Computing the fine structure of real reductive symmetric spaces

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Abstract

Much of the structure of Lie groups has been implemented in several computer algebra packages, including LiE, GAP4, Chevie, Magma, Maple and Mathematica. The structure of a reductive symmetric space is similar to that of the underlying Lie group and a computer algebra package for computations related to symmetric spaces would be an important tool for researchers in many areas of mathematics. Until recently very few algorithms existed for computations in symmetric spaces due to the fact that their structure is much more complicated then that of the underlying group.

In recent work, Daniel and Helminck (2004) gave a complete set of algorithms to compute the fine structure of Riemannian symmetric spaces. In this paper we make the first step in extending these results to general real reductive symmetric spaces and give a number of algorithms to compute some of their fine structure. This case is more complicated since it involves the intricate relations of 5 root systems and their Weyl groups instead of just two as in the Riemannian case.

Key words: Symmetric spaces, Computational Lie theory

1 Introduction

Real reductive symmetric spaces are the homogeneous spaces G/H, where G is a reductive Lie group and H is an open subgroup of the fixed point group of an involution σ of G. They are of importance in mathematics and physics and are

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[1] *Author is partially supported by N.S.F. Grant DMS-9977392

best known for their role in representation theory (see for example Harish-Chandra (1984); Ōshima and Matsuki (1984); Delorme (1998)).

It was shown in Helminck (1988) that a real symmetric space can be related to a complex reductive algebraic group with an ordered pair of commuting involutions. In section 2 we show that the fine structure of a general real symmetric space is the same as that of a complex semisimple Lie algebra with an ordered pair of involutions. The fine structure of a general real symmetric space is more complicated then that of a Riemannian symmetric space, since it involves the intricate relations of 5 root systems and their Weyl groups instead of just two as in the Riemannian case. In this paper we give a number of algorithms to compute some of the fine structure of the general real symmetric spaces. The fine structure of root systems and Weyl groups can be computed at the level of the Lie algebra. Therefore the algorithms in this paper will be presented at the Lie algebra level.

Note 1 *The notation used is mostly standard, but the following list may be of help.*

G	*a reductive algebraic group;*
$\mathfrak{g}$	*the Lie algebra of* G*;*
σ, θ	*commuting involutions of* $\mathfrak{g}$*;*
$\mathfrak{h}$	*the set of* x *in* $\mathfrak{g}$ *such that* $\sigma(x) = x$*;*
$\mathfrak{q}$	*the set of* x *in* $\mathfrak{g}$ *such that* $\sigma(x) = -x$*;*
$\mathfrak{k}$	*the set of* x *in* $\mathfrak{g}$ *such that* $\theta(x) = x$*;*
$\mathfrak{p}$	*the set of* x *in* $\mathfrak{g}$ *such that* $\theta(x) = -x$*;*
$\Phi(\mathfrak{s})$	*the set of roots of* $\mathfrak{g}$ *with respect to a toral subalgebra* $\mathfrak{s}$*;*

Remark 1 $\mathfrak{q}$ *and* $\mathfrak{p}$ *are* local compact reductive symmetric spaces *and* $\mathfrak{q} \cap \mathfrak{p}$ *is a* local real reductive symmetric space.

The isomorphy classes of pairs of involutions can be represented by two invariants, one of which is the (σ, θ)-diagram (see section 3.4) which represents the action of the involutions on the root system and determines 4 of the 5 root systems. In a forthcoming paper we will give algorithms to compute the fifth root system. The four root systems that we deal with in this paper are:

(i) $\Phi(\mathfrak{t})$, where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$
(ii) $\Phi(\mathfrak{a}_1)$, where $\mathfrak{a}_1$ is a maximal toral subalgebra of $\mathfrak{q}$
(iii) $\Phi(\mathfrak{a}_2)$, where $\mathfrak{a}_2$ is a maximal toral subalgebra of $\mathfrak{p}$
(iv) $\Phi(\mathfrak{a})$, where $\mathfrak{a}$ is a maximal toral subalgebra of $\mathfrak{q} \cap \mathfrak{p}$

For $\mathfrak{g}$ simple there are 88 types of local symmetric spaces and the corresponding (σ, θ)-diagrams are called absolutely irreducible. For $\mathfrak{g}$ not simple there are additionally 83 types types of local symmetric spaces for which the (σ, θ)-diagram is irreducible, but not absolutely irreducible. In this paper we give a number of algorithms to compute some of the fine structure of the general real symmetric spaces.

These algorithms are briefly outlined in the following steps:

(i) Recover the action of σ and θ on the original root system, $\Phi(\mathfrak{t})$, from the (σ, θ)-diagram.
(ii) Compute the restricted σ- and θ- diagrams that represent the actions of the involutions on the restricted root systems $\Phi(\mathfrak{a}_2)$ and $\Phi(\mathfrak{a}_1)$ respectively.
(iii) There are several projections (restrictions) by restricting root systems to smaller tori. In each of the 5 cases that the roots of one torus are restricted to a smaller torus determine all the positive roots that project down to each root in the base for the restricted root system.
(iv) For each case as in (iii) find representatives in the Weyl group of the larger torus for each element in the (restricted) Weyl group of the smaller torus.
(v) Use the Weyl group representatives to find a complete list of positive roots in the restricted root systems.
(vi) Determine all the positive roots of $\mathfrak{g}$ that project down to each root in the base for the restricted root system using the restricted Weyl group and the representatives in the original Weyl group as in (iv).

This fine structure can be used to compute many other aspects of these reductive symmetric spaces. We illustrate this by computing the various root spaces for the above toral subalgebras. These in turn can then be used to compute nice bases for the reductive symmetric space. All that is needed is the following additional step to the above algorithm:

(vii) Using (v) and (vi) compute the root spaces for the toral subalgebras $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}$ in terms of sums of the 1-dimensional root spaces of $\mathfrak{t}$.

For each of these steps one needs to develop algorithms to compute the corresponding structure. In this paper we setup the proper framework to compute the fine structure of the 4 restricted root systems and Weyl groups that can be derived from the (σ, θ)-diagram. We first illustrate the algorithms by working through the first few steps of an example. Finally we give an outline of the algorithms for computing the various steps above.

2 Geometric Motivation

Reductive groups over algebraically closed fields have a natural fine structure of a root system with its Weyl group coming from a maximal torus. If the group is defined over a field k which is not algebraically closed then there is a second root system and Weyl group which characterizes the k structure of the group. This additional fine structure comes from the root system of a maximal k-split torus A of G together with its Weyl group and the multiplicities of the roots. This fine structure of the two root systems with their Weyl groups and multiplicities of the roots plays a fundamental role in all studies of reductive k-groups and their representations. Reductive symmetric spaces have a similar fine structure, which plays an equally important role in the study of these symmetric spaces and their applications as their counterpart in the groups case. However this fine structure is much more complicated since it involves the intricate relations of 5 root systems instead of just one or two. To describe these we need a bit more notation. For a subgroup K of G and a torus S in K let K_k denote the set of k-rational points, let $\Phi(S, K)$ denote the set of roots of S in K and let $W(S, K) = N_K(S)/Z_K(S)$ denote the Weyl group of S in K, where $N_K(S)$ is the normalizer in K of S and $Z_K(S)$ is the centralizer in K of S. In the case that $K = G$ we will also write $\Phi(S)$ for $\Phi(S, G)$ and $W(S)$ for $W(S, G)$. If S is a k-split torus, then $\Phi(S, G_k) = \Phi(S, G)$. The 5 root systems of a symmetric space are the following. First there is the natural root system associated with these symmetric spaces. This root system is the set of roots of a maximal k-split torus of G contained in the symmetric variety $Q = \{x\sigma(x)^{-1} \mid x \in G_k\} \simeq G_k/H_k$. The k-split tori in Q are also called (σ, k)-split tori and in Helminck and Wang (1993) it was shown that the set of roots of a maximal (σ, k)-split torus A is actually a root system with Weyl group $W(A) = N_{G_k}(A)/Z_{G_k}(A)$. The second root system is that of a maximal σ-split k-torus A_1 of G (i.e. $\sigma(a) = a^{-1}$ for all $a \in A_1$). It can be chosen to contain A. Since this torus is (σ, k)-split over the algebraic closure $\bar{k}$ it follows that $\Phi(A_1)$ is a root system with Weyl group $W(A_1) = N_G(A_1)/Z_G(A_1)$. The third root system comes from a maximal k-split torus A_2 of G, which again we can choose containing A. There exists a maximal k-torus T of G which contains A, A_1 and A_2. This is called a *σ-standard* maximal k-torus. It gives us our fourth root system. Finally the fifth root system is the subsystem Φ_0 of $\Phi(A)$ consisting of those roots $\alpha \in \Phi(A)$ for which the corresponding reflection in the Weyl group has a representative in H_k. The Weyl group of Φ_0 is precisely the subgroup $W(A, H_k) = N_{H_k}(A)/Z_{H_k}(A)$ of $W(A, G_k)$. The root systems $\Phi(A)$, $\Phi(A_1)$ and $\Phi(A_2)$ can be identified with the projections of $\Phi(T)$ to A, A_1 and A_2. Similarly, the Weyl groups can be identified with the quotient of two subgroups of the Weyl group $W(T)$. The root system $\Phi(A)$ can also be identified with the projections of $\Phi(A_1)$ or $\Phi(A_2)$ on A. Similarly, the Weyl group $W(A, G_k)$ can be identified with the quotient of two subgroups of the Weyl group $W(A_1)$ or the quotient of two subgroups of the Weyl group $W(A_2, G_k)$. The intricate relations between all these root systems and their Weyl group plays a fundamental role in the study of these symmetric spaces.

It remains to show that the fine structure of the reduced root systems with Weyl groups and multiplicities for a reductive symmetric space is the same as that of a complex semisimple algebraic group with an ordered pair of commuting involutions. We use the same notation as in the introduction. For a σ-stable torus A let A_+ and A_- denote the maximal subtori of A such that $\sigma|A_+ = 1$, $\sigma|A_- = -1$, where 1 is the identity map and -1 the map $x \mapsto x^{-1}$. Then we have the decomposition $A = A_+A_-$. Let σ and θ be commuting k-involutions of G with θ a Cartan involution of G and let H be an open k-subgroup of G^σ. In (Helminck and Wang, 1993, Section 11) it was shown that any (maximal) θ-split k-torus of G is (maximal) (θ, k)-split. Moreover given any σ-stable maximal k-split torus A of G, there is $h \in H(k)$ such that hAh^{-1} is θ-stable. Two (σ, θ)-stable maximal k-split tori A_1 and A_2 which are H_k-conjugate are also $(H \cap G^{\theta})_k$-conjugate. The (σ, θ)-stable maximal k-split tori A of G with maximal A_+ (resp. A_-) parts (with respect to the involution σ) are $(G^{\sigma} \cap G^{\theta})_k^0$-conjugate and all maximal (σ, k)-split tori are H_k-conjugate. A maximal k-torus will be called a *(σ, θ)-standard maximal k-torus* if it is (σ, θ)-stable, contains a maximal (σ, θ)-split k-torus, a maximal σ-split k-torus and a maximal θ-split k-torus.

Proposition 2 *Let σ and θ be commuting k-involutions of G with θ a Cartan involution of G. Then any σ-standard maximal k-torus is $(G^{\sigma})_k^0$-conjugate with a (σ, θ)-standard maximal k-torus.*

PROOF. Let S be a maximal (σ, k)-split torus of G and let C, M_1, M_2 denote the central, anisotropic and isotropic factors of $Z_G(S)$ over k respectively. Then from (Helminck and Wang, 1993, Lemma 4.5) it follows that $M_2 \subset H$ and if A is any maximal k-split torus of $Z_G(S)$, then A is θ-stable and moreover $CM_1 \subset Z_G(A)$. From this it follows that there exists a σ-standard maximal k-torus. Then the result is immediate from the above results.

Corollary 3 *Let σ and θ be commuting k-involutions of G with θ a Cartan involution of G. If T is a (σ, θ)-standard maximal k-torus of G, then:*

(i) $T_-^{\theta} := \{t \in T \mid \theta(t) = t^{-1}\}^0$ *is a maximal θ-split torus of G, which is also maximal k-split.*
(ii) $T_-^{\sigma,\theta} := \{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^0$ *is a maximal (σ, θ)-split torus of G, which is also maximal (σ, k)-split.*
(iii) $\Phi(T_-^{\theta}) = \Phi(T_-^{\theta}(k))$ *and* $W(T_-^{\theta}) = W(T_-^{\theta}(k))$.
(iv) $\Phi(T_-^{\sigma,\theta}) = \Phi(T_-^{\sigma,\theta}(k))$ *and* $W(T_-^{\sigma,\theta}) = W(T_-^{\sigma,\theta}(k))$.

PROOF. (i) follows from the definition of a (σ, θ)-standard maximal k-torus of G and (Helminck and Wang, 1993, 11.5). A similar argument shows that $T_-^{\sigma,\theta}$ is maximal (σ, k)-split. Since T_-^{θ} and $T_-^{\sigma,\theta}$ are k-split (iii)-(iv) are immediate.

These results show that it suffices to consider (σ, θ)-standard maximal tori with all the restricted root systems and Weyl groups of the related subtori defined by the involutions and one does not need to consider the tori and root systems with Weyl groups defined by the k-structure. In other words the fine structure of the real reductive symmetric spaces is the same as that of a complex reductive algebraic group with an ordered pair of commuting involutions.

3 Root systems and Weyl groups of a symmetric space

In this section we discuss some results about these symmetric spaces and their fine structure, which will be needed for the algorithms. We start out by introducing some of the notation used throughout the remainder of this paper.

3.1 Preliminaries and Notation

Let $\sigma, \theta \in \mathrm{Aut}(G)$ be involutions, i.e. $\sigma^2 = \theta^2 = \mathrm{id}$ and write $K = \{g \in G \mid \theta(g) = g\}$ the fixed point group of θ, $P_\theta = \{g\theta(g)^{-1} \mid g \in G\}$ and similarly $H = \{g \in G \mid \sigma(g) = g\}$ the fixed point group of σ, $P_\sigma = \{g\sigma(g)^{-1} \mid g \in G\}$. The varieties P_θ and P_σ are called *symmetric varieties* or also *reductive symmetric spaces*. If G is semisimple, P_θ and P_σ are also called *semisimple symmetric spaces*.

Note 2 *$P_\theta \simeq G/K$ and $P_\sigma \simeq G/H$.*

Let $\mathfrak{g}$ denote the Lie algebra of G and denote the involutions of $\mathfrak{g}$ induced by σ and θ also by σ and θ. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ is the Lie algebra of K and $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$ is the tangent space in the identity of P_θ. Similarly $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{h} = \{x \in \mathfrak{g} \mid \sigma(x) = x\}$ is the Lie algebra of H and $\mathfrak{q} = \{x \in \mathfrak{g} \mid \sigma(x) = -x\}$ is the tangent space in the identity of P_σ. Since σ and θ commute, we can write $\mathfrak{g} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{q}$.

3.2 Root space decomposition.

For a vector space V, we denote its dual space by V^*. Let $\mathfrak{s}$ be a toral subalgebra of the Lie algebra $\mathfrak{g}$. For $\alpha \in \mathfrak{s}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{s}\}$ and let $\Phi(\mathfrak{s}) = \{\alpha \in \mathfrak{s}^* \mid \mathfrak{g}_\alpha \neq 0\}$. The elements of $\Phi(\mathfrak{s})$ are called *roots* and the subspaces $\mathfrak{g}_\alpha$ are called *root subspaces*. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi(\mathfrak{s})} \mathfrak{g}_\alpha$$

is a *root space decomposition* of $\mathfrak{g}$. Here $\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{s})$ is the root subspace for $\alpha = 0$, which is exactly the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. For $\mathfrak{s} = \mathfrak{t}$ a maximal toral subalgebra in $\mathfrak{g}$ we have $\mathfrak{t} = \mathfrak{g}_0$ and $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi(\mathfrak{t})$.

Remark 4 *For some $\mathfrak{s}$ the set of roots $\Phi(\mathfrak{s})$ is a (reduced) root system in the sense of Bourbaki (1981). Examples are $\mathfrak{s} = \mathfrak{t}$ a maximal total subalgebra, $\mathfrak{s} = \mathfrak{a}_0 \subset \mathfrak{p}$ a maximal toral subalgebra of $\mathfrak{p}$ (see Richardson (1982)) and $\mathfrak{s} = \mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ a maximal toral subalgebra of $\mathfrak{p} \cap \mathfrak{q}$.*

Let $\mathfrak{a}$ be a maximal toral subalgebra in $\mathfrak{p} \cap \mathfrak{q}$. We use the root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi(\mathfrak{a})} \mathfrak{g}_\lambda$ with respect to $\mathfrak{a}$ to find a basis for $\mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{p} \cap \mathfrak{q}$. Here $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x \text{ for all } t \in \mathfrak{a}\}$ and $\Phi(\mathfrak{a}) = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0 \text{ and } \mathfrak{g}_\lambda \neq 0\}$. These root subspaces can be decomposed as a sum of root subspaces of a maximal total subalgebra containing $\mathfrak{a}$, but also as sums of root subspaces of a maximal θ-split (or σ-split) toral subalgebra containing $\mathfrak{a}$. To handle all these decompositions at the same time we define the following:

Definition 5 *A maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called* standard *if it satisfies the following conditions:*

(i) $\mathfrak{t}$ *is (σ, θ)-stable, i.e.* $\mathfrak{t} = \mathfrak{t}_\sigma^+ \oplus \mathfrak{t}_\sigma^- = \mathfrak{t}_\theta^+ \oplus \mathfrak{t}_\theta^-$, *where* $\mathfrak{t}_\sigma^\pm = \{x \in \mathfrak{t} \mid \sigma(x) = \pm x\}$ *and* $\mathfrak{t}_\theta^\pm = \{x \in \mathfrak{t} \mid \theta(x) = \pm x\}$.
(ii) $\mathfrak{t}_\sigma^-$ *is a maximal toral subalgebra in* $\mathfrak{q}$.
(iii) $\mathfrak{t}_\theta^-$ *is a maximal toral subalgebra in* $\mathfrak{p}$.
(iv) $\mathfrak{a} = \mathfrak{t}_\sigma^- \cap \mathfrak{t}_\theta^- = \{x \in \mathfrak{t} \mid \sigma(x) = \theta(x) = -x\}$ *is a maximal toral subalgebra in* $\mathfrak{p} \cap \mathfrak{q}$.

Note 3 *To avoid confusion between roots, we reserve α for roots in $\Phi(\mathfrak{t})$, β for roots in $\Phi(\mathfrak{a}_1)$, γ for roots in $\Phi(\mathfrak{a}_2)$ and λ for roots in $\Phi(\mathfrak{a})$.*

Let $R(\mathfrak{t}) = \mathbb{Z}_{span}\{\Phi(\mathfrak{t})\}$ be the root lattice of $\Phi(\mathfrak{t})$, $X_0(\sigma) = \{\chi \in R(\mathfrak{t}) \mid \sigma(\chi) = \chi\}$, $\Phi_0(\sigma) = \Phi(\mathfrak{t}) \cap X_0(\sigma)$, and $\Delta_0(\sigma) = \Delta(\mathfrak{t}) \cap \Phi_0(\sigma)$. Similarly we define $X_0(\theta)$, $\Phi_0(\theta)$, and $\Delta_0(\theta)$. Define the projections:

(i) $\pi_1 : X \to R(\mathfrak{t})/X_0(\sigma)$ by $\pi_1(\alpha) = \frac{1}{2}(\alpha - \sigma(\alpha))$
(ii) $\pi_2 : X \to R(\mathfrak{t})/X_0(\theta)$ by $\pi_2(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$

By abuse of note we will also write π_1, resp. π_2 for the restrictions of π_1, resp. π_2 to $R(\mathfrak{t})/X_0(\theta)$, resp $R(\mathfrak{t})/X_0(\sigma)$.

Lemma 6 *The second and third root system associated with the local symmetric space can be found by*

(i) $\Phi(\mathfrak{a}_1) = \pi_1(\Phi(\mathfrak{t}) - \Phi_0(\sigma))$
(ii) $\Phi(\mathfrak{a}_2) = \pi_2(\Phi(\mathfrak{t}) - \Phi_0(\theta))$

Let $E \subset \mathfrak{t}^*$ be the Euclidean space spanned by $\Phi(\mathfrak{t})$, i.e. $E = R(\mathfrak{t}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\mathfrak{h} \cap \mathfrak{k}, \mathfrak{h} \cap \mathfrak{p}, \mathfrak{k} \cap \mathfrak{q}$, and $\mathfrak{p} \cap \mathfrak{q}$ decompose $\mathfrak{g}$ orthogonally, we are looking at the projection of E onto $\mathfrak{a}^*$. The next question is how are the roots in $\Phi(\mathfrak{t})$, $\Phi(\mathfrak{a}_1)$, $\Phi(\mathfrak{a}_2)$, and $\Phi(\mathfrak{a})$ related?

Lemma 7 *Let $\alpha \in \Phi(\mathfrak{t})$ with $\beta = \alpha|\mathfrak{a}_1, \gamma = \alpha|\mathfrak{a}_2,$ and $\lambda = \alpha|\mathfrak{a}$.*

(i) $\beta \in \Phi(\mathfrak{a}_1) \cup \{0\}$ *and* $\mathfrak{g}_\alpha \subset \mathfrak{g}_\beta$.

(ii) $\gamma \in \Phi(\mathfrak{a}_2) \cup \{0\}$ *and* $\mathfrak{g}_\alpha \subset \mathfrak{g}_\gamma$.
(iii) $\lambda \in \Phi(\mathfrak{a}) \cup \{0\}$ *and* $\mathfrak{g}_\alpha \subset \mathfrak{g}_\lambda$.
(iv) $\lambda = \alpha|\mathfrak{a} = \beta|\mathfrak{a}_2 = \gamma|\mathfrak{a}_1$.

PROOF. Let $\lambda = \alpha|\mathfrak{a}$ and $\beta = \alpha|\mathfrak{a}_1$. We have $[h,x] = \alpha(h)x = \beta(h)x \; \forall \; h \in \mathfrak{a}_1$. Now consider the restriction $\beta|\mathfrak{a}_2$. Since $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$, this is the same as considering the restriction $\beta|\mathfrak{a}$. We have $[h,x] = \alpha(h)x = \beta(h)x = \lambda(h)x \; \forall \; h \in \mathfrak{a}$. Therefore, $\lambda = \beta|\mathfrak{a}_2$. By similar arguments, $\lambda = \gamma|\mathfrak{a}_1$. The remaining statements are immediate from this.

Let $\Delta(\mathfrak{t})$, $\Delta(\mathfrak{a}_1)$, $\Delta(\mathfrak{a}_2)$, and $\Delta(\mathfrak{a})$ be bases for $\Phi(\mathfrak{t})$, $\Phi(\mathfrak{a}_1)$, $\Phi(\mathfrak{a}_2)$, and $\Phi(\mathfrak{a})$, respectively. Ideally we would like the natural projection map $\pi : \mathfrak{g}^* \to (\mathfrak{p} \cap \mathfrak{q})^*$, defined by $\pi(\alpha) = \frac{1}{4}(\alpha - \sigma(\alpha) - \theta(\alpha) + \sigma\theta(\alpha))$ to satisfy $\pi(\Delta(\mathfrak{t})) = \Delta(\mathfrak{a})$. This will only be the case with a special choice of basis for $\Phi(\mathfrak{t})$ (see 3.4).

3.3 *Root systems and Weyl groups.*

Let $W(\mathfrak{t})$ denote the Weyl group of the root system $\Phi(\mathfrak{t})$. Let $X_0(\sigma,\theta) = \{\chi \in R(\mathfrak{t}) \mid \chi - \theta(\chi) - \sigma(\chi) + \sigma\theta(\chi) = 0\}$ and $\Phi_0(\sigma,\theta) = \Phi(\mathfrak{t}) \cap X_0(\sigma,\theta)$. Define $W_0(\sigma,\theta) = \{w \in W(\mathfrak{t}) \mid w(\Phi_0(\sigma,\theta)) = \Phi_0(\sigma,\theta)\}$ and $W_1(\sigma,\theta) = \{w \in W(\mathfrak{t}) \mid w(X_0(\sigma,\theta)) = X_0(\sigma,\theta)\}$.

Let $\overline{\Phi} = \pi(\Phi(\mathfrak{t}) - \Phi_0(\sigma,\theta))$ denote the set of *restricted roots* of $\Phi(\mathfrak{t})$ relative to (σ,θ). All $w \in W_1(\sigma,\theta)$ induce a mapping $\pi(w) \in \mathrm{Aut}(R(\mathfrak{t})/X_0(\sigma,\theta))$ such that $\pi(w(\chi)) = \pi(w)(\pi(\chi))$. Define $\overline{W} = \{\pi(w) \mid w \in W_1(\sigma,\theta)\}$. Let $W(\mathfrak{a})$ denote the Weyl group of the restricted root system $\Phi(\mathfrak{a})$. It is also called the *restricted Weyl group* with respect to the action of (σ,θ) on $R(\mathfrak{t})$ due to the following result.

Proposition 8 *Let $\overline{\Phi}$, $\overline{W}$, etc. be as above. Then*

(i) $\overline{\Phi} = \Phi(\mathfrak{a})$.
(ii) $W(\mathfrak{a}) = \overline{W} \cong W_1(\sigma,\theta)/W_0(\sigma,\theta)$.

The roots in $\Phi(\mathfrak{a})$ can be computed as projections of roots in $\Phi(\mathfrak{a}_1)$, $\Phi(\mathfrak{a}_2)$, or $\Phi(\mathfrak{t})$. Let $X_0^\sigma(\theta) = \{\chi \in R(\mathfrak{a}_1) \mid \theta(\chi) - \chi = 0\}$ and $\Phi_0^\sigma(\theta) = \Phi(\mathfrak{a}_1) \cap X_0^\sigma(\theta)$. Similarly define $X_0^\theta(\sigma)$ and $\Phi_0^\theta(\sigma)$. Note that $\pi_1^{-1}(X_0^\sigma(\theta)) = X_0(\sigma,\theta)$ and $\pi_2^{-1}(X_0^\theta(\sigma)) = X_0(\sigma,\theta)$. From this we get:

Theorem 9 *The restricted root system $\Phi(\mathfrak{a})$ can be obtained in three ways.*

(i) $\Phi(\mathfrak{a}) = \pi(\Phi(\mathfrak{t}) - \Phi_0(\sigma,\theta))$
(ii) $\Phi(\mathfrak{a}) = \pi_2(\Phi(\mathfrak{a}_1) - \Phi_0^\sigma(\theta))$

(iii) $\Phi(\mathfrak{a}) = \pi_1(\Phi(\mathfrak{a}_2) - \Phi_0^{\theta}(\sigma))$

Let $W_1^{\sigma}(\sigma, \theta) = \{w \in W_1(\sigma, \theta) \mid w(X_0(\sigma)) = X_0(\sigma)\}$. Similarly define $W_1^{\theta}(\sigma, \theta)$.

Theorem 10 *The restricted Weyl group* $W(\mathfrak{a})$ *can be obtained in three ways.*

(i) $W(\mathfrak{a}) \cong W_1(\sigma, \theta)/W_0(\sigma, \theta)$.
(ii) $W(\mathfrak{a}) \cong W_1^{\theta}(\sigma, \theta)/W_0(\sigma, \theta)$.
(iii) $W(\mathfrak{a}) \cong W_1^{\sigma}(\sigma, \theta)/W_0(\sigma, \theta)$.

3.4 (σ, θ)-diagram.

We can represent the action of σ and θ on these root systems by a diagram. For this we first need to define a compatible order. A (σ, θ)-order on $(R(\mathfrak{t}), \Phi(\mathfrak{t}))$ related to the action (σ, θ) is defined by

$$\text{if } \chi \in R(\mathfrak{t}), \chi > 0 \text{ , and } \chi \notin X_0(\sigma, \theta), \text{ then } \sigma(\chi) < 0 \text{ and } \theta(\chi) < 0.$$

In other words, if χ is a positive element of the root lattice and χ is not fixed by both σ and θ then $\sigma(\chi)$ and $\theta(\chi)$ are both negative. A (σ, θ)-order on $(R(\mathfrak{t}), \Phi(\mathfrak{t}))$ induces orders on $X_0(\sigma, \theta)$ and $R(\mathfrak{t})/X_0(\sigma, \theta)$ and vice versa. A basis $\Delta(\mathfrak{t})$ for $\Phi(\mathfrak{t})$ with respect to the (σ, θ)-order will be called a (σ, θ)-basis of $\Phi(\mathfrak{t})$ and we write $\Delta_0(\sigma, \theta) = \Delta(\mathfrak{t}) \cap \Phi_0(\sigma, \theta)$.

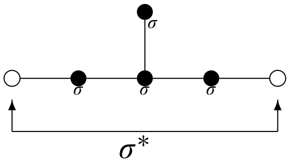
Proposition 11 *For a* (σ, θ)*-basis* $\Delta(\mathfrak{t})$*, we have* $\pi(\Delta(\mathfrak{t}) - \Delta_0(\sigma, \theta)) = \Delta(\mathfrak{a})$*, where* $\Delta(\mathfrak{a})$ *is a basis for* $\Phi(\mathfrak{a})$.

Remark 12 *Let* $\Delta(\mathfrak{t})$ *be a* (σ, θ)*-basis of* $\Phi(\mathfrak{t})$*,* $w_0(\sigma), w_0(\theta) \in W_0(\sigma, \theta)$ *the involutions such that* $w_0(\sigma)(\Delta_0(\sigma)) = -\Delta_0(\sigma)$ *and* $w_0(\theta)(\Delta_0(\theta)) = -\Delta_0(\theta)$*. Let* $\sigma^* = -\operatorname{id} \circ \sigma \circ w_0(\sigma)$ *and* $\theta^* = -\operatorname{id} \circ \theta \circ w_0(\theta)$*. Then*

(i) $w_0(\sigma)$*,* $w_0(\theta)$*,* σ^**,* θ^* *and* $-\operatorname{id}$ *commute.*
(ii) $\sigma^*, \theta^* = \begin{cases} \operatorname{id} \\ \textit{Dynkin diagram automorphism of order 2} \end{cases}$

If $(\sigma, \theta) \in \operatorname{Aut}(R(\mathfrak{t}), \Phi)$ is a pair of commuting involutions and Δ a (σ, θ)-basis of Φ, then θ is determined by the quadruple $(R(\mathfrak{t}), \Delta, \Delta_0(\theta), \theta^*)$, because $\theta = -\theta^* w_0(\theta)$. We call such a quadruple $(R(\mathfrak{t}), \Delta, \Delta_0(\theta), \theta^*)$ a θ-index. Similarly σ is determined by the quadruple $(R(\mathfrak{t}), \Delta, \Delta_0(\sigma), \sigma^*)$. The sextuple $(R(\mathfrak{t}), \Delta, \Delta_0(\sigma), \Delta_0(\theta), \sigma^*, \theta^*)$ is called a (σ, θ)-index. These indices are an invariant for the isomorphy classes of pairs of commuting involutions (σ, θ). As in Helminck (1988) we make a diagrammatic representation of the (σ, θ)-index of (σ, θ) by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\sigma) \cup \Delta_0(\theta)$, and by giving the vertices of $\Delta_0(\sigma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\sigma) \cap \Delta_0(\theta)$ a label

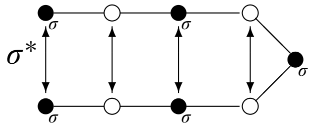
σ or θ if $\sigma(\alpha) \neq \alpha$ or $\theta(\alpha) \neq \alpha$, respectively. The actions of σ^* and θ^* are indicated by arrows. We omit the actions of σ^*, θ^* on $X_0(\sigma)$, $X_0(\theta)$, respectively since these are completely determined by the root systems $\Phi_0(\sigma)$, $\Phi_0(\theta)$. This diagram is called a (σ, θ)-diagram. An example in type E_6 is:



Remark 13 *The involutions of* $(R(\mathfrak{t}), \Phi)$ *can be recovered from the* (σ, θ)*-diagram by* $\theta = -\operatorname{id} \circ \theta^* \circ w_0(\theta)$ *and* $\sigma = -\operatorname{id} \circ \sigma^* \circ w_0(\sigma)$.

4 Example of Algorithm

In this section we illustrate the algorithm by computing the first few steps in the case of $A_9^{5,4}(\mathrm{III}_b, \mathrm{II})$. Its (σ, θ)-diagram is:



4.1 Step 1

The notation $A_9^{5,4}(\mathrm{III}_b, \mathrm{II})$ indicates that we start with the Lie algebra $\mathfrak{g} = A_9$. σ is an involution of type III_b (from Helgason (1978)) and $w_0(\sigma) = \mathrm{id}$. Likewise θ is an involution of type II and $w_0(\theta) = s_{\alpha_1} s_{\alpha_3} s_{\alpha_5} s_{\alpha_7} s_{\alpha_9}$. We also notice that σ^* is a nontrivial Dynkin diagram automorphism of order 2, while $\theta^* = \mathrm{id}$. So our involutions act as $\sigma = -\sigma^*$ and $\theta = -s_{\alpha_1} s_{\alpha_3} s_{\alpha_5} s_{\alpha_7} s_{\alpha_9}$.

4.2 Step 2

We compute bases for the restricted root systems $\Phi(\mathfrak{a}_1)$, $\Phi(\mathfrak{a}_2)$, and $\Phi(\mathfrak{a})$.

Recall that $\Phi(\mathfrak{a}_1) = \pi_1(\Phi(\mathfrak{t}) - \Phi_0(\sigma))$. The restricted base roots are
$\pi_1(\alpha_1) = \pi_1(\alpha_9) = \frac{1}{2}(\alpha_1 + \alpha_9) = \beta_1$, $\pi_1(\alpha_2) = \pi_1(\alpha_8) = \frac{1}{2}(\alpha_2 + \alpha_8) = \beta_2$, $\pi_1(\alpha_3) = \pi_1(\alpha_7) = \frac{1}{2}(\alpha_3 + \alpha_7) = \beta_3$, $\pi_1(\alpha_4) = \pi_1(\alpha_6) = \frac{1}{2}(\alpha_4 + \alpha_6) = \beta_4$, and $\pi_1(\alpha_5) = \alpha_5 = \beta_5$. So $\Delta(\mathfrak{a}_1) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$.

Recall that $\Phi(\mathfrak{a}_2) = \pi_2(\Phi(\mathfrak{t}) - \Phi_0(\theta))$. The restricted base roots are $\pi_2(\alpha_2) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) = \gamma_1$, $\pi_2(\alpha_4) = \frac{1}{2}(\alpha_3 + 2\alpha_4 + \alpha_5) = \gamma_2$, $\pi_2(\alpha_6) = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7) = \gamma_3$, and $\pi_2(\alpha_8) = \frac{1}{2}(\alpha_7 + 2\alpha_8 + \alpha_9) = \gamma_4$. So $\Delta(\mathfrak{a}_2) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.

Recall that $\Phi(\mathfrak{a}) = \pi(\Phi(\mathfrak{t}) - \Phi_0(\sigma, \theta))$. Now letting π act on $\Delta(\mathfrak{t})$, we find that $\pi(\alpha_2) = \pi(\alpha_8) = \frac{1}{4}(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_7 + 2\alpha_8 + \alpha_9) = \lambda_1$, and $\pi(\alpha_4) = \pi(\alpha_6) = \frac{1}{4}(\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_2$. So $\Delta(\mathfrak{a}) = \{\lambda_1, \lambda_2\}$.

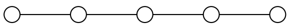
Note 4 *$\Phi(\mathfrak{a})$ can also be computed as projections of roots in $\Delta(\mathfrak{a}_1) - \Delta_0^\sigma(\theta)$ or as projections of roots in $\Delta(\mathfrak{a}_2) - \Delta_0^\theta(\sigma)$.*

(i) Letting π_2 act on all the restricted roots in $\Delta(\mathfrak{a}_1)$, we find that $\pi_2(\beta_2) = \frac{1}{2}(\beta_1 + 2\beta_2 + \beta_3) = \frac{1}{4}(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_7 + 2\alpha_8 + \alpha_9) = \lambda_1$, and $\pi_2(\beta_4) = \frac{1}{2}(\beta_3 + 2\beta_4 + \beta_5) = \frac{1}{4}(\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_2$.

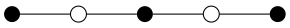
(ii) Letting π_1 act on all the restricted roots in $\Delta(\mathfrak{a}_2)$, we find that $\pi_1(\gamma_1) = \pi_1(\gamma_4) = \frac{1}{2}(\gamma_1 + \gamma_4) = \frac{1}{4}(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_7 + 2\alpha_8 + \alpha_9) = \lambda_1$, and $\pi_1(\gamma_2) = \pi_1(\gamma_3) = \frac{1}{2}(\gamma_2 + \gamma_3) = \frac{1}{4}(\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_2$.

4.3 Step 3

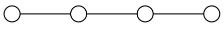
The restricted root system $\Phi(\mathfrak{a}_1)$ has basis $\Delta(\mathfrak{a}_1) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$. The Dynkin diagram of this root system is



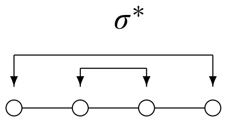
$\beta_1, \beta_3, \beta_5$ are fixed by θ so we will color these vertices black. θ^* is a trivial automorphism so the restricted θ-diagram that will complete our projection from $\Phi(\mathfrak{a}_1)$ to $\Phi(\mathfrak{a})$ is



The restricted root system $\Phi(\mathfrak{a}_2)$ has basis $\Delta(\mathfrak{a}_2) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. The Dynkin diagram of this root system is



None of the base roots are fixed by σ and σ^* is a non-trivial automorphism of order 2. The restricted σ-diagram that will complete our projection from $\Phi(\mathfrak{a}_2)$ to $\Phi(\mathfrak{a})$ is



4.4 Step 4

The computations now become lengthy and cumbersome (hence the need for symbolic computation). To illustrate the algorithms, we include only the evolution of one root $\lambda_1 \in \Delta(\mathfrak{a})$ in the remainder of the example. $\lambda_1 = \alpha_2|_{\mathfrak{a}} = \beta_2|_{\mathfrak{a}} = \gamma_1|_{\mathfrak{a}}$.

For the restriction of $\Phi(\mathfrak{t})$ to $\Phi(\mathfrak{a}_1)$, all roots that project to β_2 are $\{\alpha_2, \alpha_8\}$ and the multiplicity of β_2 in this restriction is 2.

For the restriction of $\Phi(\mathfrak{t})$ to $\Phi(\mathfrak{a}_2)$, all roots that project to γ_1 are $\{\alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ and the multiplicity of γ_1 in this restriction is 4.

For the restriction of $\Phi(\mathfrak{t})$ to $\Phi(\mathfrak{a})$, all roots that project to λ_1 are $\{\alpha_2, \alpha_8, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_7 + \alpha_8, \alpha_8 + \alpha_9, \alpha_1 + \alpha_2 + \alpha_3, \alpha_7 + \alpha_8 + \alpha_9\}$ and the multiplicity of λ_1 in this restriction is 8.

For the restriction of $\Phi(\mathfrak{a}_1)$ to $\Phi(\mathfrak{a})$, all roots that project to λ_1 are $\{\beta_2, \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3\}$ and the multiplicity of λ_1 in this restriction is 4.

For the restriction of $\Phi(\mathfrak{a}_2)$ to $\Phi(\mathfrak{a})$, all roots that project to λ_1 are $\{\gamma_1, \gamma_4\}$and the multiplicity of λ_1 in this restriction is 4.

4.5 Step 5

Here we compute representatives in $W_1(\sigma)$ for s_{β_2} in $W(\mathfrak{a}_1)$ and in $W_1(\theta)$ for s_{γ_1} in $W(\mathfrak{a}_2)$. Moreover we need to compute representatives in $W_1(\sigma, \theta)$, $W_1^{\sigma}(\sigma, \theta)$, and $W_1^{\theta}(\sigma, \theta)$ for each s_{λ_1} in $W(\mathfrak{a})$.

$\beta_2 = \frac{1}{2}(\alpha_2 + \alpha_8)$. So a representative in $W_1(\sigma)$ for s_{β_2} is $s_{\alpha_2} s_{\alpha_8}$.

$\gamma_1 = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$. So a representative in $W_1(\theta)$ for s_{γ_1} is $s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}$.

$\lambda_1 = \frac{1}{4}(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_7 + 2\alpha_8 + \alpha_9)$. So a representative in $W_1(\sigma, \theta)$ for s_{λ_1} is $s_{\alpha_2} s_{\alpha_8} s_{\alpha_1} s_{\alpha_9} s_{\alpha_3} s_{\alpha_7} s_{\alpha_2} s_{\alpha_8}$.

$\lambda_1 = \frac{1}{2}(\beta_1 + 2\beta_2 + \beta_3)$. So a representative in $W_1^{\sigma}(\theta)$ for s_{λ_1} is $s_{\beta_2} s_{\beta_1} s_{\beta_3} s_{\beta_2}$.

$\lambda_1 = \frac{1}{2}(\gamma_1 + \gamma_4)$. So a representative in $W_1^{\theta}(\sigma)$ for s_{λ_1} is $s_{\gamma_1} s_{\gamma_4}$.

Remark: To compute the results in this example we implemented the algorithms in this paper in Maple's Coxeter package for the case $A_{2n+1}^{n+1,n}(\mathrm{III}_b, \mathrm{II})$. Maple's Coxeter package uses generators and relations for the Weyl groups and this has turned out to be very inefficient for implementing our algorithms. We are currently working, together with R. Haas and D. Gagliardi, on building a new Weyl group package from scratch, which uses unique representation of Weyl group elements rather than generators and relations. This package will enable us to implement the algorithms in this paper and previous algorithms much more efficient.

5 The Algorithm

5.1 Step 1

Input: Type of (σ, θ)-diagram.

Output: $w_0(\sigma)$, $w_0(\theta)$, σ and θ

We need to recover the action of both σ and θ on $\Phi(\mathfrak{t})$. σ^* and θ^* can be recovered from the (σ, θ)-diagram. We can compute $w_0(\sigma)$ by ignoring the action of θ on $\Delta(\mathfrak{t})$ and considering the underlying (σ, id)-diagram. This diagram and the $w_0(\sigma)$ element can be found using (Daniel and Helminck, 2004, Algorithm 1). Similarly, we recover the element $w_0(\theta)$.

5.2 Step 2

Input: Type of (σ, θ)-diagram, α in $\Delta(\mathfrak{t})$

Output: β in $\Delta(\mathfrak{a}_1)$, γ in $\Delta(\mathfrak{a}_2)$, and λ in $\Delta(\mathfrak{a})$

The next step involves finding the basis roots β, γ, and λ for the root systems $\Phi(\mathfrak{a}_1)$, $\Phi(\mathfrak{a}_2)$, and $\Phi(\mathfrak{a})$, respectively. The base roots $\beta \in \Delta(\mathfrak{a}_1)$ and $\gamma \in \Delta(\mathfrak{a}_2)$ can be computed using the algorithms of Daniel and Helminck (2004) since they involve only one involution.

Algorithm 1 *We compute λ in $\Delta(\mathfrak{a})$ as the projections of roots in $\Delta(\mathfrak{t}) - \Delta_0(\sigma, \theta)$ in these steps.*

(i) If the white dot in the (σ, θ)-diagram representing $\alpha \in \Delta(\mathfrak{t}) - \Delta_0(\sigma, \theta)$ is not adjacent to a black dot, $\lambda = \pi(\alpha) = \frac{1}{4}(\alpha + \sigma^*(\alpha) + \theta^*(\alpha) + \sigma^*\theta^*(\alpha)) \in \Delta(\mathfrak{a})$.

(ii) If the white dot in the (σ, θ)-diagram representing $\alpha \in \Delta(\mathfrak{t}) - \Delta_0(\sigma, \theta)$ is adjacent to a black dot, then

$$\lambda = \pi(\alpha) = \frac{1}{4}(\alpha + w_0(\sigma)\sigma^*(\alpha) + w_0(\theta)\theta^*(\alpha) + w_0(\sigma)\sigma^* w_0(\theta)\theta^*(\alpha)) \in \Delta(\mathfrak{a}).$$

5.3 Step 3

Input: Type of (σ, θ)-diagram and Dynkin diagrams of $\Phi(\mathfrak{a}_1)$ and $\Phi(\mathfrak{a}_2)$.

Output: Restricted θ- and σ-diagrams

Next we need to find the restricted σ and θ-diagrams that represent the actions of the involutions on the restricted root systems $\Phi(\mathfrak{a}_2)$ and $\Phi(\mathfrak{a}_1)$.

Algorithm 2 *To find the restricted θ-diagram we consider the Dynkin diagram of the restricted root system $\Phi(\mathfrak{a}_1)$ with basis $\Delta(\mathfrak{a}_1) = \{\beta_1, \cdots, \beta_k\}$.*

Assuming that $|\Delta(\mathfrak{t})| = n$, we have $\beta = \sum_{j=1}^{n} c_j \alpha_{i_j}$. Create the restricted θ-diagram in these steps.

- *(i) Color black any vertex β that is fixed by θ.*
- *(ii) If θ^* is a non-trivial Dynkin diagram automorphism we transfer the action of θ^* to the restricted θ-diagram. If θ^* is non-trivial, we have that $\theta^*(\alpha_{i_j}) \neq \alpha_{i_j}$ for some i. Denote $\theta^*(\beta) = \sum_{j=1}^{n} c_j \theta^*(\alpha_{i_j})$.*

Lemma 14 $\theta^*(\beta) \in \Delta(\mathfrak{a}_1)$.

PROOF. $\beta = \pi_1(\alpha)$ for some $\alpha \in \Delta(\mathfrak{t})$. So $\beta = \frac{1}{2}(\alpha - \sigma(\alpha))$. Since θ^* commutes with $w_0(\sigma)$ and σ^* we have $\theta^*(\beta) = \frac{1}{2}(\theta^*(\alpha) - \sigma(\theta^*(\alpha))) = \pi_1(\theta^*(\alpha))$. All we have left to show is that $\theta^*(\alpha) \notin \Delta_0(\sigma)$. $\alpha \notin \Delta_0(\sigma)$ so $\alpha - \sigma(\alpha) \neq 0$ which implies that $\theta^*(\alpha) - \sigma(\theta^*(\alpha)) \neq 0$. Therefore $\theta^*(\beta) = \pi_1(\theta^*(\alpha)) \in \Delta(\mathfrak{a}_1)$.

Similarly we compute the restricted σ-diagram.

Remark 15 *Many of these restricted σ- and θ-diagrams are new to our computations, i.e., they were not dealt with in Daniel and Helminck (2004) and their fine structure must also be computed.*

Algorithm 3 *Alternatively using the restricted θ-diagram, we could compute λ in $\Delta(\mathfrak{a})$ as the projections of roots in $\Delta(\mathfrak{a}_1) - \Delta_0^{\sigma}(\theta)$ in these steps.*

- *(i) If the white dot in the restricted θ-diagram, representing $\beta \in \Delta(\mathfrak{a}_1) - \Delta_0^{\sigma}(\theta)$ is not adjacent to a black dot then $\lambda = \pi_2(\beta + \theta^*(\beta))$.*
- *(ii) If the white dot in the restricted θ-diagram, representing $\beta \in \Delta(\mathfrak{a}_1) - \Delta_0^{\sigma}(\theta)$ is adjacent to a black dot then $\lambda = \pi_2(\beta + w_0(\theta)\theta^*(\beta))$.*

Similarly, using the restricted σ-diagram, $\lambda \in \Delta(\mathfrak{a})$ can be computed as the projection of roots in $\Delta(\mathfrak{a}_2) - \Delta_0^{\theta}(\sigma)$.

5.4 Step 4

Input: Type of (σ, θ)-diagram, β in $\Delta(\mathfrak{a}_1)$, γ in $\Delta(\mathfrak{a}_2)$ and λ in $\Delta(\mathfrak{a})$.

Output: List of positive roots in $\Phi(\mathfrak{t})$ that project down to $\lambda \in \Delta(\mathfrak{a})$, $\beta \in \Delta(\mathfrak{a}_1)$, $\gamma \in \Delta(\mathfrak{a}_2)$. List of positive roots in $\Phi(\mathfrak{a}_1)$ that project down to $\lambda \in \Delta(\mathfrak{a})$. List of positive roots in $\Phi(\mathfrak{a}_2)$ that project down to $\lambda \in \Delta(\mathfrak{a})$.

We compute the multiplicity of β, γ, and λ in each basis. To do this we need to introduce a little notation.

$$\begin{aligned}
\Phi(\lambda) &= \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda\},\ m_\lambda = |\Phi(\lambda)| \\
\Phi_\sigma(\beta) &= \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}_1} = \beta\},\ m_\beta^\sigma = |\Phi_\sigma(\beta)| \\
\Phi_\theta(\gamma) &= \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}_2} = \gamma\},\ m_\gamma^\theta = |\Phi_\theta(\gamma)| \\
\Phi_1(\lambda) &= \{\beta \in \Phi(\mathfrak{a}_1) \mid \beta|_{\mathfrak{a}} = \lambda\},\ m_\lambda^1 = |\Phi_1(\lambda)| \\
\Phi_2(\lambda) &= \{\gamma \in \Phi(\mathfrak{a}_2) \mid \gamma|_{\mathfrak{a}} = \lambda\},\ m_\lambda^2 = |\Phi_2(\lambda)|
\end{aligned}$$

$\Phi_\sigma(\beta)$ and $\Phi_\theta(\gamma)$ can be computed by methods in Daniel and Helminck (2004) since they involve only one involution.

Algorithm 4 *Computing* $\Phi(\lambda)$.

(i) *If* $\lambda = \pi(\alpha)$, *where* α *corresponds to a white dot in the* (σ, θ)*-diagram, is not adjacent to a black dot then*
  (a) $\Phi(\lambda) = \{\alpha\}$ *if* $\theta^*(\alpha) = \sigma^*(\alpha) = \alpha$.
  (b) $\Phi(\lambda) = \{\alpha, \sigma^*(\alpha)\}$, *if* $\sigma^*(\alpha) \neq \alpha$.
  (c) $\Phi(\lambda) = \{\alpha, \theta^*(\alpha)\}$, *if* $\theta^*(\alpha) \neq \alpha$ *and* $\sigma^*(\alpha) = \alpha$.

(ii) *If* $\lambda = \pi(\alpha)$ *is adjacent to one black dot which corresponds to a root in a closed subroot system* $\widehat{\Phi}_1$ *of* $\Phi_0(\sigma, \theta)$, *then* $\Phi(\lambda) = \{\alpha + \widehat{\alpha}_1 \in \Phi(\mathfrak{t}) \mid \widehat{\alpha}_1 \in \widehat{\Phi}_1^+, \langle \alpha, \widehat{\alpha}_1 \rangle \geq 0\}$.

(iii) *If* $\lambda = \pi(\alpha)$ *is adjacent to two black dot which correspond to roots in closed subroot systems* $\widehat{\Phi}_1$ *and* $\widehat{\Phi}_2$, *respectively, of* $\Phi_0(\sigma, \theta)$, *then* $\Phi(\lambda) = \{\alpha + \widehat{\alpha}_1 + \widehat{\alpha}_2 \in \Phi(\mathfrak{t}) \mid \widehat{\alpha}_i \in \widehat{\Phi}_i^+, \langle \alpha, \widehat{\alpha}_i \rangle \geq 0$ *for* $i = 1, 2\}$.

Algorithm 5 *Computing* $\Phi_1(\lambda)$.

(i) *If* $\lambda = \pi_2(\beta)$, *where* β *corresponds to a white dot in the restricted* θ*-diagram, is not adjacent to a black dot then*
  (a) $\Phi_1(\lambda) = \{\beta, \theta^*(\beta)\}$ *if* $\theta^*(\beta) \neq \beta$.
  (b) $\Phi_1(\lambda) = \{\beta\}$ *if* $\theta^*(\beta) = \beta$.

(ii) *If* $\lambda = \pi_2(\beta)$ *is adjacent to one black dot which corresponds to a root in a closed subroot system* $\widehat{\Phi}_1$ *of* $\Phi_0^\sigma(\theta)$*, then* $\Phi_1(\lambda) = \{\beta + \widehat{\beta}_1 \in \Phi(\mathfrak{a}_1) \mid \widehat{\beta}_1 \in \widehat{\Phi}_1^+, \langle \beta, \widehat{\beta}_1 \rangle \geq 0\}$.

(iii) *If* $\lambda = \pi_2(\beta)$ *is adjacent to two black dots which correspond to roots in closed subroot systems* $\widehat{\Phi}_1$ *and* $\widehat{\Phi}_2$*, respectively, of* $\Phi_0^\sigma(\theta)$*, then* $\Phi_1(\lambda) = \{\beta + \widehat{\beta}_1 + \widehat{\beta}_2 \in \Phi(\mathfrak{a}_1) \mid \widehat{\beta}_i \in \widehat{\Phi}_i^+, \langle \beta, \widehat{\beta}_i \rangle \geq 0$ *for* $i = 1, 2\}$.

By similar methods, we compute $\Phi_2(\lambda)$.

5.5 Step 5

Input: Type of (σ, θ)-diagram, β in $\Delta(\mathfrak{a}_1)$, γ in $\Delta(\mathfrak{a}_2)$ and λ in $\Delta(\mathfrak{a})$.

Output: Weyl group representatives for s_β, s_γ, and s_λ

From Daniel and Helminck (2004) , we have that $W(\mathfrak{a}_1) \simeq W_1(\sigma)/W_0(\sigma)$ and $W(\mathfrak{a}_2) \simeq W_1(\theta)/W_0(\theta)$. We compute

(i) Representatives w_i^β in $W_1(\sigma)$ for s_{β_i} in $W(\mathfrak{a}_1)$.
(ii) Representatives w_i^γ in $W_1(\theta)$ for s_{γ_i} in $W(\mathfrak{a}_2)$.

From Theorem 10, we have that

$$W(\mathfrak{a}) \simeq W_1(\sigma, \theta)/W_0(\sigma, \theta) \simeq W_1^\sigma(\sigma, \theta)/W_0(\sigma, \theta) \simeq W_1^\theta(\sigma, \theta)/W_0(\sigma, \theta).$$

We compute

(i) Representatives w_i in $W_1(\sigma, \theta)$ for each s_{λ_i} in $W(\mathfrak{a})$.
(ii) Representatives w_i^σ in $W_1^\sigma(\sigma, \theta)$ for each s_{λ_i} in $W(\mathfrak{a})$.
(iii) Representatives w_i^θ in $W_1^\theta(\sigma, \theta)$ for each s_{λ_i} in $W(\mathfrak{a})$.

5.6 Step 6

Input: Type of (σ, θ)-diagram, bases for $\Delta(\mathfrak{a}_1)$, $\Delta(\mathfrak{a}_2)$ and $\Delta(\mathfrak{a})$.

Output: Complete list of roots in $\Phi(\mathfrak{a}_1)^+$, $\Phi(\mathfrak{a}_2)^+$, and $\Phi(\mathfrak{a})^+$

We compute the list of positive roots in $\Phi(\mathfrak{a}_1)$ and $\Phi(\mathfrak{a}_2)$ by using the weyl group representatives w_i^β and w_i^γ applied to all β in $\Delta(\mathfrak{a}_1)$ and all γ in $\Delta(\mathfrak{a}_2)$ and compute the list of all positive roots in $\Phi(\mathfrak{a})$ by using the weyl group representatives w_i (or w_i^σ ,w_i^θ) applied to all λ in $\Delta(\mathfrak{a})$.

5.7 Step 7

Input: Type of (σ,θ)-diagram, complete list of roots in $\Phi(\mathfrak{a}_1)^+$, $\Phi(\mathfrak{a}_2)^+$, and $\Phi(\mathfrak{a})^+$

Output: List of positive roots in $\Phi(\mathfrak{t})$ that project down to $\lambda \in \Phi(\mathfrak{a})$, $\beta \in \Phi(\mathfrak{a}_1)$, $\gamma \in \Phi(\mathfrak{a}_2)$. List of positive roots in $\Phi(\mathfrak{a}_1)$ that project down to $\lambda \in \Phi(\mathfrak{a})$. List of positive roots in $\Phi(\mathfrak{a}_2)$ that project down to $\lambda \in \Phi(\mathfrak{a})$.

The next step is to compute the multiplicity of all roots in each restriction. A nice way to do this is to note that

$$\begin{array}{ll}
\Phi(s_{\lambda_i}(\lambda_j)) = \Phi(\lambda_j - \langle\lambda_j,\lambda_i\rangle\lambda_i) = w_i(\Phi(\lambda_j)) & \\
\Phi_\sigma(s_{\beta_i}(\beta_j)) = w_i^\beta(\Phi_\sigma(\beta_j)) & \Phi_\theta(s_{\gamma_i}(\gamma_j)) = w_i^\gamma(\Phi_\theta(\gamma_j)) \\
\Phi_1(s_{\lambda_i}(\lambda_j)) = w_i^\sigma(\Phi_1(\lambda_j)) & \Phi_2(s_{\lambda_i}(\lambda_j)) = w_i^\theta(\Phi_2(\lambda_j))
\end{array}$$

Essentially, we use the same weyl group representatives that we used above in 5.6 to compute the the list of positive roots in each restriction, but this time we apply them set-wise to the projection spaces from 5.4.

5.8 Step 8

Input: Type of (σ,θ)-diagram and a restricted root in one of the restricted root systems $\Phi(\mathfrak{a})$, $\Phi(\mathfrak{a}_1)$ or $\Phi(\mathfrak{a}_2)$.

Output: Root spaces as the direct sum of subsequent root spaces.

For each restriction, we write the root spaces as the direct sum of subsequent root spaces using:

$$\begin{array}{ll}
\mathfrak{g}_\beta = \sum_{i=1}^{m_\beta^\sigma} \mathfrak{g}_{\alpha_i} \text{ where } \alpha_i \in \Phi_\sigma(\beta) & \mathfrak{g}_\gamma = \sum_{i=1}^{m_\gamma^\theta} \mathfrak{g}_{\alpha_i} \text{ where } \alpha_i \in \Phi_\theta(\gamma) \\
\mathfrak{g}_\lambda = \sum_{i=1}^{m_\lambda} \mathfrak{g}_{\alpha_i} \text{ where } \alpha_i \in \Phi(\lambda) & \mathfrak{g}_\lambda = \sum_{i=1}^{m_\lambda^1} \mathfrak{g}_{\beta_i} \text{ where } \beta_i \in \Phi_1(\lambda) \\
\mathfrak{g}_\lambda = \sum_{i=1}^{m_\lambda^2} \mathfrak{g}_{\gamma_i} \text{ where } \gamma_i \in \Phi_2(\lambda) &
\end{array}$$

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