ON THE CONJUGACY OF CARTAN SUBSPACES

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Abstract. Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k, H a k-open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. For real and $\mathfrak{p}$-adic symmetric k-varieties the space $L^2(G_k/H_k)$ of square integrable functions decomposes into several series, one for each H_k-conjugacy class of Cartan subspaces of G_k/H_k.

In this paper we give a characterization of the H_k-conjugacy classes of these Cartan subspaces in the case that there exists a splitting extension of order 2 and (G,σ) satisfies the additional condition that all tori which are maximal σ-split and k-split are H_k-conjugate. This condition is satisfied for k the the real numbers and many of the $\mathfrak{p}$ symmetric k-varieties with a splitting extension of order 2. For $k=\mathbb{R}$ we prove a number of additional results as well.

1. Introduction

Let G be a connected reductive linear algebraic group defined over a field k of characteristic not 2, σ an involution of G defined over k, $G_\sigma = \{g \in G \mid \sigma(g) = g\}$ the set of fixed points of σ and H a k-open subgroup of G_σ. Denote the set of k-rational points of G (resp. H) by G_k (resp. H_k). The variety G/H is called a symmetric variety and the variety G_k/H_k a symmetric k-variety. The symmetric k-varieties over the real and $\mathfrak{p}$-adic numbers are also called reductive symmetric spaces. Symmetric k-varieties occur in many problems in representation theory and geometry. For real and $\mathfrak{p}$-adic symmetric k-varieties one mainly studies the decomposition into irreducible components of the regular representation of G_k on the Hilbert space $L^2(G_k/H_k)$ of square integrable functions on G_k/H_k (also called the harmonic analysis of the reductive symmetric space). Especially the harmonic analysis of real symmetric k-varieties has been studied extensively in the last few decades. (see for example [HC84, BD92, CD94, Del97, FJ80, OM84, OS80, BS97]).

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For real symmetric k-varieties $L^2(G_k/H_k)$ decomposes into several series, one for each H_k-conjugacy class of Cartan subspaces of $\mathfrak{q} := \mathrm{Lie}(G_k/H_k)$. The most extreme of these are respectively the most discrete series (in the group case called fundamental series) corresponding to the conjugacy class of Cartan subspaces with maximal compact part and the most continuous series, corresponding to the conjugacy class with maximal non-compact part. For the $\mathfrak{p}$-adic symmetric k-varieties one gets a similar decomposition of $L^2(G_k/H_k)$. The equivalent of the "Cartan subspaces of $\mathfrak{q} = \mathrm{Lie}(G_k/H_k)$" for general symmetric k-varieties are the maximal σ-split k-tori. These are those k-tori A of G, satisfying $\sigma(a) = a^{-1}$ for all $a \in A$. By abuse of notation we will also call these tori Cartan subspaces.

For real symmetric k-varieties a characterization of the conjugacy classes of Cartan subspaces can be obtained using the dual pair correspondence and identifying them as maximal split tori for the dual pair (see [Mat79]). Unfortunately this dual pair correspondence only exists for real groups and depends on the existence of a unique compact real form for each complex reductive group. So for symmetric k-varieties over other base fields, like the $\mathfrak{p}$-adic numbers and finite fields, one will need a characterization of the conjugacy classes of Cartan subspaces which is independent of this dual pair correspondence.

This paper is the first in a series of papers in which we characterize the conjugacy classes of the Cartan subspaces for a number of base fields. In this paper we consider the special case that the Galois group Γ of a splitting extension $K \supset k$ for the maximal σ-split k-tori is of order 2 and the pair (G, σ) satisfies the extra condition that all maximal σ-split k-tori containing a maximal k-split part are H_k-conjugate. This condition is satisfied in the case that $k = \mathbb{R}$ and for many of the $\mathfrak{p}$-adic symmetric k-varieties. The main reason that we treat this case separately is because in this case we can give a much more detailed characterization of the conjugacy classes of the Cartan subspaces in terms of involutions in a Weyl group. When Γ has order n one needs to consider conjugacy classes of elements of order > 2 in the Weyl group and much less is known about these.

The characterization we give of the conjugacy classes of Cartan subspaces is relatively short and direct and does not use the dual pair correspondence. Consequently it holds for a number of other base fields as well. For real symmetric k-varieties the characterization we obtain is different from the one given by Matsuki in [Mat79] via the dual pair correspondence and we are also able to give more precise results. In a future paper we hope to expend these results to arbitrary base fields.

The results we obtained are as follows. If T is a σ-stable torus of G, then we write: $T^+ = (T \cap H)^0$ and $T^- = \{t \in T \mid \sigma(t) = t^{-1}\}^0$. The torus T^- is called a *σ-split* torus of G. Denote the set of characters and the set of roots of T with respect to G by respectively, $X^*(T)$ and $\Phi(T)$. For a closed subgroup M of G we denote the Weyl group of M relative to T by $W(A, M) = N_M(A)/Z_M(A)$. If $M = G$, then we will also write $W(A) = W(A, G)$.

Let $\mathcal{A}_k$ denote the set of maximal σ-split k-tori of G. An extension $K \supset k$ will be called a *splitting extension of* (G, σ) if all maximal σ-split k-tori of G are K-split. Assume $\Gamma = \mathrm{Gal}(K/k)$ is of order 2, i.e $\Gamma = \{\mathrm{id}, \delta\}$. If A is a maximal σ-split k-torus of G, then we can write $A = A^a A^d$, where A^a is the anisotropic part of A and A^d is the k-split part of A. Since Γ is of order 2 we can also write these as eigenspaces under the action of Γ. Let $X_0(\Gamma) = \{\chi \in X^*(A) \mid \delta(\chi) = -\chi\}$ and $X^\Gamma = \{\chi \in X^*(A) \mid \delta(\chi) = \chi\}$. Then A^d is the annihilator of $X_0(\Gamma)$ and A^a is the annihilator of X^Γ.

If we fix a maximal σ-split k-torus A of G, then since all σ-split tori of G are H-conjugate, any other maximal σ-split k-torus A_1 of G is of the form $A_1 = gAg^{-1}$ with $g \in H$ and $n = g^{-1}\delta(g) \in N_G(A)$. Denote the image of n in $W(A)$ by w_g. Then $\delta(w_g)w_g = \mathrm{id}$, i.e., w_g is a δ-twisted involution in $W(A)$. The representative g of the H_k-conjugacy class of gAg^{-1} is unique up to left translations from H_k and right translations from $N_{G_k}(A)$. If $x = hgn$ is another representative ($h \in H_k$, $n \in N_{G_k}(A)$, $w \in W(A)$ the corresponding Weyl group element), then $w_x = w^{-1}w_g\delta(w)$. So this leads to δ-twisted conjugacy classes of δ-twisted involutions. In the case that A contains a maximal (σ, k)-split torus (i.e. a torus which is both σ-split and k-split) we show that we can choose g in such a way that w_g becomes an involution (see Lemma 3.4). This involutions in $W(A)$ is called A_1-standard or k-singular. Unfortunately in general the k-singular involutions are not uniquely determined by the H_k conjugacy class of maximal σ-split k-tori. For this an additional condition is required. We will say that the pair (G, σ) is (σ, k)*-split conjugate* (resp. (σ, k)*-anisotropic conjugate*) if all maximal σ-split k-tori of G containing a maximal (σ, k)-split (resp. σ-split k-anisotropic) torus of G are H_k-conjugate. We note that both these conditions are satisfied in the case that $k = \mathbb{R}$ or a real closed field and (G, σ) is (σ, k)-split conjugate in the case of most $\mathfrak{p}$-adic fields with a splitting extension of order 2.

Fix an element $A_0 \in \mathcal{A}_k$ such that A_0^d is a maximal (σ, k)-split torus of G and let $\mathcal{I}(A_0)$ denote the set of involutions in $W(A_0)$. In Theorem 3.11 and Proposition 4.4 we show that if (G, σ) is (σ, k)-split conjugate, then there exists a map $\varphi : \mathcal{A}_k/H_k \to \mathcal{I}(A_0)/W(A_0, H_k)$. If (G, σ) is also (σ, k)-anisotropic conjugate, then φ is in fact one-to-one. In the other cases we need a second invariant to characterize the H_k-conjugacy classes in $\mathcal{A}_k$. In section 5 we characterize $\varphi(\mathcal{A}_k/H_k) \subset \mathcal{I}(A_0)/W(A_0, H_k)$. This is precisely the set of k-singular involutions in $W(A_0)$.

Since A_0^d is maximal (σ, k)-split the k-singular involutions in $W(A_0)$ are also involutions in $W(A_0^d)$. The characterization of $W(A_0, H_k)$-conjugacy classes of k-singular involutions can be reduced to $W(A_0^d, H_k)$-conjugacy classes. The Weyl group $W(A_0^d)$ is the natural Weyl group of the symmetric k-variety and is known from the classification in [Hel98b]. The subgroup $W(A_0^d, H_k)$ is often easier to determine than $W(A_0, H_k)$ and in fact in many cases $W(A_0^d) = W(A_0^d, H_k)$. When this happens we have even a stronger characterization of the conjugacy classes, see Corollary 4.6 and Proposition 6.29. For $k = \mathbb{R}$ a characterization of the pairs for which these Weyl groups are the same is given in Proposition 6.32.

In section 6 we look in more detail at the real symmetric k-varieties. In this case (G, σ) is both (σ, k)-split conjugate and (σ, k)-anisotropic conjugate, so the above results give a characterization of the $H_{\mathbb{R}}$-conjugacy classes of Cartan subspaces in terms of $\mathbb{R}$-singular involutions. One can improve this characterization by using the fact that G has a Cartan involution θ, which commutes with σ. Since every maximal σ-split $\mathbb{R}$-torus of G is $H_{\mathbb{R}}^0$-conjugate with one which is θ-stable (see Proposition 6.3), one can reduce in this case to θ-stable maximal σ-split $\mathbb{R}$-tori. Denote the set of these tori by $\mathcal{A}_{\mathbb{R}}^{\theta}$. Using the Cartan decomposition of $G_{\mathbb{R}}$, it then follows that $A_1, A_2 \in \mathcal{A}_{\mathbb{R}}^{\theta}$ are $H_{\mathbb{R}}$-conjugate if and only if they are conjugate under $H_{\mathbb{R}}^+ := H_{\mathbb{R}} \cap G_{\theta}(\mathbb{R})$ (see Corollary 6.5). Similar as in Theorem 3.11 the $H_{\mathbb{R}}^+$-conjugacy classes can be characterized by $W(A_0^-, H_{\mathbb{R}}^+)$-conjugacy classes of $\mathbb{R}$-singular involutions, see Theorem 6.14. In fact in this case one can even extend this result to $G_{\mathbb{R}}$-conjugacy classes of Cartan subspaces using a result of [HHNÓ98] (see Corollary 6.16).

In the real case we can also refine the characterization of the $\mathbb{R}$-singular involutions using the involutions θ and $\sigma\theta$ (see 6.19). Combined with the classification of pairs of commuting involutions in [Hel88] it is then easy to determine whether an involution of $W(A_0^-)$ is $\mathbb{R}$-singular or not. Finally the conjugation of the $\mathbb{R}$-singular involutions is under the group $W(A_0^-, (H_{\mathbb{R}}^+)^0)$ which is the Weyl group of A_0^- with respect to $G_{\sigma\theta}$. This Weyl group again easily follows from the classification in [Hel88].

We conclude the paper by making the connection of the above results for real reductive symmetric spaces with those one can obtain using the dual pair correspondence. The characterization in this paper gives a number of additional results including an easier way to classify the conjugacy classes.

2. Preliminaries and Recollections

In this section we set the notations and recall a few results from [HW93], [Hel88] and [Hel91]. We use as our basic reference for reductive groups the papers of Borel and Tits [BT65, BT72] and also the books of Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations and terminology.

2.1. Given an algebraic group G, the identity component is denoted by G^0. We use $L(G)$ (resp. $\mathfrak{g}$, the corresponding lower case German letter) for the Lie algebra of G. If S is a subset of G and H a closed subgroup of G, then we write $N_H(S)$ (resp. $Z_H(S)$) for the normalizer (resp. centralizer) of S in H. We write $Z(G)$ for the center of G. The commutator subgroup of G is denoted by $D(G)$ or $[G, G]$.

Let k be a field. An algebraic group defined over k shall also be called an algebraic k-group. For an extension K of k, the set of K-rational points of G is denoted by G_K or $G(K)$.

If G is a reductive k-group and A a torus of G then we denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of A (resp. one-parameter subgroups of A) and by $\Phi(A) = \Phi(G, A)$ the set of the roots of A in G. The group $X^*(A)$ can be put in duality with $X_*(A)$ by a pairing $\langle \cdot\ ,\ \cdot \rangle$ defined as follows: if $\chi \in X^*(A),\ \lambda \in X_*(A)$, then $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $t \in k^*$.

For a closed subgroup H of G we denote the Weyl group of H relative to A by $W_H(A) = W(A, H) = N_H(A)/Z_H(A)$. If $H = G$, then we will also write $W(A) = W(A, G) = N_G(A)/Z_G(A)$. If $\alpha \in \Phi(A, G)$, then let U_α denote the unipotent subgroup of G corresponding to α. If A is a maximal torus, then U_α is one-dimensional. Given a quasi-closed subset ψ of $\Phi(A, G)$, the group G_ψ (resp. G^*_ψ) is defined in [BT65, 3.8]. If G^*_ψ is unipotent, ψ is said to be unipotent and often one writes U_ψ for G^*_ψ.

If T is a torus of G defined over k, then there are subtori T^a and T^d of T, where T^a is the largest anisotropic subtorus of T and T^d is the largest k-split subtorus of T defined over k. These tori satisfy: $T = T^a \cdot T^d$ and $T^a \cap T^d$ is finite (see [Bor91, 8.15]).

If T_1 and T_2 are tori and ϕ is a homomorphism of T_1 into T_2, then the mapping ${}^t\phi$ of $X^*(T_2)$ into $X^*(T_1)$, defined by

$$ {}^t\phi(\chi_2) = \chi_2 \circ \phi, \qquad \chi_2 \in X^*(T_2) \tag{2.1.1} $$

is a module homomorphism. If ϕ is an isomorphism, then ${}^t\phi^{-1}$ is a module isomorphism from $(X^*(T_1), \Phi(T_1))$ onto $(X^*(T_2), \Phi(T_2))$. Instead of ${}^t\phi$ we will also write ϕ^*.

Throughout the paper G will denote a connected reductive algebraic k-group.

2.2. **Involutions of G.**

Let k be a field of characteristic not two, G a connected algebraic k-group, σ an automorphism of G of order two and $G_\sigma = \{g \in G \mid \sigma(g) = g\}$ the set of fixed points of σ. This is a subgroup of G which is reductive if G is reductive. If G is semisimple and simply connected, then G_σ is connected, but in general G_σ is not necessarily connected. When G and σ are defined over k, the automorphism σ will also be called a k-involution of G.

If G is reductive and H a k-open subgroup of G_σ, then we call the variety G/H a *symmetric variety* and the variety G_k/H_k a symmetric k-variety. Symmetric varieties are spherical.

Given $g,\ x \in G$, the *twisted action* associated to σ is given by $(g, x) \mapsto g * x = gx\sigma(g)^{-1}$. This action will also be called the *σ-twisted action*. Let $Q = \{g^{-1}\sigma(g) \mid g \in G\}$ and $Q' = \{g \in G \mid \sigma(g) = g^{-1}\}$. The set Q is contained in Q'. Both Q and Q' are invariant under the twisted action associated to σ. There are only a finite number of twisted G-orbits in Q' and each such orbit is closed (see [Ric82]). In particular, Q is a connected closed k-subvariety of G. Define a morphism $\tau_\sigma : G \to G$ by

$$\tau_\sigma(x) = \sigma(x)x^{-1}, \quad (x \in G). \tag{2.2.1}$$

We will omit the subscript σ from this map if there is no ambiguity about the involution involved. The image $\tau(G) = Q$ is a closed k-subvariety of G and τ induces an isomorphism of the coset space G/G_σ onto $\tau(G)$. Note that $\tau(x) = \tau(y)$ if and only if $y^{-1}x \in G_\sigma$ and $\sigma(\tau(x)) = \tau(x)^{-1}$ for $x \in G$.

2.3. If $T \subset G$ is a torus and $\phi \in \mathrm{Aut}(G, T)$ an involution, then we write $T_\phi^+ = (T \cap G_\phi)^0$ and $T_\phi^- = \{x \in T \mid \phi(x) = x^{-1}\}^0$. It is easy to verify that the product map

$$\mu : T_\phi^+ \times T_\phi^- \to T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular $T = T_\phi^+ T_\phi^-$ and $T_\phi^+ \cap T_\phi^-$ is a finite group. (In fact it is an elementary abelian 2-group.) The automorphisms of $\Phi(T, G)$ and $W(T, G)$ induced by ϕ will also be denoted by ϕ.

Recall from [Hel88] that a torus A of G is called *σ-split* if $\sigma(a) = a^{-1}$ for every $a \in A$. If A is a maximal σ-split torus of G, then $\Phi(A, G)$ is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$ (see [Ric82]). This is the root system associated with the symmetric variety G/H. To the symmetric k-variety G_k/H_k one can also associate a natural root system. This is the root system of a maximal (σ, k)-split torus, which are those tori that are both σ-split and k-split. In [HW93, 5.9] it was shown that the set of roots $\Phi(A, G)$ of a maximal (σ, k)-split torus A in G is a root system with Weyl group $N_{G_k}(A)/Z_{G_k}(A)$. We can also obtain this root system by restricting the root system of G_k. Namely let $A_0 \supset A$ be a σ-stable maximal k-split torus of G. Then $A = (A_0)_\sigma^-$ and $\Phi(A, G)$ can be identified with $\bar{\Phi}_\sigma = \{\alpha|A \neq 0 \mid \alpha \in \Phi(A_0, G)\}$.

2.4. An extension $K \supset k$ will be called a *splitting extension of* (G, σ) if all maximal σ-split k-tori of G are K-split. We will call K a *minimal splitting extension of* (G, σ) if K is a splitting extension and any subfield K_0 with $k \subset K_0 \subsetneq K$ is not a splitting extension of (G, σ). We will denote the Galois group of $K \supset k$ by $\Gamma = \mathrm{Gal}(K/k)$.

2.5. Let $\mathcal{A}$ denote the set of maximal σ-split tori of G and $\mathcal{A}_k$ the set of maximal σ-split k-tori of G. If $k = \mathbb{R}$ and $A \in \mathcal{A}_k$ is a maximal σ-split $\mathbb{R}$-torus of G, then the Lie algebra $\mathfrak{a}_{\mathbb{R}}$ of $A_{\mathbb{R}}$ is also called a *Cartan subspace* of $\mathfrak{g}$. By abuse of notation we will sometimes also call the maximal σ-split k-torus of G a Cartan subspace of G.

As for the conjugacy of the Cartan subspaces we first recall the following result by Vust [Vus74]:

Proposition 2.6 ([Vus74]). *All maximal σ-split tori of G are conjugate under H^0.*

3. Standard pairs

In this section we assume that there exists a minimal splitting extension $K \supset k$ such that every maximal σ-split k-torus of G splits over K and that $\Gamma = \mathrm{Gal}(K/k)$ is of order 2. We first note that this is also a splitting extension for all reductive subgroups of G which are centralizers of tori:

Lemma 3.1. *Let $K \supset k$ be a minimal splitting extension of (G, σ) and S a σ-stable k-torus of G. Then every maximal σ-split k-torus of $Z_G(S)$ splits over K.*

Proof. Since any maximal σ-split k-torus of $Z_G(S)$ is contained in a maximal σ-split k-torus of G, the result is clear. ☐

3.2. If A is a maximal σ-split k-torus of G, then we can write, similarly as in 2.1 $A = A^a A^d$, where A^a is the anisotropic part of A and A^d is the k-split part of A. Since Γ is of order 2 we can also write these as eigenspaces under the action of Γ. Let $\Gamma = \{\mathrm{id}, \delta\}$, $X_0(\Gamma) = \{\chi \in X^*(A) \mid \delta(\chi) = -\chi\}$ and $X^\Gamma = \{\chi \in X^*(A) \mid \delta(\chi) = \chi\}$. Then $A^d = \{a \in A \mid \chi(a) = e \text{ for all } \chi \in X_0(\Gamma)\}$ is the annihilator of $X_0(\Gamma)$ and $A^a = \{a \in A \mid \chi(a) = e \text{ for all } \chi \in X^\Gamma\}$ is the annihilator of X^Γ. Since $X_0(\Gamma)$ and X^Γ are the eigenspaces of δ (which has order 2) we will write by a slight abuse of notation $A^a = A^+_\delta$ and $A^d = A^-_\delta$, although in a strict sense A^a and A^d are not eigenspaces of δ.

Definition 3.3. For $A_1, A_2 \in \mathcal{A}_k$, the pair (A_1, A_2) is called *standard* if $A^d_1 \subset A^d_2$. In this case, we also say that A_1 is *standard* with respect to A_2.

A standard pair (A_1, A_2) of maximal σ-split k-tori of G gives rise to an involution in $W(A_2)$.

Lemma 3.4. *Let (A_1, A_2) be a standard pair of maximal σ-split k-tori of G. Then we have the following conditions:*

(1) *There exists $g \in Z_{H^0}(A^d_1)$ such that $gA_1g^{-1} = A_2$.*
(2) *If $n = \delta(g)g^{-1}$, then $n \in N_{H^0}(A_2)$.*
(3) $\mathrm{Int}(n)^* = -\mathrm{id} \circ \delta : X^*(A_2) \to X^*(A_2)$.

(4) *Let w be the image of n in $W(A_2)$. Then $w^2 = e$, and $(A_2)^+_w = A_1^d A_2^a$ which characterizes w.*

Proof. (1) is immediate from Proposition 2.6. Let $g \in Z_{H^0}(A_1^d)$ such that $gA_1g^{-1} = A_2$ and let $n = \delta(g)g^{-1}$. Since A_1, A_2 are k-tori, it follows that $A_2 = \delta(A_2) = \delta(g)A_1\delta(g)^{-1}$, hence $n \in N_{H^0}(A_2)$, what shows (2).

Since $g \in Z_{H^0}(A_1^d)$ we may assume that $A_1^d = \{e\}$, i.e. A_1 is k-anisotropic and consequently δ acts as $-\operatorname{id}$ on $X^*(A_1)$. The map $\operatorname{Int}(g)$ maps A_1 onto A_2, so by (2.1.1) the dual map $\operatorname{Int}(g)^* : X^*(A_2) \to X^*(A_1)$. Since also $\operatorname{Int}(\delta(g))^* : X^*(A_2) \to X^*(A_1)$ it follows that $\operatorname{Int}(n)^* = \operatorname{Int}(\delta(g)g^{-1})^* = \operatorname{Int}(g^{-1})^* \operatorname{Int}(\delta(g))^* : X^*(A_2) \to X^*(A_2)$ is an isomorphism. Let $Y \in X^*(A_2)$ and let $Y_1, Y_2 \in X^*(A_2)$ be such that $Y = Y_1 + Y_2$ and $\delta(Y_1) = -Y_1$, $\delta(Y_2) = Y_2$. Let $X_1 = \operatorname{Int}(g)^*(Y_1)$ and $X_2 = \operatorname{Int}(g)^*(Y_2)$. Then

$$\begin{aligned}\operatorname{Int}(n)^*(Y) &= \operatorname{Int}(g^{-1})^* \operatorname{Int}(\delta(g))^*(Y_1 + Y_2) = \operatorname{Int}(g^{-1})^*\delta \operatorname{Int}(g)^*\delta(Y_1 + Y_2)\\ &= \operatorname{Int}(g^{-1})^*\delta \operatorname{Int}(g)^*(-Y_1 + Y_2) = \operatorname{Int}(g^{-1})^*\delta(-X_1 + X_2)\\ &= \operatorname{Int}(g^{-1})^*(X_1 - X_2) = Y_1 - Y_2 = -\operatorname{id}\circ\delta(Y).\end{aligned}$$

This proves (3). Finally (4) is immediate from (3). □

Remark. By (4) of Lemma 3.4, w is independent of our choice of $g \in Z_{H^0}(A_1^d)$ with $gA_1g^{-1} = A_2$.

Definition 3.5. Let A_1, A_2, $w \in W(A_2)$ be as in Lemma 3.4. We call w the A_1-*standard involution of* $W(A_2)$. We will call an involution $w \in W(A_2)$ a standard involution if there exists $A_1 \in \mathcal{A}_k$ such that (A_1, A_2) is standard, $g \in Z_{H^0}(A_1^d)$ such that $gA_1g^{-1} = A_2$ and $n = \delta(g)g^{-1}$ is a representative of w.

Remarks 3.6. (1). In the case that G has a Cartan involution as in [HW93, 11.8] we can replace $\delta \in \Gamma$ in the above discussion by a Cartan involution θ of G commuting with σ. The A_1-standard involution of $W(A_2)$ etc. can then be defined as above and similar results hold. We discuss this case in more detail in section 6.

(2) If Γ is cyclic of order n and δ is a generator of Γ, then the element $n = \delta(g)g^{-1}$ as in the above result is again contained in $N_{H^0}(A_2)$, but the corresponding Weyl group element no longer needs to be an involution, but has order $\leq n$. In this case we get similar results as in the case that $|\Gamma| = 2$, but the standard involutions are replaced by elements of order $\leq n$. A characterization of this case will be discussed in a forthcoming paper. We give a separate treatment of the case that $|\Gamma| = 2$ since the corresponding standard involutions can be described in much more detail and it also leads to a detailed description of the real case.

Definition 3.7. We will say that the pair (G, σ) is (σ, k)-*split conjugate* if for any σ-stable k-torus S all maximal σ-split k-tori of $Z_G(S)$ containing a maximal (σ, k)-split torus are $H_k \cap Z_G(S)$-conjugate. Similarly we will say that the pair (G, σ)

is (σ, k)-*anisotropic conjugate* if for any σ-stable k-torus S all maximal σ-split k-tori of $Z_G(S)$ containing a maximal σ-split k-anisotropic torus are $H_k \cap Z_G(S)$-conjugate.

In these two special cases we will be able to prove a number of additional results. We note that both these conditions are satisfied in the case that $k = \mathbb{R}$ and in the case of many $\mathfrak{p}$-adic fields with a splitting extension of order 2.

To show that all maximal σ-split k-tori of G can be put in a standard position we need the condition that all maximal (σ, k)-split of G are H_k-conjugate, i.e. (G, σ) is (σ, k)-split conjugate. In the following, fix an element $A_0 \in \mathcal{A}_k$ such that A_0^d is a maximal (σ, k)-split torus of G. Using similar arguments as in the case of maximal tori stable under a single involution as in [Hel91] we get now the following result:

Lemma 3.8. *Assume that (G, σ) is (σ, k)-split conjugate. Then every $A \in \mathcal{A}_k$ is H_k-conjugate to one which is standard with respect to A_0.*

Proof. Let $A \in \mathcal{A}_k$ and $S \supset A^d$ a maximal (σ, k)-split torus of G. Since (G, σ) is (σ, k)-split conjugate and A_0^d is also a maximal (σ, k)-split torus of G there exists $h \in H_k$ such that $hSh^{-1} = A_0^d$. But then $hA^dh^{-1} \subset A_0^d$, what proves the result. ☐

Remark 3.9. If $A \in \mathcal{A}_k$ such that (A, A_0) is standard and $w \in W(A_0^a)$ the A-standard involution, then $(A_0)_w^+ = A_0^a A^d$, $(A_0)_w^- = (A_0^d)_w^-$ and $A_0^d = A^d (A_0^d)_w^-$ with $A^d \cap (A_0^d)_w^-$ finite.

The following result will be useful in the sequel:

Lemma 3.10. *Assume that (G, σ) is (σ, k)-split conjugate and $A_1, A_2 \in \mathcal{A}_k$. Then we have the following:*

1. *If $A_1^a \supset A_2^a$, then there exists $x \in Z_{H_k}(A_2^a)$ such that (A_1, xA_2x^{-1}) is standard.*
2. *If in addition (G, σ) is (σ, k)-anisotropic conjugate, then A_1^d and A_2^d are H_k-conjugate if and only if A_1 and A_2 are H_k-conjugate.*

Proof. Let $S \subset Z_G(A_2^a)$ be a maximal (σ, k)-split torus of $Z_G(A_2^a)$ with $S \supset A_1^d$. Since A_2^d is a maximal (σ, k)-split torus of $Z_G(A_2^a)$ and as well and (G, σ) is (σ, k)-split conjugate it follows from Lemma 3.1 that there exists $x \in H_k \cap Z_G(A_2^a)$ such that $xA_2^dx^{-1} = S \supset A_1^d$. This proves (1).

(2). If A_1 and A_2 are conjugate under $h \in H_k$, then since A_1^d and A_2^d are the k-split parts of A_1 and A_2 we also have $hA_1^dh^{-1} = A_2^d$.

Conversely assume that $h \in H_k$ such that $hA_1^dh^{-1} = A_2^d$. Reducing to $Z_G(A_2^d)$ we may assume that $A_1 = A_1^a$ and $A_2 = A_2^a$. But then, since (G, σ) is (σ, k)-anisotropic conjugate it follows that A_1 and A_2 are H_k-conjugate. ☐

Theorem 3.11. *Assume that* (G, σ) *is* (σ, k)*-split conjugate and that* $A_1,\ A_2 \in \mathcal{A}_k$ *are standard with respect to* A_0*. Let* w_1 *and* w_2 *be the* A_1*-standard and* A_2*-standard involutions in* $W(A_0)$*. Then we have the following:*

(1) *If* A_1 *and* A_2 *are* H_k*-conjugate (resp.* H_k^0*-conjugate), then* w_1 *and* w_2 *are conjugate under* $W(A_0^d, H_k)$ *(resp.* $W(A_0^d, H_k^0)$*).*

(2) *If* (G, σ) *is also* (σ, k)*-anisotropic conjugate, then* A_1 *and* A_2 *are* H_k*-conjugate if and only if* w_1 *and* w_2 *are conjugate under* $W(A_0, H_k)$*.*

Proof. (1). Assume $h_1 \in H_k$ such that $h_1 A_1 h_1^{-1} = A_2$. Then A_0^d and $h_1 A_0^d h_1^{-1}$ are maximal (σ, k)-split tori of $Z_G(A_2^d)$. Since (G, σ) is (σ, k)-split conjugate there exists $h_2 \in Z_G(A_2^d) \cap H_k$ such that $h_2 h_1 A_0^d h_1^{-1} h_2^{-1} = A_0^d$. Let $h = h_2 h_1$. Then $h \in N_{H_k}(A_0^d)$ and $h(A_0^d)^-_{w_1} h^{-1} = (A_0^d)^-_{w_2}$, what proves the first statement.

(2). Assume first that A_1 and A_2 are H_k-conjugate. From (1) and Remark 3.9 it follows that there exists $h \in N_{H_k}(A_0^d)$ such that $h(A_0^d)^-_{w_1} h^{-1} = (A_0^d)^-_{w_2}$ and $hA_1^d h^{-1} = A_2^d$. Then A_0^a and $hA_0^a h^{-1}$ are maximal σ-split k-anisotropic tori of $Z_G(A_0^d)$. Since (G, σ) is (σ, k)-split conjugate it follows that there exists $h_0 \in Z_G(A_0^d) \cap H_k$ such that $h_0 h A_0^a h^{-1} h_0^{-1} = A_0^a$. Then $h_0 h \in N_{H_k}(A_0)$ and the corresponding Weyl group element in $W(A_0)$ maps w_1 to w_2.

Assume next that w_1 and w_2 are conjugate under $w \in W(A_0, H_k)$. Let $h \in N_{H_k}(A_0)$ be a representative of w. Then $h(A_0^d)^-_{w_1} h^{-1} = (A_0^d)^-_{w_2}$. Since A_0^d is the k-split part of A_0 and $h \in H_k$ we also have $h(A_0^d)h^{-1} = A_0^d$. From Remark 3.9 it follows now that $hA_1^d h^{-1} = A_2^d$. But then by Lemma 3.10 A_1 and A_2 are H_k-conjugate. ☐

4. $W(A_0^d, H_k)$ and $W(A_0^d, H_k)$-conjugacy classes

The Weyl groups $W(A_0^d, H_k)$ and $W(A_0^d, H_k)$ play an essential role in the above results. If (G, σ) is (σ, k)-split conjugate, then the conjugacy of the standard involutions under $W(A_0^d, H_k)$ in Theorem 3.11(1) can be extended to $W(A_0, H_k)$. However in many cases it is in fact easier to determine the $W(A_0^d, H_k)$-conjugacy classes. In this section we first characterize $W(A_0^d, H_k)$ and then look in more detail at the relation between these Weyl groups and the corresponding conjugacy classes.

4.1. If $w \in W(A_0, H_k)$ and $h \in N_{H_k}(A_0)$ a representative, then $hA_0^d h^{-1}$ is a k-split torus contained in A_0, so since A_0^d is the k-split part of A_0 it follows that $h \in N_{H_k}(A_0^d)$. This means that the restriction map induces a natural map $\phi : W(A_0, H_k) \to W(A_0^d, H_k)$. In general this map does not need to be surjective, but if (G, σ) is (σ, k)-split conjugate this map is surjective as follows from Proposition 4.4 below.

In the following we recall and prove a few more results about these Weyl groups and also their relation to $W(A_0, H)$. We first recall the following results from [Hel97]:

Lemma 4.2 ([Hel97, Corollary 6.6])**.** *Let A_0 be a maximal σ-split k-torus of G such that A_0^d is a maximal (σ, k)-split torus of G, $W_0(A_0) = \{w \in W(A_0) \mid w(a) = a,$ for all $a \in A_0^d\}$ and $W_1(A_0) = \{w \in W(A_0) \mid w(A_0^d) \subset A_0^d\}$. Then $W(A_0, H^0) = W_1(A_0)$ and $W(A_0^d) \simeq W_1(A_0)/W_0(A_0)$.*

Proposition 4.3 ([Hel97, Proposition 6.7])**.** *Let A_0 be a maximal σ-split k-torus of G such that A_0^d is a maximal (σ, k)-split torus of G and $w_1, w_2 \in W(A_0)$ involutions such that $(A_0)^-_{w_i} \subset A_0^d$. Then w_1 and w_2 are conjugate under $W(A_0)$ if and only if w_1 and w_2 are conjugate under $W_1(A_0) = \{w \in W(A_0) \mid w(A_0^d) \subset A_0^d\}$.*

This result shows that instead of looking at $W(A_0)$-conjugacy classes of standard involutions it suffices to look at $W(A_0^d)$-conjugacy classes of standard involutions. The Weyl group $W(A_0^d)$ is known from the classification of k-involutions in [Hel98b], while not much is known about $W(A_0)$. For the subgroups of these Weyl groups which have representatives in H_k we have a similar result in the case that (G, σ) is (σ, k)-anisotropic conjugate:

Proposition 4.4. *Let (G, σ) be (σ, k)-split conjugate, let A_0 be a maximal σ-split k-torus of G such that A_0^d is a maximal (σ, k)-split torus of G and $w_1, w_2 \in W(A_0)$ involutions such that $(A_0)^-_{w_i} \subset A_0^d$, $i = 1, 2$. Then w_1 and w_2 are conjugate under $W(A_0, H_k)$ if and only if w_1 and w_2 are conjugate under $W(A_0^d, H_k)$.*

Proof. Assume first that $w \in W(A_0, H_k)$ such that $ww_1w^{-1} = w_2$ and $h \in N_{H_k}(A_0)$ a representative of w. Then $hA_0^dh^{-1}$ is a k-split torus contained in A_0 and since A_0^d is the k-split part of A_0 it follows that $h \in N_{H_k}(A_0^d)$. So w_1 and w_2 are also conjugate under $W(A_0^d, H_k)$.

Conversely assume that $w \in W(A_0^d, H_k)$ such that $ww_1w^{-1} = w_2$ and $h \in N_{H_k}(A_0^d)$ a representative of w. Then $hA_0h^{-1} \supset A_0^d$ is a σ-split k-torus. Since (G, σ) is (σ, k)-split conjugate, there exists $h_1 \in H_k \cap Z_G(A_0^d)$ such that $h_1hA_0h^{-1}h_1^{-1} = A_0$. Since h_1h is also a representative of w the result follows. □

Remark 4.5. The Weyl group $W(A_0^d, H_k)$ is easier to determine than $W(A_0, H_k)$. For an example of this see 6.27. Ideally one would like $W(A_0^d, H_k)$ be identical to $W(A_0^d) = W(A_0, H)$, but unfortunately this is not the case. However in many cases, like almost standard pairs for real symmetric spaces (see [Hel88] and [HS97, Theorem 11.11]) these Weyl groups are the same. In the case that these Weyl groups are the same we have the following result:

Corollary 4.6. *Let (G,σ) be (σ,k)-split conjugate, let $A_0\in\mathcal{A}_k$ be a maximal σ-split k-torus of G with A_0^d maximal and assume $W(A_0^d)$ has representatives in H_k. Let A_1, $A_2\in\mathcal{A}_k$ be standard with respect to A_0 and w_1, w_2 the A_1-standard and A_2-standard involutions in $W(A_0)$. Then the following are equivalent.*

(1) *w_1 and w_2 are conjugate under $W(A_0,H_k)$.*
(2) *w_1 and w_2 are conjugate under $W(A_0^d)$.*

If in addition (G,σ) is (σ,k)-anisotropic conjugate, then the above is equivalent to:

(3) *A_1 and A_2 are H_k-conjugate.*

Proof. (1) ⇒ (2) is immediate from Proposition 4.4 and since $W(A_0^d)$ has representatives in H_k (2) ⇒ (1) follows from Proposition 4.4 as well.

Finally if (G,σ) is also (σ,k)-anisotropic conjugate, then by Theorem 3.11(2) statement (2) is equivalent to (3), what proves the result. ☐

4.7. Let $\mathcal{I}(A_0)$ denote the set of involutions in $W(A_0)$. Theorem 3.11 and Proposition 4.4 show that if (G,σ) is (σ,k)-split conjugate, then there exists a map $\varphi:\mathcal{A}_k/H_k\to\mathcal{I}(A_0)/W(A_0,H_k)$. If (G,σ) is also (σ,k)-anisotropic conjugate, then φ is in fact one-to-one. In the other cases we need a second invariant to characterize the H_k-conjugacy classes in $\mathcal{A}_k$. In section 5 we will characterize $\varphi(\mathcal{A}_k/H_k)\subset\mathcal{I}(A_0)/W(A_0,H_k)$.

5. k-singular involutions

Theorem 3.11 provides a sound criterion when elements in $\mathcal{A}_k$ are H_k-conjugate in the case that (G,σ) is (σ,k)-split conjugate. To complete the characterization of the H_k-conjugacy classes of $\mathcal{A}_k$, it reduces to single out those $w\in W(A_0)$ which are the A-standard involutions for some $A\in\mathcal{A}_k$. We will discuss a characterization of these involutions in this section.

Definition 5.1. A k-involution τ of a connected reductive k-group M is called *(τ,k)-split* if there exists a maximal τ-split torus of G which is k-split. Similarly τ is called *(τ,k)-anisotropic* if there exists a maximal τ-split torus of G which is k-anisotropic.

5.2. Let $A\in\mathcal{A}_k$ and $w\in W(A)$ satisfying $w^2=e$ and $w\sigma=\sigma w$. Set $G_w=Z_G(A_w^+)$. Let n be a preimage of w in $N_G(A)$. Then $n\in Z_G(A_w^+)$ and $A_w^-\cap Z(G_w)$ is finite. As a consequence, A_w^- is a maximal σ-split k-torus of $[G_w,G_w]$.

Definition 5.3. Let $A\in\mathcal{A}_k$ and $w\in W(A)$. Then w is *k-singular* if

(1) $w^2=e$
(2) $\sigma w=w\sigma$
(3) the involution $\sigma|[G_w,G_w]$ is (σ,k)-split and (σ,k)-anisotropic.

A root $\alpha \in \Phi(A)$ is called k-singular if the corresponding reflection $s_\alpha \in W(A)$ is k-singular.

Lemma 5.4. *If A is a maximal σ-split k-torus of G such that A^d is a maximal (σ, k)-split torus of G and $w \in W(A)$ a k-singular involution, then $A_w^- \subset A^d$.*

Proof. Assume $w \in W(A)$ is k-singular and $S = A_w^-$. Then S is a maximal σ-split k-torus of $[G_w, G_w]$. Since A^d is maximal (σ, k)-split it follows that $S = S^d$ is (σ, k)-split. For if not, then since w is k-singular the group $[G_w, G_w]$ would contain a (σ, k)-split torus $A_1 \supsetneq S^d$ and then $A_1 A^d \supsetneq A^d$ is a larger (σ, k)-split torus, what contradicts the maximality of A^d. ☐

We have the following characterization of the H_k-conjugacy classes of maximal σ-split k-tori of G.

Proposition 5.5. *Let A_0 be a maximal σ-split k-torus of G such that A_0^d is a maximal (σ, k)-split torus of G. Then the standard involutions of $W(A_0)$ are precisely the k-singular involutions of $W(A_0)$.*

Proof. If $w \in W(A_0)$ is a standard involution, then there exists $A_1 \in \mathcal{A}_k$ such that (A_1, A_0) is standard and a $g \in Z_{H^0}(A_1^d)$ such that $gA_1g^{-1} = A_0$. Then $n = \delta(g)g^{-1} \in N_{H^0}(A_0)$ is a representative of w. From Lemma 3.4 it follows that $\sigma|[G_w, G_w]$ is both (σ, k)-split and (σ, k)-anisotropic, hence w is k-singular.

Conversely assume $w \in W(A_0)$ is k-singular. By Lemma 5.4 we have $(A_0)_w^- \subset A_0^d$. Let $S \subset Z_G((A_0)_w^+)$ be a maximal σ-split k-torus of $Z_G((A_0)_w^+)$ with S^a maximal. Clearly (S, A_0) is standard. From Proposition 2.6 it follows that there exists $h \in Z_G((A_0)_w^+) \cap H$ such that $hSh^{-1} = A_0$. Then $n = \delta(g)g^{-1} \in N_{H^0}(A_0)$ is a representative of w, what proves the result. ☐

Corollary 5.6. *Assume that (G, σ) is (σ, k)-split conjugate and (σ, k)-anisotropic conjugate. Let A_0 be a maximal σ-split k-torus of G such that A_0^d is a maximal (σ, k)-split torus of G. Then there is a one to one correspondence between the H_k-conjugacy classes of $\mathcal{A}_k$ and the $W(A_0, H_k)$-conjugacy classes of k-singular involutions of $W(A_0)$.*

Proof. This result is immediate from Proposition 5.5 and Theorem 3.11. ☐

Proposition 5.7. *For $A \in \mathcal{A}_k$ and a k-singular involution w of $W(A)$, there exist orthogonal k-singular roots $\alpha_1, \dots, \alpha_\ell$ of A such that $w = s_{\alpha_1} \dots s_{\alpha_\ell}$.*

Proof. This result follows using similar arguments as in [HW93, 12.16]. The details are left to the reader. ☐

6. Real reductive symmetric spaces

In the case that $k = \mathbb{R}$ we can improve on the above characterization of the H_k-conjugacy classes of Cartan subspaces by using the fact that one has a Cartan involution as well. In this section we discuss this refinement and also discuss the relation of these results with those following from the dual pair correspondence. We note that all the arguments in this section also hold in the more general setting of "Groups with a Cartan involution", which were introduced in [HW93, §11] (see also [HH98]). This includes for example groups defined over a real closed field, i.e. k is formally real, but has no formally real proper algebraic extension field (see [Pre84, 3.2]). We leave it to the reader to verify that all the results and arguments also hold in this more general setting.

6.1. In the case that $k = \mathbb{R}$ the real points of G are given by a conjugation ρ of G. Then $G_\rho = \{x \in G \mid \rho(x) = x\}$ is a real form of G. By choosing a maximal compact subgroup of G_ρ we get a Cartan involution θ of G_ρ which lifts to an involution of G by extending the base field from $\mathbb{R}$ to $\mathbb{C}$. It is well known that one can choose this maximal compact subgroup to be invariant under σ, i.e. the Cartan involution θ commutes with σ (see [Hel88, 10.3] or [Ber57]). One can also construct this Cartan involution by choosing a compact real form of G which is both ρ and σ invariant (see [Hel88, 10.3]). If τ is the conjugation of this compact real form, then $\theta = \tau\rho$ is a Cartan involution of G_ρ and G, which commutes with σ.

For the remainder of this paper we assume that $k = \mathbb{R}$, ρ is a conjugation of G commuting with σ such that $G_{\mathbb{R}} = G_\rho$, τ is a conjugation of a compact real form G which is both ρ and σ invariant and $\theta = \tau\rho$ is the Cartan involution of G commuting with σ. We will write $K = G_\theta$ and let H denote a open subgroup of G_σ such that $G_\sigma(\mathbb{R}) \subset H_{\mathbb{R}} \subset G_\sigma(\mathbb{R})$. We will assume that $H_{\mathbb{R}}/(H_{\mathbb{R}} \cap Z(G_{\mathbb{R}}))$ is noncompact. Let $H_{\mathbb{R}}^+ = H_{\mathbb{R}} \cap G_\theta(\mathbb{R})$. It is well known that $(H_{\mathbb{R}}^+)^0 = H_{\mathbb{R}}^0 \cap G_\theta(\mathbb{R})$ (see for example [HS97]).

6.2. Let $\mathcal{A}(G)$ denote the set of maximal σ-split tori of G, $\mathcal{A}_{\mathbb{R}}(G)$ the set of maximal σ-split $\mathbb{R}$-tori of G, $\mathcal{A}^\theta(G)$ the set of θ-stable maximal σ-split tori of G and $\mathcal{A}_{\mathbb{R}}^\theta(G) \subset \mathcal{A}^\theta(G)$ the set of θ-stable maximal σ-split $\mathbb{R}$-tori of G. We will say that a torus A of G is a maximal σ-split $\mathbb{R}$-torus of G if its complexification in G is a σ-split $\mathbb{R}$-torus of G and we will denote these by $\mathcal{A}_{\mathbb{R}}(G)$. The sets $\mathcal{A}_{\mathbb{R}}(G)$ and $\mathcal{A}_{\mathbb{R}}(G)$ are isomorphic and the same for $\mathcal{A}_{\mathbb{R}}^\theta(G)$ and $\mathcal{A}_{\mathbb{R}}^\theta(G)$. We will use the notation $\mathcal{A}_{\mathbb{R}}$ for both $\mathcal{A}_{\mathbb{R}}(G)$ and $\mathcal{A}_{\mathbb{R}}(G)$. Similarly we will write $\mathcal{A}_{\mathbb{R}}^\theta$ for both $\mathcal{A}_{\mathbb{R}}^\theta(G)$ and $\mathcal{A}_{\mathbb{R}}^\theta(G)$.

If $A \in \mathcal{A}_{\mathbb{R}}$ is a maximal σ-split $\mathbb{R}$-torus of G, then the Lie algebra $\mathfrak{a}$ of A is also called a *Cartan subspace* of $\mathfrak{g}$. By abuse of notation we will also call the maximal σ-split $\mathbb{R}$-torus of G a Cartan subspace of G.

The study of the $H_{\mathbb{R}}$-conjugacy classes of maximal σ-split $\mathbb{R}$-tori of G can be reduced to $(H_{\mathbb{R}}^{+})^0$-conjugacy classes of θ-stable maximal σ-split $\mathbb{R}$-tori as follows:

Proposition 6.3. *Every Cartan subspace A of G is $H_{\mathbb{R}}^0$-conjugate to a θ-stable Cartan subspace.*

Proof. Let A be a Cartan subspace of G. Since A is a maximal σ-split $\mathbb{R}$-torus of G we can write, similarly as in 2.1 $A = A^a A^d$, where A^a is the anisotropic part of A and A^d is the $\mathbb{R}$-split part of A. Then A^d is a $(\sigma, \mathbb{R})$-split torus of G. Let $S \supset A^d$ be a maximal $(\sigma, \mathbb{R})$-split torus of G. By [HW93, 11.20] there exists $h \in H_{\mathbb{R}}^0$ such that hSh^{-1} is θ-split. But then $hA^d h^{-1}$ is θ-split as well. Reducing to $[Z_G(hA^d h^{-1}), Z_G(hA^d h^{-1})]$ we may assume that $A^d = \{e\}$. Then $A = A^a$ is a σ-split anisotropic $\mathbb{R}$-torus of G. Let $S_1 \subset Z_G(A)$ be a maximal $\mathbb{R}$-split torus. Since A is maximal σ-split it follows from [Vus74] that $S_1 \subset H \cap Z_G(A)$. Since $\theta | H$ is a Cartan involution of H (see [HW93, 11.9]) it follows from [HW93, 11.18] that there exists $h \in H_{\mathbb{R}}^0 \cap Z_G(A)$ such that hS_1h^{-1} is $(\theta, \mathbb{R})$-split. Reducing to $Z_G(hS_1h^{-1})$ we may assume that $Z_G(A)$ contains no $\mathbb{R}$-split tori. Let $T \supset A$ be a σ-stable maximal k-torus. Since $Z_G(A)$ contains no $\mathbb{R}$-split tori it follows that $T = T^a$ is $\mathbb{R}$-anisotropic. By [Hel78, Ch. III, Theorem 7.1] (see also [Hel88, 10.3]) there exists a compact real form of G with conjugation τ' such that $\tau'(T) = T$ and τ' commutes with δ and σ. Then $\theta' = \tau' \delta$ is a Cartan involution of G commuting with σ and $T \subset G_{\theta'}$. By [HW93, 11.17] there exists $x \in G_{\mathbb{R}}$ such that $\operatorname{Int}(x) \theta' \operatorname{Int}(x)^{-1} = \theta$. Since σ commutes with θ and θ' it follows that we can take $x \in H_{\mathbb{R}}^0$. But then $xAx^{-1} \subset G_\theta$ is θ-stable, what proves the result. □

This result can also be derived from [OM80, Lemma 5], [Mat79, Lemma 3] and [Loo69, Theorem 2.1]. However the above proof is relatively short and easily generalizes to groups with a Cartan involution as well.

The conjugacy under $G_{\mathbb{R}}$ can be restricted to $K_{\mathbb{R}}$:

Lemma 6.4 ([HHNÓ98, Lemma 7]). *Let G_1 be a θ-invariant $\mathbb{R}$-subgroup of G and denote the Lie algebra of G_1 by $\mathfrak{g}_1$. Assume that $A_1, A_2 \in \mathcal{A}_{\mathbb{R}}^{\theta}$ and $g \in G_1(\mathbb{R})$ such that $gA_1g^{-1} = A_2$. If $g = k \exp X$ with $k \in K_{\mathbb{R}} \cap G_1(\mathbb{R})$ and $X \in (\mathfrak{p} \cap \mathfrak{g}_1)_{\mathbb{R}}$ is the Cartan decomposition of g, then $kA_1k^{-1} = A_2$.*

Applying this result to $G_1 = H$ it follows now that we can restrict to $H_{\mathbb{R}}^{+}$-conjugacy:

Corollary 6.5. *Assume that $A_1, A_2 \in \mathcal{A}_{\mathbb{R}}^{\theta}$. Then A_1 and A_2 are $H_{\mathbb{R}}$-conjugate if and only if they are $H_{\mathbb{R}}^{+}$-conjugate.*

6.6. The notions of standard pair and standard involution can be slightly refined for the $H_{\mathbb{R}}^{+}$-conjugacy classes in $\mathcal{A}_{\mathbb{R}}^{\theta}$. In particular we can replace the role of $\delta \in$

Γ by the Cartan involution θ. The main change is that what we called the $+1$-eigenspaces for δ are $+1$-eigenspaces for θ and conversely. So in particular if A is a θ-stable maximal σ-split $\mathbb{R}$-torus of G, then $A^-_\theta = A^-_\delta = A^d$ and $A^+_\theta = A^+_\delta = A^a$. Since in the remainder we will mainly consider eigenspaces for θ (instead of δ) we will write A^+ for A^+_θ and A^- for A^-_θ. In the following we show how the results in the previous two sections can be modified in this setting of the Cartan involution and θ-stable maximal σ-split $\mathbb{R}$-tori of G. Essential in this analyzes will be the following result:

Lemma 6.7. *Let G, σ and θ be as above. Then we have the following:*

1. *All θ-stable maximal σ-split $\mathbb{R}$-tori A of G with A^-_θ maximal are $(H^+_{\mathbb{R}})^0$-conjugate.*
2. *All θ-stable maximal σ-split $\mathbb{R}$-tori A of G with A^+_θ maximal are $(H^+_{\mathbb{R}})^0$-conjugate.*

Proof. If A is a θ-stable maximal σ-split $\mathbb{R}$-torus of G with A^-_θ maximal, then A^-_θ is a maximal $(\sigma, \mathbb{R})$-split torus of G, which by [HW93, 11.19] are all conjugate under $(H^+_{\mathbb{R}})^0$. So it suffices to show that all θ-stable maximal σ-split $\mathbb{R}$-tori S of G containing A^-_θ are $(H^+_{\mathbb{R}})^0$-conjugate. Passing to $Z_G(A^-_\theta)$ we may assume that $A \subset K$. Switching to the real form given by the conjugation $\rho\sigma$, then $\sigma|K$ is a Cartan involution of $G_{\rho\sigma} \cap K$ and any maximal σ-split $\mathbb{R}$-torus of G is a maximal $(\sigma, \mathbb{R})$-split torus of $G_{\rho\sigma} \cap K$. By [HW93, 11.19] are all these are conjugate under $(K \cap H)^0_R = (H^+_{\mathbb{R}})^0$, what proves the first statement.

Since all θ-stable maximal σ-split $\mathbb{R}$-tori A of K with A^+_θ maximal are maximal $(\sigma\theta, \mathbb{R})$-split tori of G the second statement follows from the first applied to the involution $\sigma\theta$ of G. ☐

6.8. In the following we generalize the notions of standard pair and standard involution to include the the Cartan involution θ.

Definition 6.9. For $A_1, A_2 \in \mathcal{A}^\theta_{\mathbb{R}}$, the pair (A_1, A_2) is called *standard* if $A^-_1 \subset A^-_2$ and $A^+_1 \supset A^+_2$. In this case, we also say that A_1 is *standard* with respect to A_2.

The θ-stable maximal σ-split $\mathbb{R}$-tori of G can be put in a standard position.

Lemma 6.10. *Let $A_1, A_2 \in \mathcal{A}^\theta_{\mathbb{R}}$ such that $A^+_1 \supset A^+_2$ (resp. $A^-_1 \subset A^-_2$). Then there exists $x \in Z_{H^+_{\mathbb{R}}}(A^+_2)$ (resp. $Z_{H^+_{\mathbb{R}}}(A^-_1)$) such that (A_1, xA_2x^{-1}) is standard. In particular if A^+_1 and A^+_2 (resp. A^-_1 and A^-_2) are $H^+_{\mathbb{R}}$-conjugate, so are A_1 and A_2.*

Proof. This result is immediate from Lemma 6.7 using an argument similar as in Lemma 3.10. ☐

A standard pair (A_1, A_2) of θ-stable maximal σ-split $\mathbb{R}$-tori of G gives rise to an involution in $W(A_1)$ (resp. $W(A_2)$).

Lemma 6.11. *Let (A_1, A_2) be a standard pair of θ-stable maximal σ-split $\mathbb{R}$-tori of G. Then we have the following conditions:*

(1) *There exists $g \in Z_{H^0_{\mathbb{C}}}(A_1^- A_2^+)$ such that $gA_1g^{-1} = A_2$.*

(2) *If $n_1 = \theta(g)^{-1}g$ and $n_2 = \theta(g)g^{-1}$, then $n_1 \in N_{H^0_{\mathbb{C}}}(A_1)$ and $n_2 \in N_{H^0_{\mathbb{C}}}(A_2)$.*

(3) *Let w_1 and w_2 be the images of n_1 and n_2 in $W(A_1)$ and $W(A_2)$ respectively. Then $w_1^2 = e$, $w_2^2 = e$, and $(A_1)^+_{w_1} = (A_2)^+_{w_2} = A_1^- A_2^+$ which characterizes w_1 and w_2.*

Proof. This result follows with a similar argument as in Lemma 3.4. □

Remark. By (3) of Lemma 6.11, w_1 and w_2 are independent of our choice of $g \in Z_{H^0_{\mathbb{C}}}(A_1^- A_1^+)$ with $gA_1g^{-1} = A_2$.

Definition 6.12. Let A_1, A_2, $w_1 \in W(A_1)$ and $w_2 \in W(A_2)$ be as in Lemma 6.11. We call w_1 (resp. w_2) the A_2-standard involution (resp. A_1-standard involution) of $W(A_1)$ (resp. $W(A_2)$).

In the following, fix an element $A_0 \in \mathcal{A}^\theta_{\mathbb{R}}$ such that A_0^- is a maximal $(\sigma, \mathbb{R})$-split torus of G. Similarly we choose $S \in \mathcal{A}^\theta_{\mathbb{R}}$ standard with respect to A_0 such that S^+ is a maximal σ-split torus of K.

Lemma 6.13. *Every $A \in \mathcal{A}^\theta_{\mathbb{R}}$ is $H^+_{\mathbb{R}}$-conjugate to one which is standard with respect to both A_0 and S.*

Proof. Let $A \in \mathcal{A}^\theta_{\mathbb{R}}$. By embedding A^- in a maximal (σ, θ)-split $\mathbb{R}$-torus of G and using Lemma 6.7 we may assume that $A^- \subset A_0^-$. But then by Lemma 6.10 we may assume that A is standard with respect to A_0. Let $A' \supset A^+$ be a maximal σ-split $\mathbb{R}$-torus of K. Then both S^+ and A' are maximal σ-split $\mathbb{R}$-tori of $K \cap Z_G(A_0^+)$. By Lemma 6.7 they are conjugate under $H^+_{\mathbb{R}} \cap Z_G(A_0^+)$. So we may assume that $A^+ \subset S$. Passing to $Z_G(A^+)$ we may assume that $A = A^-$ is $(\sigma, \mathbb{R})$-split. Again by Lemma 6.7 it follows that A is conjugate under $H^+_{\mathbb{R}} \cap Z_G(A^+)$ with A_0, what proves the result. □

Theorem 6.14. *Assume that $A_1, A_2 \in \mathcal{A}^\theta_{\mathbb{R}}$ such that they are standard with respect to A_0 (resp. S). Let w_1 and w_2 be the A_1-standard and A_2-standard involutions in $W(A_0)$ (resp. $W(S)$) respectively. Then A_1 and A_2 are $H^+_{\mathbb{R}}$-conjugate if and only if w_1 and w_2 are conjugate under $W(A_0, H^+_{\mathbb{R}})$ (resp. $W(S, H^+_{\mathbb{R}})$).*

Proof. We prove the result for A_1, A_2 standard with respect to A_0. The result for S follows with a similar argument.

Assume first that $h \in H^+_{\mathbb{R}}$ such that $hA_1h^{-1} = A_2$. Then A_0^- and $hA_0^-h^{-1}$ are maximal (σ, θ)-split tori of $Z_G(A_2^-)$. By Lemma 6.7 they are conjugate under $Z_G(A_2^-) \cap H^+_{\mathbb{R}}$. So we may assume that $h \in N_{H^+_{\mathbb{R}}}(A_0^-)$. But then A_0 and hA_0h^{-1}

are θ-stable maximal σ-split $\mathbb{R}$-tori of $Z_G(A_0^-)$ with A_0^+ and $hA_0^+h^{-1}$ maximal. By Lemma 6.7 they are conjugate under $Z_G(A_0^-) \cap H_{\mathbb{R}}^+$, hence we may assume that $h \in N_{H_{\mathbb{R}}^+}(A_0)$. Since $h(A_0)_{w_1}^- h^{-1} = (A_0)_{w_2}^-$ it follows that w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}}^+)$.

The converse follows with a similar argument as in Theorem 3.11(2). □

Combined with Theorem 3.11 we get now the following result:

Corollary 6.15. *Assume that $A_1, A_2 \in \mathcal{A}_{\mathbb{R}}^\theta$ such that they are standard with respect to A_0. Let w_1 and w_2 be the A_1-standard and A_2-standard involutions in $W(A_0)$ respectively. Then the following are equivalent:*

(1) *A_1 and A_2 are $H_{\mathbb{R}}$-conjugate.*
(2) *A_1 and A_2 are $H_{\mathbb{R}}^+$-conjugate.*
(3) *w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}})$.*
(4) *w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}}^+)$.*

In [HHNÓ98, Theorem 10] it was shown that two Cartan subspaces of G are $G_{\mathbb{R}}$-conjugate if and only if they are $H_{\mathbb{R}}^0$-conjugate. This leads to the following refinement of the above result:

Corollary 6.16. *Assume that $A_1, A_2 \in \mathcal{A}_{\mathbb{R}}^\theta$ such that they are standard with respect to A_0. Let w_1 and w_2 be the A_1-standard and A_2-standard involutions in $W(A_0)$ respectively. Then A_1 and A_2 are $G_{\mathbb{R}}$-conjugate if and only if w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}}^+)$.*

The above result provides a sound criterion when elements in $\mathcal{A}_{\mathbb{R}}^\theta$ are $H_{\mathbb{R}}$-conjugate. To complete the characterization of the $H_{\mathbb{R}}$-conjugacy classes of $\mathcal{A}_{\mathbb{R}}^\theta$, it reduces to single out those $w \in W(A_0)$ which are the A-standard involutions for some $A \in \mathcal{A}_{\mathbb{R}}^\theta$.

Definition 6.17. A $\mathbb{R}$-involution ϕ of a connected reductive $\mathbb{R}$-group $M_{\mathbb{C}}$ is called $(\sigma, \mathbb{R})$-split if there exists a ϕ-split maximal σ-split $\mathbb{R}$-torus of $M_{\mathbb{C}}$.

6.18. Let $A \in \mathcal{A}_{\mathbb{R}}^\theta$ and $w \in W(A)$ satisfying $w^2 = e$ and $w\sigma = \sigma w$. Set $G_w = Z_{G_{\mathbb{C}}}(A_w^+)$. Let n be a preimage of w in $N_{G_{\mathbb{C}}}(A)$. Then $n \in Z_{G_{\mathbb{C}}}(A_w^+)$ and $A_w^- \cap Z(G_w)$ is finite. As a consequence, A_w^- is a (σ, θ)-stable maximal σ-split $\mathbb{R}$-torus of $[G_w, G_w]$.

Definition 6.19. Let $A \in \mathcal{A}_{\mathbb{R}}^\theta$ and $w \in W(A)$. Then w is θ-*singular* if

(1) $w^2 = e$
(2) $\sigma w = w\sigma$
(3) the involutions $\theta \,|\, [G_w, G_w]$ and $\sigma\theta \,|\, [G_w, G_w]$ are $(\sigma, \mathbb{R})$-split.

A root $\alpha \in \Phi(A)$ is called θ-singular if the corresponding reflection $s_\alpha \in W(A)$ is θ-singular.

We have the following characterization of the $H_{\mathbb{R}}^{+}$-conjugacy classes of θ-stable maximal σ-split $\mathbb{R}$-tori of G.

Proposition 6.20. *Let A_0 be a θ-stable maximal σ-split $\mathbb{R}$-torus of G with maximal A_0^-. Then there is a one to one correspondence between the $H_{\mathbb{R}}^{+}$-conjugacy classes of $\mathcal{A}_{\mathbb{R}}^{\theta}$ and the $W(A_0, H_{\mathbb{R}}^{+})$-conjugacy classes of θ-singular involutions of $W(A_0)$.*

Proof. This result follows from Theorem 6.14 and Corollary 5.6. ☐

With a similar argument as in Proposition 5.7 we have the following characterization for the θ-singular involutions:

Proposition 6.21. *For $A \in \mathcal{A}_{\mathbb{R}}^{\theta}$ and a θ-singular involution w of $W(A)$, there exist orthogonal θ-singular roots $\alpha_1, \dots, \alpha_\ell$ of A such that $w = s_{\alpha_1} \dots s_{\alpha_\ell}$.*

Remark 6.22. All θ-singular involutions of $W(A_0)$ are k-singular in the sense of 5.3. The condition for θ-singular is somewhat easier to check from the classification of commuting involutions in [Hel88].

The Weyl groups $W(A_0, H_{\mathbb{R}})$ and $W(A_0^-, H_{\mathbb{R}})$ have representatives in $H_{\mathbb{R}}^{+}$:

Lemma 6.23. *Let let A_0 be a θ-stable maximal σ-split $\mathbb{R}$-torus of G such that A_0^- is a maximal $(\sigma, \mathbb{R})$-split torus of G. Then $W(A_0, H_{\mathbb{R}}) = W(A_0, H_{\mathbb{R}}^{+})$ and $W(A_0^-, H_{\mathbb{R}}) = W(A_0^-, H_{\mathbb{R}}^{+})$.*

Proof. We prove the result for $W(A_0, H_{\mathbb{R}})$. The proof for $W(A_0^-, H_{\mathbb{R}})$ follows with a similar argument.

Let $w \in W(A_0, H_{\mathbb{R}})$ and $h \in N_{H_{\mathbb{R}}}(A_0)$ a representative of w. Since $\theta | H_{\mathbb{R}}$ is also a Cartan involution of $H_{\mathbb{R}}$ we can write $h = h_1 \exp(X)$, where $h \in H_{\mathbb{R}}^{+}$ and $X \in \mathfrak{h} \cap \mathfrak{p}$, where $\mathfrak{h}$ is the Lie algebra of $H_{\mathbb{R}}$ and $\mathfrak{p} = \{Y \in \mathfrak{g} \mid \theta(Y) = -Y\}$ the Lie algebra of $\tau_\theta(G_{\mathbb{R}})$ with τ_θ as in (2.2.1). With the same argument as in [HHNÓ98, Lemma 7] it follows that $\exp(X)$ commutes with A_0, hence $h_1 \in N_{H_{\mathbb{R}}^{+}}(A_0)$ is a representative of w as well. ☐

Combining this result with Proposition 4.4 it follows that the conjugacy of the θ-singular involutions of $W(A_0)$ can be restricted to $W(A_0^-, H_{\mathbb{R}}^{+})$:

Proposition 6.24. *Let let A_0 be a θ-stable maximal σ-split $\mathbb{R}$-torus of G such that A_0^- is a maximal $(\sigma, \mathbb{R})$-split torus of G and $w_1, w_2 \in W(A_0)$ involutions such that $(A_0)_{w_i}^- \subset A_0^-$, $i = 1, 2$. Then the following are equivalent:*

1. *w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}})$.*
2. *w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}}^{+})$.*
3. *w_1 and w_2 are conjugate under $W(A_0^-, H_{\mathbb{R}})$.*
4. *w_1 and w_2 are conjugate under $W(A_0^-, H_{\mathbb{R}}^{+})$.*

The Weyl group $W(A_0^-)$ is known from the classification of $\mathbb{R}$-involutions in [Hel98b]. Unfortunately for general H the Weyl group $W(A_0^-, H_{\mathbb{R}}^+)$ is difficult to determine. However the subgroup $W(A_0^-, (H_{\mathbb{R}}^+)^0)$ is completely known, since it is the Weyl group of A_0^- in $G_{\sigma\theta}$:

Lemma 6.25 ([HS97, 4.4])**.** $W(A_0^-, (H_{\mathbb{R}}^+)^0) = W(A_0^-, G_{\sigma\theta})$.

Combined with Corollary 6.16 it follows that the $W(A_0, H_{\mathbb{R}}^+)$-conjugacy classes and $W(A_0, (H_{\mathbb{R}}^+)^0)$-conjugacy classes are the same:

Proposition 6.26. *There is a one to one correspondence between the $W(A_0, H_{\mathbb{R}}^+)$-conjugacy classes of θ-singular involutions and the $W(A_0, (H_{\mathbb{R}}^+)^0)$-conjugacy classes of θ-singular involutions.*

Remark 6.27. The Weyl group $W(A_0^-, (H_{\mathbb{R}}^+)^0) = W(A_0^-, G_{\sigma\theta})$ easily follows from the classification of commuting involutions in [Hel88], see also [Hel98a]. So the above results give a simple algorithm to compute the $W(A_0, (H_{\mathbb{R}}^+)^0)$-conjugacy classes of θ-singular involutions. The conjugacy classes of θ-singular involutions also follows from the classification of maximal $\mathbb{R}$-split tori in [Hel98a] using the dual pair correspondence.

6.28. For real groups the Weyl group $W(A_0^-)$ has representatives in H. So combined with Corollary 4.6 it follows that in the case that the Weyl group $W(A_0^-)$ has representatives in $H_{\mathbb{R}}$ we can reduce to conjugacy classes of involutions under the full Weyl group:

Proposition 6.29. *Let $A_0 \in \mathcal{A}_{\mathbb{R}}^{\theta}$ be a θ-stable maximal σ-split $\mathbb{R}$-torus of G with A_0^- maximal and assume $W(A_0^-)$ has representatives in $H_{\mathbb{R}}$. Let $w_1, w_2 \in W(A_0)$ involutions such that $(A_0)_{w_i}^- \subset A_0^-$, $i = 1, 2$. Then the following are equivalent.*

1. *w_1 and w_2 are conjugate under $W(A_0, H_{\mathbb{R}}^+)$.*
2. *w_1 and w_2 are conjugate under $W(A_0)$.*
3. *w_1 and w_2 are conjugate under $W(A_0^-) = W(A_0^-, H_{\mathbb{R}})$.*

Proof. This result is immediate from Proposition 6.24, Corollary 4.6 and Proposition 4.3. ☐

6.30. A remaining question is for which pairs $(G_{\mathbb{R}}, H_{\mathbb{R}})$ has the Weyl group $W(A_0^-)$ representatives in $H_{\mathbb{R}}$? In [HS97] the pairs $(G_{\mathbb{R}}, H_{\mathbb{R}})$ with this property were called (σ, θ)-split. Fortunately most pairs $(G_{\mathbb{R}}, H_{\mathbb{R}})$ are (σ, θ)-split, however for general H it is difficult to determine if a pair is (σ, θ)-split or not. On the other hand the subset of pairs $(G_{\mathbb{R}}, H_{\mathbb{R}})$ for which $W(A_0^-)$ has representatives in $(H_{\mathbb{R}}^+)^0$ can easily be determined from the classification of pairs of involutions in [Hel88], since it is the Weyl group of the root system $\Phi(A_0^-, (G^{\sigma\theta})^0)$. Pairs $(G_{\mathbb{R}}, H_{\mathbb{R}})$ with this property were called *weakly-standard* in [HS97]. In this case we will also say that the pair of involutions (σ, θ) is weakly-standard. We note that the Weyl group $W(A_0^-, (H_{\mathbb{R}}^+)^0) = W(A_0^-, (H \cap K)_{\mathbb{R}}^0)$ and $W(A_0^-, (H \cap K)^0)$ are the same (see [HS97, 11.11]).

Clearly weakly standard pairs are split, but these notions are different as can be seen from [HS97, Example 7.6].

6.31. The weakly standard pairs are completely determined by the signatures which are given in [Hel88]. In the following we briefly review the characterization by these signatures, which was given in [HS97].

Let A_0 be a θ-stable maximal σ-split $\mathbb{R}$-torus, such that A_0^- is a maximal $(\sigma, \mathbb{R})$-split torus and $T \supset A_0$ a maximal $\mathbb{R}$-torus, such that T^d is a maximal $\mathbb{R}$-split torus of G. Let $\mathfrak{g}(A_0^-, \lambda)$ denote the root space corresponding to $\lambda \in \Phi(A_0^-)$. Since $\sigma(\lambda) = \theta(\lambda) = -\lambda$, $\sigma\theta$ stabilizes $\mathfrak{g}(A_0^-, \lambda)$. Set

$$\mathfrak{g}(A_0^-, \lambda)_{\pm}^{\sigma\theta} = \{X \in \mathfrak{g}(A_0^-, \lambda) \mid \sigma\theta(X) = \pm X\}$$
$$m^{\pm}(\lambda, \sigma\theta) = \dim \mathfrak{g}(A_0^-, \lambda)_{\pm}^{\sigma\theta}$$

For $\lambda \in \Phi(A_0^-)$ call $(m^+(\lambda, \sigma\theta), m^-(\lambda, \sigma\theta))$ the *signature* of λ.

Proposition 6.32 ([HS97, 9.13]). *Let A_0, (σ, θ) etc. be as above and let Δ be a basis of $\Phi(A_0^-)$. Then the following are equivalent:*

(1) *(σ, θ) is weakly-standard (i.e. $W_{(H\cap K)_{\mathbb{R}}^0}(A_0^-) = W_{(H\cap K)^0}(A_0^-) = W(A_0^-)$).*
(2) *$m^+(\lambda, \sigma\theta) \neq 0$ or $m^+(2\lambda, \sigma\theta) \neq 0$ for all $\lambda \in \Delta$.*

Remark 6.33. The signatures of the roots λ and 2λ with $\lambda \in \Delta$ for all pairs of commuting involutions are given in [Hel88]. From this classification it follows that more than $2/3$ of the pairs of commuting involutions are in fact weakly standard. In these cases the conjugacy of the θ-singular involutions is under the action of the full Weyl group. The conjugacy classes of involutions in Weyl groups were classified in [Hel91].

6.34. **Dual Pair correspondence.** We conclude this paper by showing how the results in this section relate to those one obtains through the dual pair correspondence. The dual pair of $(G_{\mathbb{R}}, H_{\mathbb{R}})$ can be obtained as follows. Assume first that $H_{\mathbb{R}} = G_\sigma(\mathbb{R})$. The real form $G_{\mathbb{R}}$ corresponds to the conjugation $\rho = \tau\theta$. The involution σ defines another real form via the conjugation $\tau\sigma$. Then $\sigma|G_{\tau\sigma}$ is a Cartan involution of $G_{\tau\sigma}$. The dual pair of $(G_\rho, H_{\mathbb{R}})$ is the pair $(G_{\tau\sigma}, K \cap G_{\tau\sigma})$. Using the Cartan decomposition of $H_{\mathbb{R}}$ one can define a dual pair also in the case that $H_{\mathbb{R}}$ is a open and closed subgroup of $G_\sigma(\mathbb{R})$. Now if A is a θ-stable maximal σ-split $\mathbb{R}$-torus of G with respect to the real form G_ρ, then A is a θ-stable maximal $\mathbb{R}$-split (and σ-split) torus of G with respect to the real form G_ρ. So we have the following result:

Proposition 6.35. *There exists a one to one correspondence between the $H_{\mathbb{R}}^{+}$-conjugation classes of the θ-stable maximal σ-split $\mathbb{R}$-tori of G_ρ and the $H_{\mathbb{R}}^{+}$-conjugation classes of the θ-stable maximal $\mathbb{R}$-split (and σ-split) tori of $G_{\tau\sigma}$.*

Remark 6.36. Since by Proposition 6.3 the characterization of $H_{\mathbb{R}}$-conjugation classes of maximal σ-split $\mathbb{R}$-tori of G_ρ can be reduced to $H_{\mathbb{R}}^{+}$-conjugation classes of θ-stable maximal σ-split $\mathbb{R}$-tori of G_ρ, the above result enables one to obtain a characterization of the conjugacy classes of Cartan subspaces via a characterization of conjugation classes of θ-stable maximal $\mathbb{R}$-split tori. A first characterization of these were given in [Mat79]. A different characterization can be found in [Hel98a], what includes a classification as well.

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