{\partial t} = \partial^3(v) - 6v^2\partial(v), \tag{3.3.8}$$

then u satisfied the KdV-equation. This last equation is equivalent to the first non-trivial equation in the KdV-hierarchy

$$\frac{\partial \mathcal{L}}{\partial t} = [(\mathcal{L}^{\frac{3}{2}})_+, \mathcal{L}]. \tag{3.3.9}$$

The modified KdV or shortly MKdV-equation, has also a description in Lax-form, see [14]. Take the matrix differential operator

$$\underline{\mathcal{L}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xi + \begin{pmatrix} 0 & -v \\ v & 0 \end{pmatrix}. \tag{3.3.10}$$

Then a direct computation shows

$$\mathcal{L}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi^2 + \begin{pmatrix} \partial(v) - v^2 & 0 \\ 0 & -\partial(v) - v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi^2 + \begin{pmatrix} u & 0 \\ o & \widetilde{u} \end{pmatrix}.$$

Let Q now be the operator in $M_2(R)[\xi,\xi^{-1}]$ defined by

$$Q=(\underline{\mathcal{L}}^2)^{\frac{1}{2}}=\begin{pmatrix}(\xi^2+u)^{\frac{1}{2}} & \\ 0 & (\xi^2+\widetilde{u})^{\frac{1}{2}}\end{pmatrix}.$$

One directly verifies then that the Lax equation.

$$\frac{\partial\underline{\mathcal{L}}}{\partial t}=[4(Q^{\frac{3}{2}})_+,\underline{\mathcal{L}}] \tag{3.3.11}$$

is equivalent to the MKdV-equation. This Lax equation is again the simplest non-trivial equation for a hierarchy, the socalled *MKdV-hierarchy*. This hierarchy can not only be introduced for the Schrödinger operator, but also for higher order differential operators. Consider thereto, in $M_n(R)[\xi]$ the operator

$$\underline{\mathcal{L}}=\begin{pmatrix}0 & \cdots & 0 & 1\\ 1 & \ddots & & 0\\ \vdots & \ddots & & \vdots\\ 0 & \cdots & 1 & 0\end{pmatrix}\xi+\begin{pmatrix}0 & \cdots & 0 & v_{n-1}\\ v_0 & \ddots & & 0\\ 0 & \ddots & & \vdots\\ \vdots & \ddots & & \vdots\\ 0 & \cdots 0 & v_{n-2} & 0\end{pmatrix}, \tag{3.3.12}$$

with $\sum_{i=0}^{n-1}v_i=0$ Then one computes directly that

Lemma 3.6. *We have* $\underline{\mathcal{L}}^n=\begin{pmatrix}\mathcal{L}_0 & & 0\\ & \ddots & \\ 0 & & \mathcal{L}_{n-1}\end{pmatrix}$, *where* $\mathcal{L}_i$ *in* $R[\xi]$ *is given by* $\mathcal{L}_i=(\xi+v_{n+i-1})(\xi+v_{n+i-2})\cdots(\xi+v_i)$ *and the subscripts in the* $\mathcal{L}_i$ *are read* $\mod n$.

The conditions $\sum_{i=0}^{n-1}v_i=0$ implies that the coefficient of ξ^{n-1} in each $\mathcal{L}_i$ is zero. If we write

$$\mathcal{L}_0=(\xi+v_{n-1})(\xi+v_{n-2})\cdots(\xi+v_0)=\xi^n+\sum_{i=0}^{n-2}u_i\xi^i$$

then the $\{u_i\}$ are polynomials in the $\{\partial^s(v_j)|s\geq 0,0\leq j\leq n-1\}$. The transformation

$$\{v_i\}\mapsto\{u_i\} \tag{3.3.13}$$

we call again the *Miura transformation*. To $\underline{\mathcal{L}}$ we associate the operator

$$Q=\begin{pmatrix}\mathcal{L}_0^{\frac{1}{n}} & & 0\\ & \ddots & \\ 0 & & \mathcal{L}_{n-1}^{\frac{1}{n}}\end{pmatrix} \tag{3.3.14}$$

in $M_n(R)[\xi, \xi^{-1}]$. For each $k \geq 1$, we consider the Lax equation

$$\frac{\partial}{\partial_{t_k}} \underline{\mathcal{L}} = [(Q^k)_+, \underline{\mathcal{L}}]. \tag{3.3.15}$$

We call this the *k-th equation* of the *n-th MKdV-hierarchy* and the operator $\underline{\mathcal{L}}$ that satisfies all these equations is called a solution of this hierarchy. Since (3.3.15) also holds for arbitrary powers of a solution, we see that we have

Proposition 3.7. *The Miura transformation maps solutions of the n-th MKdV-hierarchy to solutions of the n-th KdV-hierarchy.*

4. Solutions

4.1. **Solutions of the KP-hierarchy.** Here we recall the basic facts from [21] for the construction of solutions of the KP-hierarchy such as we need them. One considers the case $m = 2$, i.e. $\mathcal{F}(\underline{s})$ is the Grassmann manifold. Because of (2.2.2) we may assume that $\underline{s} = (s_1) = (0)$. Take any $W \in \mathcal{F}^{(\underline{0})}$ and consider

$$\Gamma_W = \{\gamma \in \Gamma_+ \mid \Lambda^k \circ p_1 \circ \Lambda^{-k} : \gamma^{-1} W \to \Lambda^k H_1 \text{ is a bijection } \}$$

Then we have

Lemma 4.1. *The set Γ_W is non-empty and open in Γ_+.*

Let R be the ring of analytic functions on Γ_W. For each $\gamma \in \Gamma_W$, we define

$$\begin{aligned} \hat{\psi}_W(\gamma) &= (\Lambda^k \circ p_1 \circ \Lambda^{-k})^{-1}(z^k) \in \gamma^{-1} W \\ &= z^k + \textstyle\sum_{i<k} a_i(\gamma) z^i \quad \text{and} \\ \psi_W(\gamma) &= \gamma(\hat{\psi}_W(\gamma)) \in W. \end{aligned}$$

Then ψ_W is by construction a function of type z^k in $M(g(\lambda))$ and the central result is

Theorem 4.2. *If we write $\psi_W = K_W \cdot g(z)$, with $K_W \in R[\xi, \xi^{-1}]$. Then ψ_W is a Baker function of type z^k for $L_W = K_W \xi K_W^{-1}$.*

From the fact that

$$L_W = K_W \xi K_W^{-1} = K_W (a_0 + \sum_{i<0} a_i \xi^i) \xi (a_0 + \sum_{i<0} a_i \xi^i)^{-1} K_W^{-1},$$

if $\partial(a_i) = 0$ for all i and the actual form of the action of Γ_-, one obtains moreover

Theorem 4.3. *(i) For each γ in Γ_- and $\delta \in \Delta$, we have $L_{\gamma\delta W} = L_W$.*
(ii) Let $\overline{\Gamma\Delta}$ be the closure of the image of $\Gamma_-\Delta$ in $Gl_{\text{res}}(H)$. The set of solutions of the KP-hierarchy obtained in this way can be identified with $\overline{\Gamma_-\Delta} \setminus Gl_{\text{res}}(H)/P(\underline{s})$.

That Δ acts trivially on the solutions shows that each connected component of $\mathcal{F}(\underline{0})$ renders the same set of solutions of the KP-hierarchy. On the other hand the triviality of the Γ_--action tells you that the commuting flows from Γ_+ give you practically a maximal set of independent directions.
One can also characterize geometrically the solutions from 4.2 that correspond to solutions of the n-th KdV-hierarchy.

Theorem 4.4. *For each $W \in \mathcal{F}(\underline{0})$ the following properties are equivalent*

(i) $(L_W^n)_+ = L_W^n$.
(ii) $\Lambda^n W \subset W$.

Proof. (i) $\Rightarrow$ (ii)
If $(L_W^n)_+ = L_W^n$, then we have

$$\partial_n(\psi_W)(\gamma) = z^n \psi_W(\gamma) \quad \text{for all } \gamma \in \Gamma_W.$$

Since the $\{\psi_W(\gamma) \mid \gamma \in \Gamma_W\}$ are dense in W and the left hand side of this equation belongs to W, we get $\Lambda^n W \subset W$.
(ii) $\Rightarrow$ (i)
For the Baker function ψ_W we have assume $W \in \mathcal{F}(0)^{(-k)}$

$$\begin{aligned} \partial_n(\psi_W) &= \{\partial_n(K_W)K_W^{-1} + L_W^n\} \cdot \psi_W \\ &= z^n \cdot \psi_W + \partial_n(K_W)K_W^{-1} \cdot \psi_W = (L_W^n)_+ \cdot \psi_w \end{aligned}$$

If $\wedge^n W \subset W$, then we know that $z^n \psi_W(\gamma) \in W$. This implies $\partial_n(K_w) \cdot g(z) \in W \cap (\gamma(z^k H_1)^\perp = \{0\}$ and hence $(L_W^n)\psi_W = (L_W^n)_+\psi_W$. This gives the required equality. □

4.2. **Darboux transforms.** We start with the fundamental observation that shows how flags make their appearance in this framework.

Theorem 4.5. *Let W_1 and W_2 be elements of $\mathcal{F}(\underline{0})$ belonging to respectively $\mathcal{F}(\underline{0})^{(m)}$ and $\mathcal{F}(\underline{0})^{(m+n)}$, with $n \geq 1$. Then the following 2 points are equivalent:*

(i) *There is a polynomial P in $R[\xi]$ of order n such that $P \cdot \psi_{W_2} = \psi_{W_1}$.*
(ii) $W_1 \subset W_2$.

Proof. (i) $\Rightarrow$ (ii) for each γ in $\Gamma_{W_1} \cap \Gamma_{W_2}$ the vector $(P \cdot \psi_{W_2})(\gamma)$ belongs by construction to W_2. Since the vectors

$$\{\psi_{W_1}(\gamma) \mid \gamma \in \Gamma_{W_1} \cap \Gamma_{W_2}\}$$

are dense in W_1, we see that the equality in (i) implies that $W_1 \subset W_2$.
(ii) $\Rightarrow$ (i) As we have seen at the construction of the wavefunctions we have for each $i \geq 0$.

$$\xi^i \cdot \psi_{W_2}(\gamma) = \{z^{-m-n+i} + \sum_{j<-m-n+i} a_j(\gamma) z^j\} g(z) \quad \text{and}$$

$$\psi_{W_1}(\gamma) = \{z^{-m} + \sum_{l<-m} b_l(\gamma) z^l\} g(z).$$

Hence we can always find a polynomial P in $R[\xi]$ of order n such that

$$P \cdot \psi_{W_2}(\gamma) - \psi_{W_1}(\gamma) = \{\sum_{r<-m-n} c_r(\gamma) z^r\} g(z).$$

If $W_1 \subset W_2$, then the left hand side of this equation belongs to W_2 and to the space $\gamma((z^{-m-n}H_1)^{\perp})$. By construction, this has to be zero and we obtain the desired equality. □

Take $\underline{s} = (0, -n)$ and consider the manifold $\mathcal{F}(\underline{s})$. The foregoing theorem shows that the map

$$(W_1, W_2) \to (L_{W_1}, L_{W_2})$$

assigns to each point in $\mathcal{F}(\underline{s})$ a pair of solutions of the KP-hierarchy that are coupled by a Darboux transformation of order n, i.e.

$$L_{W_1} = P L_{W_2} P^{-1}, \text{ with } P \in R[\xi] \text{ of order } n.$$

By combining this with theorem 4.3 we get analogously

Corollary 4.6. *Consider the collection of pairs $(L, \widetilde{L})$, where L and $\widetilde{L}$ are solutions of the KP-hierarchy as constructed in theorem 4.1.1 and such that L and $\widetilde{L}$ are coupled by $L = P\widetilde{L}P^{-1}$, with $P \in R[\xi]$ of order n. Then this set can be identified with*

$$\overline{\Gamma_- \Delta} \setminus Gl_{\text{res}}(H)/P(\underline{s}).$$

Next we consider the subvariety $\mathcal{F}(\underline{0})_m$ in $\mathcal{F}(\underline{0})$, consisting of all W in $\mathcal{F}(\underline{0})$ satisfying

$$\Lambda^m W \subset W.$$

Since the points of $\mathcal{F}(\underline{0})_m$ yield exactly the m-th roots of differential operators, we look for Darboux transforms that are also of this form. For $n \leq m$ and $\underline{s} = (0, -n)$, consider the subvariety $\mathcal{F}_m(\underline{s})$ of $\mathcal{F}(\underline{s})$ given by

$$\mathcal{F}_m(\underline{s}) = \{(\widetilde{W}, W) \mid W \in \mathcal{F}(\underline{0})_m(H) \text{ and } \Lambda^m W \subset \widetilde{W} \subset W\}.$$

Since $\Lambda^m \widetilde{W} \subset \Lambda^m W \subset \widetilde{W}$ we see that for each pair $(\widetilde{W}, W)$ in $\mathcal{F}_m(\underline{s})$ both requirements hold, i.e.

$$L^m_{\widetilde{W}} = (L^m_{\widetilde{W}})_+ \text{ and } L_{\widetilde{W}} = P L_W P^{-1},$$

with $P \in R[\xi]$ of order n.

4.3. **Solutions of the n-th MKdV-hierarchy.** In this subsection we consider the class of so-called periodic flags that is an intersection of the flagvarieties $\mathcal{F}_m(\underline{s})$ from subsection 4.2. Again we start with a W in $\mathcal{F}(\underline{0})_n$. A *periodic flag* associated to W is a chain

$$z^n(W) = W_m \subsetneq W_{m-1} \subsetneq \cdots W_0 = W.$$

If $\underline{s} = (s_1, s_1 - 1, \cdots, s_1 - n + 1)$, then the collection of all these flags is a subvariety $\mathcal{F}_{\text{per}}(\underline{s})$ of $\mathcal{F}(\underline{s})$. According to theorem 4.2.1 the Baker functions of the $\{W_i\}$ satisfy

$$\psi_{W_{i+1}} = (\xi + v_i)\psi_{W_i} \quad \text{for all } \ 1 \leq i \leq n-1.$$

If we use the operator $\underline{\mathcal{L}}$ in $M_n(R)[\xi]$ from subsection 3.4 and the fact that $\psi_{W_m} = z^n \psi_{W_0}$, these relations can be written as

$$\mathcal{L} \begin{pmatrix} \psi_{W_0} \\ \vdots \\ \psi_{W_{n-1}} \end{pmatrix} = \begin{pmatrix} z^n \psi_{W_0} \\ \psi_{W_1} \\ \vdots \\ \psi_{W_{n-1}} \end{pmatrix}.$$

By iterating this action we get form

$$\underline{\mathcal{L}}^n \begin{pmatrix} \psi_{W_0} \\ \vdots \\ \psi_{W_{n-1}} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_0 & & 0 \\ & \cdots & \\ 0 & & \mathcal{L}_{n-1} \end{pmatrix} \begin{pmatrix} \psi_{W_0} \\ \vdots \\ \psi_{W_{n-1}} \end{pmatrix} = z^n \begin{pmatrix} \psi_{W_0} \\ \vdots \\ \psi_{W_{n-1}} \end{pmatrix}.$$

In particular this implies that for all i, $0 \leq i \leq n-1$,

$$L^n_{W_i} = \mathcal{L}_i$$

and that $\underline{\mathcal{L}}^n$ satisfies the equations

$$\frac{\partial}{\partial t_k} \underline{\mathcal{L}}^n = [(Q^k)_+, \underline{\mathcal{L}}^n] \quad , k \geq 1,$$

where $Q = (\underline{\mathcal{L}}^n)^{\frac{1}{n}} \in M_n(R)[\xi, \xi^{-1}]$ as in subsection 3.4. We want to show now the following result.

Theorem 4.7. *The operator $\underline{\mathcal{L}}$ constructed above is a solution of the m-th MKdV-hierarchy, i.e. for all $k \geq 1$ it satisfies*

$$\frac{\partial \underline{\mathcal{L}}}{\partial t_k} = [(Q^k)_+, \underline{\mathcal{L}}]$$

Proof. Because of proposition 3.3 in [16], it suffices to show that $\underline{\mathcal{L}}$ and Q commute. We have that $[\underline{\mathcal{L}}, \underline{\mathcal{L}}^n] = [\underline{\mathcal{L}}, Q^n] = 0$ and since $\text{ad}(\underline{\mathcal{L}})$ is a derivation we get

$$[\underline{\mathcal{L}}, Q^n] = \sum_{i=0}^{n-1} Q^i [\underline{\mathcal{L}}, Q] \cdot Q^{n-i-1} = 0.$$

If $[\underline{\mathcal{L}}, Q] = \alpha\xi^s$ + "lower order", then there must hold

$$[\underline{\mathcal{L}}, Q^n] = n\alpha\xi^{n+s-1} + \text{ "lower order" } = 0.$$

This shows $[\mathcal{L}, Q] = 0$. $\square$

If $W \in \mathcal{F}(\underline{0})_n$, then we have seen that each $(W_i) \in \mathcal{F}_{\mathrm{per}}(\underline{s})$, with $W_0 = W$, gives you a decomposition of L^n_W into linear factors

$$L^n_W = \xi^n + \sum_{i=0}^{n-2} u_i\xi^i = (\xi + v_{n-1}) \cdots (\xi + v_0).$$

Because of theorem 4.3.1 we may say that the Miura transformation $\{v_i\} \to \{u_i\}$ corresponds geometrically to the natural projection $(W_i) \to W_0$ of $\mathcal{F}_{\mathrm{per}}(\underline{s})$ onto $\mathcal{F}(\underline{0})_n$.

Remark 4.8. The periodic flags also turn up in a natural way in algebraic geometric situations, see e.g. [1] and [20].

Remark 4.9. The KP-hierarchy and its modificed versions can also be formulated in the so-called Hirota bilinear form. The equations are then expressed in quadratic relations for a finite number of so-called τ-functions. The geometric interpretation of these functions requires the transition to the bundle *Det**, see [21]. For the geometric description of the relations, we refer to [8].

5. Deformations

5.1. **Deformations of meromorphic equations on $\mathbb{P}_1(\mathbb{C})$.** It is a well-known fact, see e.g. [11], that deformations of linear meromorphic differential equations on $\mathbb{P}^1(\mathbb{C})$ may lead to solutions of non-linear differential equations that possess interesting features, like the Painlévé-property. We start with a description of the setting. Consider a linear differential equation on $\mathbb{P}^1(\mathbb{C})$ that is meromorphic over $Y_0 = \{a^0_1, \cdots, a^0_m, \infty\}$. By this, we mean the set of data

1.3. (i) a holomorphic vector bundle E^0 over $\mathbb{P}^1(\mathbb{C})$.
(ii) An integrable connection ∇^0 of the bundle $E^0 \mid \mathbb{P}^1(\mathbb{C}) - Y_0$ that is meromorphic over Y_0.

If x is the meromorphic function that identifies $\mathbb{P}_1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$, then the connection form Ω^0 of ∇^0 can be written as

$$\Omega^0 = \{\sum_{1\le k\le m} \{\sum_{l\ge 1} \frac{A^0_{kl}}{(x - a^0_k)^l}\} + \sum_{l\ge 0} A^0_{\infty l} x^l\} dx,$$

where the A^0_{kl} and $A^0_{\infty l}$ are complex $n \times n$ matrices that are zero for sufficiently large l.
Now we are interested in deformations that move the poles $\{a^0_i\}$ of Ω^0 in a holomorphic way inside $\mathbb{C}$. For simplicity, we assume that the parameterspace of the deformation,

shortly called **deformation space**, is a connected complex variety T. The way the poles are moved is determined by a set of holomorphic functions $a_i : T \to \mathbb{C}$, $1 \le i \le m$, the so-called **deformation functions**. Furthermore there has to be a base point t_0 in T such that $a_i(t_0) = a_i^0$ for all i. To avoid a priori topological obstructions, we will restrict ourselves to deformations for which the poles never coincide, i.e. the deformation functions satisfy

$$a_i(t) \neq a_j(t) \text{ for all t in T and all i} \neq \text{j.}$$

Let X be $\mathbb{P}^1(\mathbb{C}) \times T$. We can introduce a smooth codimension one subvariety Y of X by

$$Y = Y_1 \cup \ldots \cup Y_m \cup Y_\infty \text{ with}$$
$$Y_i = \{(x,t) \mid (x,t) \in X, x = a_i(t)\} \text{ and } Y_\infty = \{(\infty,t) \mid (\infty,t) \in X\}.$$

The object one is interested in, is then

Definition 5.1. An **integrable deformation** (E,∇) of the pair (E^0,∇^0) with deformation space T, deformation functions $\{a_i\}$ and base point t_0 in T consists of

(i) a holomorphic vector bundle E over $X = \mathbb{P}_1(\mathbb{C}) \times T$ of rank n.
(ii) an integrable connection ∇ of $E \mid X - Y$, that is meromorphic over Y and that is such that the restriction of (E,∇) to $\mathbb{P}_1(\mathbb{C}) \times \{t_0\}$ is isomorphic to (E^0,∇^0).

Given the deformation space, the deformation functions $\{a_i\}$ and the base point $\{t_0\}$, a relevant question is if there exists an integrable deformation with these deformation data. Even if there is no fusion of the poles, one might hit at topological obstructions. Let $i : \mathbb{P}_1(\mathbb{C}) \to T$ be the embedding $x \mapsto (x,t_0)$. It induces a natural map $i^* : \pi_1(\mathbb{P}_1(\mathbb{C}) - Y_0) \to \pi_1(X - Y)$. If there exists an integrable deformation (E,∇) of (E^0,∇^0), then the representation ρ of $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{a_1^0,\ldots,a_m^0,\infty\})$ corresponding to (E^0,∇^0) has to factorize through i^*. Consider the projection $p_2 : X - Y \to T$, given by $p_2((x,t))$ = t. The fiber over t is equal to $\mathbb{C} - \{a_1(t),\ldots,a_m(t)\}$. Thus $(X - Y, T, p_2)$ is a fiber bundle and we have the long exact sequence

$$\pi_2(T) \to \pi_1(\mathbb{P}_1(\mathbb{C}) - \{a_1^0,\ldots,a_m^0,\infty\}) \xrightarrow{i^*} \pi_1(X - Y) \to \pi_1(T) \to 1.$$

Now ρ has to be trivial on the image of $\pi_2(T)$ and should be extendable to $\pi_1(X - Y)$. Such problems do not occur if $\pi_2(T) = \pi_1(T) = \{1\}$. In that case the monodromy is preserved by the deformation. An example of such a deformation space occurred in [5] and is given by

Example 5.2. Let Z be the space

$$Z = \mathbb{C}^m - \bigcup_{i \neq j} D_{ij}, \text{ where } D_{ij} = \{(x_k) \in \mathbb{C}^m, x_i = x_j\}.$$

As deformation space we take the universal covering space $\tilde{Z}$ of Z and for $a_i : \tilde{Z} \to \mathbb{C}$ we take the composition of the natural projection $\pi : \tilde{Z} \to Z$ with the projection on the i-th coordinate.

Remark 5.3. From the construction one sees directly that in a neighbourhood of the singular point $(a_j^0, t_0) \in Y_j$, the singular part of the connection form Ω has w.r.t. any trivializing basis of local sections of E, the form

$$\sum_{l\geq 1} \frac{B_{jl}(t)}{(x-a_j(t))^l} d(x-a_j),$$

where all the B_{jl} are holomorphic. By applying a proper coordinate transformation, one may moreover assume

$$B_{jl}(t_0) = A_{jl}(t_0) \text{ for all } l.$$

Since we considered integrable deformations the functions B_{jl} satisfy the non-linear compatibility conditions and those are the non-linear equations we refered to at the beginning. E.g., if ∇^0 has a logarithmic pole over Y_0, i.e. $A_{kl}^0 = 0$ for all $l > 1$ and $A_{\infty l}^0 = 0$ for all $l \geq 0$, then the B_{j1} satisfy the so-called "Schlesinger equations"

$$dB_{i1} = -\sum_{j\neq i} [B_{i1}, B_{j1}] \frac{d(a_i - a_j)}{a_i - a_j}.$$

For the space $\tilde{Z}$ with the deformation functions $\{a_i\}$ as in the foregoing example we had no topological obstructions and this turns out to be sufficient, for following carefully the lines of proof in [15], one can show

Theorem 5.4. *For each pair* (E^0, ∇^0) *there exists an integrable deformation* (E, ∇) *of* (E^0, ∇^0) *with* $\tilde{Z}$ *as deformation space and the* $\{a_i\}$ *as deformation functions.*

Let (E, ∇) be the integrable deformation from theorem 5.4. and assume E^0 was a trivial vector bundle. Then we consider as in [15], the set

$$\Theta = \{t \in \tilde{Z} | E|_{\mathbb{P}^1(\mathbb{C})\times\{t\}} \text{ is non-trivial}\}$$

Since all holomorphic line bundles over Z are trivial, one can show

Proposition 5.5. *There is a holomorphic* $\underline{\tau} : \tilde{Z} \to \mathbb{C}$ *such that*

$$\Theta = \{t \in \tilde{Z}, \underline{\tau}(t) = 0\}.$$

Choose open balls D_1 and D_2 in $\mathbb{P}^1(\mathbb{C})$ such that $D_1 \cup D_2 = \mathbb{P}^1(\mathbb{C})$ and that $E|D_1 \times \tilde{Z}$ and $E|D_2 \times \tilde{Z}$ are trivial. The comparison of trivializing sections of these two bundles, gives you a holomorphic transfer funtion

$$S : D_1 \cap D_2 \times \tilde{Z} \to Gl_n(\mathbb{C}).$$

Let H be $L^2(S^1, \mathbb{C}^n)$ with the decomposition $H = H_+ \oplus H_+^\perp$, where

$$H_+ = \{\sum_{i\geq 0} b_i x^i \in H, b_i \in \mathbb{C}^n\}.$$

Then S determines a holomorphic family of operators $\{\mathbb{S}(t) | t \in \tilde{Z}\}$ in $Gl_{\text{res}}(H)$, such that $\mathbb{S}(t)$ is in the big cell for all $t \in \tilde{Z} - \Theta$. Locally one can fit the $\mathbb{S}(t)$ into the group G and this suffices to prove that

Proposition 5.6. *(i) There exist holomorphic maps* $S_+ : D_2 \times \{\tilde{Z} - \Theta\} \to Gl_n(\mathbb{C})$ *and* $S_- : D_1 \times \{\tilde{Z} - \Theta\} \to Gl_n(\mathbb{C})$ *with* $S_-(\infty, t) = Id$ *such that* $S = S_-^{-1} S_+$.
(ii) The maps S_+ *and* S_- *are meromorphic over* $D_2 \times \Theta$ *resp.* $D_1 \times \Theta$.

In the case that ∇^0 has a logarithmic pole at infinity this proposition is used to show that the connection form of $\nabla | D_i \times \tilde{Z} - Y \cap \{D_i \times \tilde{Z}\}$ w.r.t. suitable trivializing basis extends meromorphically to $D_i \times \tilde{Z}$. Thus one obtains the following result.

Theorem 5.7. *(i) Assume that* E^0 *is trivial and that* ∇^0 *has a logarithmic pole at infinity . Let* (E, ∇) *be the integrable deformation from 5.4. Then there is a neighbourhood* U *of* t_0 *such that* $E \mid \mathbb{P}^1(\mathbb{C}) \times U$ *is trivial and a basis of sections* $\{h_i\}$ *such that the connection form* Ω *w.r.t. the* $\{h_i\}$ *looks like*

$$\Omega = \sum_{i=1}^{m} \sum_{l\geq 1} \frac{B_{il}(t)}{(x - a_i(t))^l} d(x - a_i),$$

with $B_{il}(t_0) = A_{il}^0$ *for all i and l.*

(ii) These solutions $\{B_{il}\}$ *of the integrability equations extend holomorphically to* $\tilde{Z} - \Theta$ *and are meromorphic on* $\tilde{Z}$*.*

Remark 5.8. In the case that ∇^0 has logarithmic poles over Y_0, Malgrange has shown in [15] that there is a holomorphic $\tau : \tilde{Z} \to \mathbb{C}$ with zero-set Θ such that $\omega = \frac{d\tau}{\tau}$ is the differential form presented in [11]. This τ-function can also be interpreted as the determinant of an associated Cauchy-Riemann operator on the spin bundle over $\mathbb{P}^1(\mathbb{C})$, see [17].

Remark 5.9. The bundle Det* plays also an important role at monodromy preserving deformations of Dirac equations, see [18] and [19].

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