COMPUTING ORBITS OF MINIMAL PARABOLIC *k*-SUBGROUPS ACTING ON SYMMETRIC *k*-VARIETIES

A.G. HELMINCK

Abstract. The orbits of a minimal parabolic k-subgroup acting on a symmetric k-variety are essential to the study of symmetric k-varieties and their representations. This paper gives an algorithm to compute these orbits and most of the combinatorial structure of the orbit decomposition. There are several ways to describe these orbits, see for example [22, 28, 35]. Fundamental in all these descriptions are the associated twisted involutions in the restricted Weyl group. These describe the combinatorial structure of the orbit decomposition in a similar manner to the special case of orbits of a Borel subgroup acting on a symmetric variety, see [19]. However, the orbit structure in the general case is much more complicated than the special case of orbits of a Borel subgroup.

In this paper we first modify the characterization of the orbits of minimal parabolic k-subgroups acting on a symmetric k-varieties given in [22], to illuminate the similarity to the one for orbits of a Borel subgroup acting on a symmetric variety in [19]. Using this characterization we show how the algorithm in [19] can be adjusted and extended to compute these twisted involutions as well.

1. Introduction

Symmetric varieties are defined as the spherical homogeneous spaces G/H with G a reductive algebraic group and H the fixed point group of an involution θ. They occur in many problems in representation theory, including the study of Harish-Chandra modules with $k = \mathbb{C}$ (see for example [44]), geometry (see [31], [21] and [1]) and singularity theory (see [26] and [37]).

When G and θ are defined over a field k which is not necessarily algebraically closed, then the variety G_k/H_k is called a *symmetric k-variety*. Here G_k (resp. H_k) denotes the set of k-rational points of G (resp. H). These also occur in several areas including the cohomology of arithmetic subgroups (see [43]) and representation theory. It is in this last area that these symmetric k-varieties are best known. The representation theory and Plancherel formulas of symmetric k-varieties over the real numbers (also called reductive symmetric spaces) has been studied extensively in the last few decades. Most of the early work was done by Harish-Chandra, who gave the Plancherel formulas for the Riemannian symmetric spaces and the groups

1991 *Mathematics Subject Classification.* 68Q40, 20G15, 20G20, 22E15, 22E46, 43A85.

case. Expanding on Harish-Chandra's ideas the representation theory for the general reductive symmetric spaces has been carried out by a number of mathematicians including Flensted-Jensen, Oshima, Sekiguchi, Matsuki, Brylinski, Delorme, Schlichtkrul and van den Ban (see [9, 30, 29, 8, 2, 3]). Once the work on the real symmetric k-varieties and their representations has been completed, a natural next case to study will be the $\mathfrak{p}$-adic symmetric k-varieties and their representations. Some people have already started on this with a number of interesting results (see [24, 32, 20]). Another case which has been studied is the representations associated to symmetric k-varieties defined over a finite field (see [25] and [10]).

The one essential tool in all of the above studies of symmetric k-varieties and their representations is a description of the geometry of the orbits of a minimal parabolic k-subgroup acting on the symmetric k-variety. A description of these orbits naturally leads to a description of most of the fine structure of these symmetric k-varieties including the restricted root system of the symmetric k-variety. Besides representation theory, these orbits are also of importance in a number of other areas, including geometry (see [31] and [26]) and the cohomology of arithmetic subgroups (see [43]).

The orbits of a minimal parabolic k-subgroup acting on the symmetric k-variety have been studied for various base fields by many mathematicians. For $k = \bar{k}$ a description was given by Springer [39]. For $k = \mathbb{R}$ descriptions were given by Matsuki [28] and Rossmann [35]. For arbitrary fields of characteristic not 2 a description was given by Helminck and Wang [22]. In practice it is quite difficult and cumbersome to actually compute these orbits and their closures. Fortunately most of the geometry of these orbits and their closures can be described combinatorially and therefore most of this work could be done by a computer. For k algebraically closed (of characteristic not 2) we gave in [19] an efficient algorithm to compute the orbits and their closures. In this paper we show how the algorithm in [19] can be modified and extended to compute the orbits for a number of base fields, including $k = \mathbb{R}$. Although the structure of symmetric k-varieties and the related k-groups is much more complicated than that of symmetric varieties over algebraically closed fields the algorithm to compute the above orbits is similar to the one for symmetric varieties over algebraically closed fields given in [19]. Most of the code developed for that situation can be modified to compute the orbits in this case as well.

Essential in the algorithm to compute the orbits in [19] was the description of the image and fibers of the natural map $\varphi_k : V_k \to \mathcal{I}$, where V_k denotes the set of orbits of a minimal parabolic k-subgroup acting on the symmetric k-variety and $\mathcal{I}$ the set of twisted involutions in the Weyl group. Most of the combinatorial structure of the orbits in V_k depended only on properties of the associated twisted involutions in $\varphi_k(V_k) \subset \mathcal{I}$. For arbitrary base fields we can define this map again and also in this case most of the combinatorial structure can be adapted from the twisted involutions to the orbits and their closures. As can be expected there are a few additional difficulties in describing the twisted involutions contained in $\varphi_k(V_k)$, but the main problems occur when one tries to compute the orbits in V_k. For most base fields the fibers of φ_k are infinite, which makes it impossible to compute V_k. For a number of base fields, including local fields and finite fields there are finitely many orbits and a computation of the orbits becomes feasible. The algorithm to compute the orbits in V_k necessarily also depends on a classification of the k-involutions involved. These have been fully classified for $k = \bar{k}$ and $k = \mathbb{R}$. For these fields the algorithm will be able to compute the orbits. For k a $\mathfrak{p}$-adic field, a finite field or a number field, partial classifications of the k-involutions exist, which actually enable us to compute the twisted involutions and the corresponding combinatorial structure in these cases. For $k = \mathbb{Q}_\mathfrak{p}$ a classification of the k-involutions is nearly finished, which will enable us to compute the orbits in V_k in that case as well.

In this paper we will use the description of the orbit structure in [22] and and prove a number of additional results about these orbits which are useful for the actual computation. Using these results we can modify the algorithm in [19] to compute the orbits for a number of other base fields as well. This algorithm can be implemented again on a computer using existing symbolic manipulation programs or by writing an independent program. Probably the easiest would be to write an extension to the program LiE. This would also give a relatively efficient implementation. To obtain a more efficient implementation one will need to write an independent program. Note that most of the remaining structure of a symmetric variety, like the restricted root system, will also follow from the above structure.

A brief outline of this paper is as follows. In section 2 we introduce notation and review a few generalities about the orbits of a minimal parabolic k-subgroup P acting on the symmetric k-variety G_k/H_k. We also discuss the action of the Weyl group W on these orbits. In section 3 we discuss the connection of these orbits with twisted involutions in the Weyl group. In particular we analyze the natural maps $\varphi_k : V_k \to \mathcal{I} \subset W$ and $\varphi : V \to \mathcal{I}$, where V_k is the set of orbits $P_k\backslash G_k/H_k$, V the set of orbits $P\backslash G/H$ and $\mathcal{I}$ the set of twisted involutions in the Weyl group W. The image $\varphi_k(V_k)$ is contained in $\varphi(V)$ and the combinatorial structure from $\varphi(V)$ carries over to $\varphi_k(V_k)$. We will use these maps φ and φ_k to classify the sets of orbits V and V_k for a number of base fields. For k algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers, a finite field or a number field we will compute $\varphi(V)$. For $k = \bar{k}$ and $k = \mathbb{R}$ we will compute $\varphi_k(V_k)$ and V_k as well. Most of these results are from [22] and [17].

In section 4 we discuss the Γ-, θ and (Γ, θ)-indices related to the k-structure and the k-involutions. In section 5 we introduce the θ-singular and (θ, k)-singular involutions in the restricted Weyl group and prove a number of results about these. These involutions will give us a set of representatives for the W-orbits in $\varphi(V)$

and $\varphi_k(V_k)$. For this see section 6 and 8. In section 6 we also look at the Weyl group $W(A)$ of a θ-stable maximal k-split torus containing a maximal (θ, k)-split torus. In this case we can show that there is a bijective correspondence between the twisted W-orbits in $\varphi(V)$ and the W-conjugacy classes of θ-singular involutions in the Weyl group $W(A)$.

In section 7 we show how the algorithm in [19] can be modified to compute $\varphi(V)$ for k algebraically closed, the real numbers, a $\mathfrak{p}$-adic field, a finite field or a number field. This computation only depends on the (Γ, θ)-index of the k-involution.

Finally in section 8 we show how the algorithm in section 7 can be extended to compute $\varphi_k(V_k)$ and V_k as well, provided one has a classification for the k-involutions for that field. This algorithm can be used to compute $\varphi_k(V_k)$ and V_k for $k = \mathbb{R}$. A discussion of this case is included as well.

2. Preliminaries and Recollections

In this section we set the notations and discuss the relation between the orbits of minimal parabolic k-subgroup acting on a symmetric k-variety and the H_k-conjugacy classes of θ-stable maximal k-split tori. For this we will rephrase the characterization of these orbits in [22, 17] by giving another characterization of the orbits, which is geared more toward the conjugacy classes of θ-stable maximal k-split tori. We will also prove a number of additional results. Our basic reference for reductive groups will be the papers of Borel and Tits [5, 6] and also the books of Borel [4], Humphreys [23] and Springer [38]. We shall follow their notations and terminology.

2.1. **Notations.** Given an algebraic group G, the identity component is denoted by G^0. We use $L(G)$ (resp. $\mathfrak{g}$, the corresponding lower case German letter) for the Lie algebra of G. If H is a subset of G, then we write $N_G(H)$ (resp. $Z_G(H)$) for the normalizer (resp. centralizer) of H in G. We write $Z(G)$ for the center of G. The commutator subgroup of G is denoted by $D(G)$ or $[G, G]$.

An algebraic group defined over k shall also be called an algebraic k-group. For an extension K of k, the set of K-rational points of G is denoted by G_K or $G(K)$.

If G is a reductive k-group and A a torus of G then we denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of A (resp. one-parameter subgroups of A) and by $\Phi(A) = \Phi(G, A)$ the set of the roots of A in G. The group $X^*(A)$ can be put in duality with $X_*(A)$ by a pairing $\langle \cdot\, , \,\cdot \rangle$ defined as follows: if $\chi \in X^*(A)$, $\lambda \in X_*(A)$, then $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $t \in k^*$. Let $W(A) = W(G, A) = N_G(A)/Z_G(A)$ denote the Weyl group of G relative to A. If $\alpha \in \Phi(G, A)$, then let U_α denote the unipotent subgroup of G corresponding to α. If A is a maximal torus, then U_α is one-dimensional. Given a quasi-closed subset ψ of $\Phi(G, A)$, the group G_ψ (resp.

G^*_ψ) is defined in [5, 3.8]. If G^*_ψ is unipotent, ψ is said to be unipotent and often one writes U_ψ for G^*_ψ.

2.2. Throughout the paper G will denote a connected reductive algebraic k-group, θ an involution of G defined over k, $G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of θ and H a k-open subgroup of G_θ. The involution θ is also called a k-involution of G. The variety G/H is called a symmetric variety and the variety G_k/H_k is called a symmetric k-variety.

Given $g, x \in G$, the *twisted action* associated to θ is given by $(g, x) \mapsto g * x = gx\theta(g)^{-1}$. Let $Q = \{g^{-1}\theta(g) \mid g \in G\}$ and $Q' = \{g \in G \mid \theta(g) = g^{-1}\}$. The set Q is contained in Q'. Both Q and Q' are invariant under the twisted action associated to θ. There are only a finite number of twisted G-orbits in Q' and each such orbit is closed (see [33]). In particular, Q is a connected closed k-subvariety of G. Define a morphism $\tau : G \to G$ by

$$\tau(x) = x\theta(x^{-1}), \quad (x \in G). \tag{1}$$

The image $\tau(G) = Q$ is a closed k-subvariety of G and τ induces an isomorphism of the coset space G/G_θ onto $\tau(G)$. Note that $\tau(x) = \tau(y)$ if and only if $y^{-1}x \in G_\theta$ and $\theta(\tau(x)) = \tau(x)^{-1}$ for $x \in G$.

2.3. If $T \subset G$ is a torus and $\sigma \in \mathrm{Aut}(G, T)$ an involution, then we write $T^+_\sigma = (T \cap G_\sigma)^0$ and $T^-_\sigma = \{x \in T \mid \sigma(x) = x^{-1}\}^0$. It is easy to verify that the product map

$$\mu : T^+_\sigma \times T^-_\sigma \to T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular $T = T^+_\sigma T^-_\sigma$ and $T^+_\sigma \cap T^-_\sigma$ is a finite group. (In fact it is an elementary abelian 2-group.) The automorphisms of $\Phi(G, T)$ and $W(G, T)$ induced by σ will also be denoted by σ. If $\sigma = \theta$ we reserve the notation T^+ and T^- for T^+_θ and T^-_θ respectively. For other involutions of T, we shall keep the subscript.

Recall from [11] that a torus A is called θ-*split* if $\theta(a) = a^{-1}$ for every $a \in A$. If A is a maximal θ-split torus of G, then $\Phi(G, A)$ is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$ (see [33]). This is the root system associated with the symmetric variety G/H. To the symmetric k-variety G_k/H_k one can also associate a natural root system. To see this we consider the following tori:

Definition 2.4. A k-torus A of G is called (θ, k)-split if it is both θ-split and k-split.

Consider a maximal (θ, k)-split torus A in G. In [22, 5.9] it was shown that $\Phi(G, A)$ is a root system and $N_{G_k}(A)/Z_{G_k}(A)$ is the Weyl group of this root system. We can also obtain this root system by restricting the root system of G_k. Namely let $A^0 \supset A$ be a θ-stable maximal k-split torus of G. Then $A = (A^0)^-_\theta$ and $\Phi(G, A)$ can be identified with $\overline{\Phi}_\theta = \{\alpha|A \neq 0 \mid \alpha \in \Phi(G, A^0)\}$.

2.5. ***P_k*-orbits on *G_k/H_k*.** Let P be a minimal parabolic k-subgroup of G. There are several way's in which one can characterize the double cosets $P_k\backslash G_k/H_k$. One can characterize them as the P_k-orbits on the symmetric k-variety G_k/H_k (using the θ-twisted action), one can take the H_k-orbits on the flag variety G_k/P_k or one can consider the $P_k \times H_k$-orbits on G_k. All these characterizations are essentially the same. For more details see [22]. We will use the $P_k \times H_k$-orbits on G_k to characterize $P_k\backslash G_k/H_k$. In this subsection we will briefly review this characterization.

Let A be a θ-stable maximal k-split torus of P, $N = N_G(A)$, $Z = Z_G(A)$ and $W = W(A) = N_G(A)/Z_G(A)$ the corresponding Weyl group. As in [22, 6.7] set $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_k\}$. The group $Z_k \times H_k$ acts on $\mathcal{V}_k$ by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z_k \times H_k$, $y \in \mathcal{V}_k$. Let V_k be the set of $(Z_k \times H_k)$-orbits on $\mathcal{V}_k$. Similarly let $\mathcal{V} = \{x \in G \mid \tau(x) \in N\}$. Then $Z \times H$ acts on $\mathcal{V}$ by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z \times H$, $y \in \mathcal{V}$. Denote the set of $(Z \times H)$-orbits on $\mathcal{V}$ by V. If $v \in V_k$ (resp. V), we let $x(v) \in \mathcal{V}_k$ (resp. V) be a representative of the orbit v in $\mathcal{V}_k$ (resp. V). These sets V_k (resp. V) are essential in the study of orbits of minimal parabolic subgroups on the symmetric k-variety G_k/H_k. The inclusion map $\mathcal{V}_k \to G_k$ induces a bijection of the set V_k of $(Z_k \times H_k)$-orbits on $\mathcal{V}_k$ onto the set of $(P_k \times H_k)$-orbits on G_k (see [22]). Similarly the set V is associated with the set of $H \times P$ orbits on G (see [17]). The set V is finite, but the set V_k is in general infinite. In a number of cases one can show that there are only finitely many $(P_k \times H_k)$-orbits on G_k. If k is algebraically closed, the finiteness of V_k was proved by Springer [39]. The finiteness of the orbit decomposition for $k = \mathbb{R}$ was discussed by Wolf [45], Rossmann [35] and Matsuki [28]. For general local fields this result can be found in Helminck-Wang [22]. An example that in most cases the set V_k is infinite can be found in [22, 6.12].

2.6. ***W*-action on *V* and *V_k*.** The Weyl group W acts on both V and V_k. The action of W on V (resp. V_k) is defined as follows. Let $v \in V$ and let $x = x(v)$. If $n \in N$, then $nx \in \mathcal{V}$ and its image in V depends only on the image of n in W. We thus obtain a (left) action of W on V, denoted by $(w, v) \to w \cdot v$ $(w \in W,\ v \in V)$.

To get another description of $\mathcal{V}_k$ and $\mathcal{V}$ we define first the following.

Definition 2.7. A torus A of G is called a *quasi k-split torus* if A is conjugate under G with a k-split torus of G.

Since all maximal k-split tori of G are conjugate, also all maximal quasi k-split tori of G are conjugate. If A is a maximal quasi k-split torus of G, then $\Phi(G, A)$ is a root system in $(X^*(A), X_0)$ in the sense of [5, §2.1] where X_0 is the set of characters of A which are trivial on $(A \cap [G, G])^0$; moreover the Weyl group of $\Phi(G, A)$ is $N_G(A)/Z_G(A)$.

2.8. Let $\mathcal{A}$ be the variety of maximal quasi k-split tori of G. This is an affine variety, isomorphic to $G/N_G(A)$, on which θ acts. Let $\mathcal{A}^\theta$ be the fixed point set of

θ, i.e. the set of θ-stable maximal quasi k-split tori. It is an affine variety on which H acts by conjugation.

Similarly let $\mathcal{A}_k$ denote the set of maximal k-split tori of G and let $\mathcal{A}_k^\theta$ be the fixed point set of θ i.e., the set of θ-stable maximal k-split tori. The group H_k acts on $\mathcal{A}_k^\theta$ by conjugation.

If $x \in \mathcal{V}_k$, then $x^{-1}Ax$ is again a maximal k-split torus and conversely any θ-stable maximal k-split torus in $\mathcal{A}_k^\theta$ can be written as $x^{-1}Ax$ for some $x \in \mathcal{V}_k$. Similarly any θ-stable maximal quasi k-split torus is of the form $x^{-1}Ax$ for some $x \in \mathcal{V}$.

If $v \in V_k$ (resp. $v \in V$), then $x(v)^{-1}Ax(v) \in \mathcal{A}_k^\theta$ (resp. $\mathcal{A}^\theta$). This determines maps of V_k, resp. V to the orbit sets $\mathcal{A}_k^\theta/H_k$ resp. $\mathcal{A}^\theta/H$. It is easy to check that these maps are independent of the choice of the representative $x(v)$ for v and are constant on W-orbits. So we also get maps of orbit sets: $\gamma_k : V_k/W \to \mathcal{A}_k^\theta/H_k$ and $\gamma : V/W \to \mathcal{A}^\theta/H$. In fact we have a bijection:

Proposition 2.9. *Let G, $\mathcal{A}^\theta$, $\mathcal{A}_k^\theta$, γ and γ_k be as above. Then we have the following.*

(i) $\gamma_k : V_k/W \to \mathcal{A}_k^\theta/H_k$ is bijective.
(ii) $\gamma : V/W \to \mathcal{A}^\theta/H$ is bijective.

Proof. (i). Since any θ-stable maximal k-split torus is of the form $x^{-1}Ax$ for some $x \in \mathcal{V}_k$, the map γ_k is surjective.

Let $v_1, v_2 \in V_k$ with $\gamma_k(v_1) = \gamma_k(v_2)$. Let $x_1 = x(v_1)$, $x_2 = x(v_2) \in \mathcal{V}_k$ be representatives of v_1, v_2 and let $A_1 = x_1^{-1}Ax_1$, $A_2 = x_2^{-1}Ax_2$. Since $\gamma_k(v_1) = \gamma_k(v_2)$ there exists $h \in H_k$ such that $hA_2h^{-1} = hx_2^{-1}Ax_2h^{-1} = x_1^{-1}Ax_1 = A_1$. So we may assume that $A_1 = A_2$. But then by [22, 6.10] there exists $n \in N_G(A_1)$ such that $P_knx_1H_k = P_kx_2H_k$. If $w \in W(A_1)$ is the image of n in $W(A_1)$, then $w \cdot v_1 = v_2$, what proves (i).

The proof of (ii) follows using a similar argument as in (i). □

2.10. Let $\mathcal{A}_0^\theta$ denote the set of θ-stable quasi k-split tori of G, which are H-conjugate to a θ-stable maximal k-split torus. Then $\mathcal{A}_0^\theta/H \subset \mathcal{A}^\theta/H$ can be identified with the set of H-conjugacy classes of θ-stable maximal k-split tori of G. There is a natural map

$$\zeta : \mathcal{A}_k^\theta/H_k \to \mathcal{A}^\theta/H, \tag{2}$$

sending the H_k-conjugacy class of a θ-stable maximal k-split torus onto its H-conjugacy class. Then $\mathcal{A}_0^\theta/H$ is precisely the image of ζ. On the other hand the inclusion map $\mathcal{V}_k \to \mathcal{V}$ induces a map $\eta : V_k \to V$, where η maps the orbit Z_kgH_k onto ZgH. This map is W-equivariant. Denote the corresponding orbit map by $\delta : V_k/W \to V/W$ and write V_0 for the image of η in V. Denote the restriction of γ to V_0/W by γ_0. Then γ_0 maps V_0/W onto $\mathcal{A}_0^\theta/H$. This all leads to the following diagram:

$$\begin{array}{ccc} V_k/W & \xrightarrow{\gamma_k} & \mathcal{A}_k^\theta/H_k \\ \big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle\zeta} \\ V_0/W & \xrightarrow{\gamma_0} & \mathcal{A}_0^\theta/H \end{array}$$

Since γ_0 and γ_k are bijections, it follows that there is a bijection between the fibers of δ and ζ. For a θ-stable maximal k-split torus A, the fiber $\zeta^{-1}(A)$ consists of all θ-stable maximal k-split tori, which are H-conjugate to A, but not H_k-conjugate. That these fibers can be infinite can be seen from example [22, 6.12].

3. Twisted involutions

Another way to characterize the W-orbits in V_k and V is by characterizing the image and fibers of the natural map from V into W, induced by the map $\tau|\mathcal{V} : \mathcal{V} \to N_G(T)$. This map also enables us to port the natural combinatorial structure on the image (contained in the set of twisted involutions in W) to V and V_k. Similar to the case of orbits of a Borel subgroup acting on a symmetric variety as in [19, 34] this will enable us to describe most of the combinatorial structure involved.

3.1. Recall that an element $a \in W(A)$ is a *twisted involution* if $\theta(a) = a^{-1}$ (see [39, §3] or [22, §7]). Let

(3) $$\mathcal{I} = \mathcal{I}_\theta = \mathcal{I}(W(A), \theta) = \{w \in W \mid \theta(w) = w^{-1}\}$$

be the set of twisted involutions in $W(A)$. If $v \in V$, then $\varphi(v) = \tau(x(v))Z_G(A) \in W(A)$ is a twisted involution. The element $\varphi(v) \in \mathcal{I}$ is independent of the choice of representative $x(v) \in \mathcal{V}$ for v. So this defines a map $\varphi : V \to \mathcal{I}$.

We can define a map $\varphi_k : V_k \to \mathcal{I}$ in a similar manner. Namely if $v \in V_k$, then let $\varphi_k(v) = \tau(x(v))Z_G(A) \in W(A)$. Again this is a twisted involution. From the above observations we get the following relation between φ_k and φ:

(4) $$\varphi_k = \varphi \circ \eta$$

The maps φ (resp. φ_k) play an important role in the study of the Bruhat order on V (resp. V_k). For more details, see [34] and [17].

3.2. The Weyl group W acts also on $\mathcal{I}$. This action comes from the *twisted action* of W on (the set) W, which is defined as follows: if $w, w_1 \in W$, then $w * w_1 = ww_1\theta(w)^{-1}$. If $w_1 \in W$, then $W * w_1 = \{w * w_1 \mid w \in W\}$ is the *twisted W-orbit* of w_1. Now $\mathcal{I}$ is stable under the twisted action, so that we get a twisted action of W on $\mathcal{I}$. The images of φ and φ_k in $\mathcal{I}$ are unions of twisted W-orbits, as follows from the following result:

Lemma 3.3. *Let $w \in W$ and $v \in V$ (resp. V_k). Then $\varphi(w \cdot v) = w * \varphi(v)$ (resp. $\varphi_k(w \cdot v) = w * \varphi_k(v)$).*

Proof. The proof is immediate from the above observations. □

3.4. From this result it follows now that the maps $\varphi : V \to \mathcal{I}$ (resp. $\varphi_k : V_k \to \mathcal{I}$) are equivariant with respect to the action of W on V (resp. V_k) and the twisted action of W on $\mathcal{I}$. So there are natural orbit maps $\phi : V/W \to \mathcal{I}/W$ and $\phi_k : V_k/W \to \mathcal{I}/W$. From (4) and 2.10 we get the following relation between ϕ and ϕ_k.

$$\phi_k = \phi \circ \delta$$

Since γ and γ_k are one-to-one, we also get embeddings of $\mathcal{A}_0^\theta/H$ and $\mathcal{A}_k^\theta/H_k$ into $\mathcal{I}/W$. This indicates that the W-orbits of twisted involutions can be used as an invariant to characterize the conjugacy classes in $\mathcal{A}_0^\theta/H$ and $\mathcal{A}_k^\theta/H_k$. In fact in Proposition 6.11 we will show that we can use conjugacy classes of involutions in the Weyl group W, instead of W-orbits of twisted involutions.

Remark 3.5. If k is algebraically closed, then it follows from [34, 2.5] that the map ϕ_k is one-to-one. So in this case the classes in $\varphi_k(V_k)/W$ completely characterize the H_k-conjugacy classes of θ-stable maximal k-split tori. The map ϕ_k is also one-to-one in a number of other cases, like the standard pairs (G_k, H_k) for $k = \mathbb{R}$. For a further discussion of this see [13, 14].

An example that the map ϕ_k is not always one-to-one can be found in 5.8.

3.6. In this paper we will need several properties of the twisted involutions. In the remainder of this section we will collect some of their properties from [22, 19] and prove some additional results. A first description of these twisted involutions, in the case that $\theta(\Phi^+) = \Phi^+$, was given by Springer in [39]. For $k = \bar{k}$ there exists a θ-stable Borel subgroup (see [40]), so then this condition is satisfied. However if $k \neq \bar{k}$ then G does not necessarily have θ-stable minimal parabolic k-subgroups. One can easily generalize the description in [39] and give a similar description of the twisted involutions, when $\theta(\Phi^+) \neq \Phi^+$. This follows essentially from the results in [22], although some of these results are not explicitly stated there. In the following we will review this characterization and prove a few additional results. First we need a few facts about real, complex and imaginary roots.

3.7. In the remainder of this section, let A be a θ-stable maximal quasi k-split torus of G, $\Phi = \Phi(A)$ the root system of A with respect to G, Φ^+ a set of positive roots of Φ, Δ the corresponding basis, $W = W(A)$ the Weyl group of A and $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$. The Weyl group W is generated by Σ. Let $E = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. If $\sigma \in \mathrm{Aut}(\Phi)$, then we denote the eigenspace of σ for the eigenvalue ξ, by $E(\sigma, \xi)$. For a subset Π of Δ denote the subset of Φ consisting of integral combinations of Π by Φ_Π. Then Φ_Π is a subsystem of Φ with Weyl group W_Π. Let w_Π^0 denote the longest element of W_Π with respect to Π.

The involution θ of G induces an automorphism of W, also denoted by θ, given by

$$\theta(w) = \theta \circ w \circ \theta, \quad w \in W.$$

If s_α is the reflection defined by α, then $\theta(s_\alpha) = s_{\theta(\alpha)}$, $\alpha \in \Phi$.

3.8. The roots of Φ can be divided in three subsets, related to the action of θ, as follows.

(a) $\theta(\alpha) \neq \pm\alpha$. Then α is called *complex* (relative to θ).
(b) $\theta(\alpha) = -\alpha$. Then α is called *real* (relative to θ).
(c) $\theta(\alpha) = \alpha$. Then α is called *imaginary* (relative to θ).

These definitions carry over to the Weyl group W and the set V. We define first real, complex and imaginary elements for the Weyl group, which then will imply similar definitions for V.

Definition 3.9. Given $w \in \mathcal{I}_\theta$, an element $\alpha \in \Phi$ is called *complex* (resp. *real*, *imaginary*) relative to w if $w\theta\alpha \neq \pm\alpha$ (resp. $w\theta\alpha = -\alpha$, $w\theta\alpha = \alpha$). We use the following notation,

$$\begin{aligned} C'(w,\theta) &= \{\alpha \in \Phi^+ \mid -\alpha \neq w\theta\alpha < 0\} \\ C''(w,\theta) &= \{\alpha \in \Phi^+ \mid \alpha \neq w\theta\alpha > 0\} \\ R(w,\theta) &= \{\alpha \in \Phi^+ \mid -\alpha = w\theta\alpha\} \\ I(w,\theta) &= \{\alpha \in \Phi^+ \mid \alpha = w\theta\alpha\}. \end{aligned}$$

We will omit θ from this notation if there is no ambiguity as to which involution we consider.

3.10. To get a similar description of the twisted involutions as in [39], we pass to another involution, which leaves Φ^+ invariant. Let $w_0 \in W$ such that

$$\theta(\Phi^+) = w_0(\Phi^+). \tag{5}$$

and let $\theta' = \theta w_0$. Instead of working with θ we can work again with θ' and w'. Similar as in [19, 4.6-4.8] we get the following results for w_0, θ' and the sets of twisted involutions $\mathcal{I}_\theta$ and $\mathcal{I}_{\theta'}$:

Proposition 3.11. *Let Φ, Φ^+, θ, w_0 and θ' be as above. Then we have the following properties:*

(i) $w_0 \in \mathcal{I}_\theta$.
(ii) $\theta'(\Phi^+) = \Phi^+$.
(iii) *θ' is an involution of Φ.*
(iv) $\mathcal{I}_{\theta'} = \mathcal{I}_\theta \cdot w_0$.

For the real, complex and imaginary roots we have again the following:

Lemma 3.12. *If $w \in \mathcal{I}_\theta$ and $w' = ww_0$, then $w'\theta' = w\theta$. In particular we have*

$$\begin{array}{rclcrcl} I(w,\theta) & = & I(w',\theta'), & & R(w,\theta) & = & R(w',\theta') \\ C'(w,\theta) & = & C'(w',\theta'), & & C''(w,\theta) & = & C''(w',\theta') \end{array}$$

We also have again the following characterization of twisted involutions:

Proposition 3.13. [22, 7.9] *If $w \in \mathcal{I}_\theta$ and $w' = ww_0 \in \mathcal{I}_{\theta'}$, then there exist $s_1, \dots, s_h \in \Sigma$ and a θ'-stable subset Π of Δ satisfying the following conditions:*

(i) $w' = s_1 \dots s_h w^0_\Pi \theta'(s_h) \dots \theta'(s_1)$ *and* $l(w') = 2h + l(w^0_\Pi)$.

(ii) $w^0_\Pi \theta' \alpha = -\alpha$, $\alpha \in \Phi_\Pi$ (*i.e.* $\Phi^+_\Pi \subset R(w^0_\Pi, \theta')$).

Moreover if $w' = t_1 \dots t_m w^0_\Lambda \theta'(t_m) \dots \theta'(t_1)$, where $t_1, \dots, t_m \in \Sigma$ and Λ a θ'-stable subset of Δ satisfying conditions (i) and (ii), then $m = h$, $s_1 \dots s_h \Pi = t_1 \dots t_h \Lambda$ and

$$s_1 \dots s_h \theta'(s_h) \dots \theta'(s_1) = t_1 \dots t_h \theta'(t_h) \dots \theta'(t_1).$$

3.14. In the case of orbits of a Borel subgroup acting on a symmetric variety as in [19] the involution θ' could be lifted to an involution of G conjugate to θ. In this case θ' can again be lifted to an involution of G conjugate to θ, but this involution is mostly not a k-involution. To solve this question we will have to look at the following class of parabolic subgroups of G:

Definition 3.15. A parabolic subgroup P in G is called a *quasi parabolic k-subgroup* if there exist $x \in \mathcal{V}$ such that xPx^{-1} is a parabolic k-subgroup.

Using a similar argument as in [22, 2.4] one can show that every quasi parabolic k-subgroup of G contains a θ-stable maximal quasi k-split torus of G.

Let P be a quasi parabolic k-subgroup of G, $A \subset P$ a θ-stable maximal quasi k-split torus, $W = W(A)$, $\Phi = \Phi(A)$, and $\mathcal{I} = \mathcal{I}_\theta$ the set of twisted involutions in W. Let $w \in \mathcal{I}$ and $\xi = w\theta$. Then ξ is an involution of Φ. In the following we show when ξ can be lifted to an involution of G and when that involution is conjugate to θ. Using a similar argument as in [19, 5.1] we get the following results:

Lemma 3.16. *Let $w \in W$ and $n \in N_G(A)$ a representative. Then* $\mathrm{Int}(n)\theta$ *is an involution of G if and only if $\theta(n) = n^{-1}z$, with $z \in Z(G)$.*

Lemma 3.17. *Let $\mathcal{V}$, V, $\varphi : V \to \mathcal{I}$ be as above. Let $w \in \mathcal{I}$, $n \in N_G(A)$ a representative of w and $\xi = \mathrm{Int}(n)\theta$. Assume that $\theta(n) = n^{-1}z$, with $z \in Z(G)$. Then ξ is conjugate to θ if and only if $n \in \tau(\mathcal{V})Z(G)$.*

Proof. Assume first that $x \in G$ such that $\xi = \mathrm{Int}(x)\theta\,\mathrm{Int}(x^{-1})$. Then $\xi = \mathrm{Int}(n)\theta = \mathrm{Int}(x\theta(x)^{-1})\theta$, so $n \in \tau(\mathcal{V})Z(G)$. Conversely if $n \in \tau(\mathcal{V})Z(G)$, then let $x \in \mathcal{V}$ and $z \in Z(G)$ such that $n = x\theta(x)^{-1}z$. Then $\xi = \mathrm{Int}(n)\theta = \mathrm{Int}(x)\theta\,\mathrm{Int}(x^{-1})$ is conjugate to θ. □

Combining the above results we get:

Lemma 3.18. *Let $\mathcal{V}$, V, $\varphi : V \to \mathcal{I}$ be as above and let $w \in \mathcal{I}$. Then the following are equivalent.*

(1) *There exists a representative $n \in N_G(A)$ for w, such that $\xi = \mathrm{Int}(n)\theta$ is an involution of G conjugate to θ.*
(2) $w \in \varphi(V) \subset \mathcal{I}$.

3.19. Let P and A be as above. As in (5), let $w_0 \in W = W(A)$ be such that $\theta(\Phi^+) = w_0(\Phi^+)$ and let $\theta' = \theta w_0 = w_0^{-1}\theta$. We can solve now the question of when the involution θ' can be lifted to a conjugate of θ. By Lemma 3.18 it suffices to show that $w_0 \in \varphi(V)$. This is equivalent to the following:

Proposition 3.20. *Let $\mathcal{V}$, V, $\varphi : V \to \mathcal{I}$ and w_0 be as above. Then we have the following:*

1. *$w_0 \in \varphi(V)$ if and only if G contains a θ-stable quasi parabolic k-subgroup.*
2. *$w_0 \in \varphi_k(V_k)$ if and only if G contains a θ-stable parabolic k-subgroup.*

Proof. Assume first $w_0 \in \varphi(V)$. Let $v_0 \in V$ be such that $\varphi(v_0) = w_0$ and let $x_0 = x(v_0) \in \mathcal{V}$ be a representative of v_0. Then $\tau(x_0)$ is a representative of w_0 in $N_G(A)$. Since $\theta(\Phi^+) = w_0(\Phi^+)$, we have $\theta(P) = \tau(x_0)P\tau(x_0^{-1})$. But then $P_1 = \theta(x_0^{-1})P\theta(x_0)$ is a θ-stable quasi parabolic k-subgroup of G.

Conversely, assume $P_0 \subset G$ is a θ-stable quasi parabolic k-subgroup. By [13, 3.11] there exists $x \in \mathcal{V}$ such that $P_0 = xPx^{-1}$. Since $\theta(P_0) = P_0$ it follows that

$$\theta(P) = \theta(x)^{-1}xPx^{-1}\theta(x) = \tau(\theta(x)^{-1})P\tau(\theta(x))$$

Now $\tau(\theta(x)^{-1}) \in N_G(A)$. Let $w \in W$ be the corresponding Weyl group element. Then $\theta(\Phi^+) = w(\Phi^+)$, so $w = w_0 \in \varphi(V)$. This shows (1). The proof of (2) follows with a similar argument replacing V and $\mathcal{V}$ by V_k and $\mathcal{V}_k$. □

Remark 3.21. By [13, 3.11] there always exists a θ-stable quasi parabolic k-subgroup, but θ-stable minimal parabolic k-subgroups of G do not necessarily exist as can be seen from example 5.8. See [22, sect. 3] for a discussion of θ-stable parabolic k-subgroups.

If $\xi \in \mathrm{Aut}(G)$ with $\xi(A) = A$, then by abuse of notation we will write $\xi|\Phi$ for the action of ξ on Φ. Summarizing the above results we get now the following result.

Corollary 3.22. *Let w_0, θ' be as above. There exists a representative $n \in N_G(A)$ of w_0, such that $\xi = \mathrm{Int}(n)\theta$ is an involution of G conjugate to θ satisfying $\xi|\Phi = \theta'$. The involution $\theta' \in \mathrm{Aut}(\Phi)$ can be lifted to a k-involution if and only if G has a θ-stable parabolic k-subgroup*

3.23. **Relation between θ and θ'.** Now we have shown that θ' can be lifted to G and that it is conjugate to θ' we can show that corresponding orbit decompositions for V and $\varphi(V)$ are again similar. This all can be seen as follows.

Let P, A, $\mathcal{V}$, V, $\mathcal{I}$ and $\varphi : V \to \mathcal{I}$ be as above. Write $\Phi = \Phi(A)$ and $W = W(A)$. Take $w_0 \in W$ such that $\theta(\Phi^+) = w_0(\Phi^+)$, $n_0 = x_0\theta(x_0)^{-1} \in N_G(A) \cap \tau(G)$ a representative of w_0^{-1}, $\theta' = \mathrm{Int}(n_0)\theta$ and $H' = x_0 H x_0^{-1}$. Then H' is a closed reductive subgroup of G satisfying

$$G^0_{\theta'} \subset H' \subset G_{\theta'}.$$

Denote the actions of θ and θ' on Φ also by θ and θ'. Then $\theta' = \theta w_0 = w_0^{-1}\theta$. As for θ let $\tau' : G \to G$ be the map defined by $\tau'(x) = x\theta'(x)^{-1}$, $\mathcal{V}' = \{x \in G \mid \tau'(x) \in N_G(A)\}$, V' the set of $(Z_G(A) \times H')$-orbits in $\mathcal{V}'$, $\mathcal{I}_{\theta'}$ the set of twisted involutions of W with respect to θ' and $\varphi' : V' \to \mathcal{I}_{\theta'}$ as in 3.1.

Let $\iota_{w_0} : \mathcal{I} \to \mathcal{I}_{\theta'}$ be the right translation by w_0 and let $\delta : V \to V'$ be the map induced by the map $g \to g x_0^{-1}$ from $\mathcal{V}$ to $\mathcal{V}'$. Then $\varphi' \circ \delta = \iota_{w_0} \circ \varphi$. So we obtained the following relation between the sets $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{I}$ and $\mathcal{I}_{\theta'}$.

Lemma 3.24. *Let $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{I}$, $\mathcal{I}_{\theta'}$, x_0, n_0 and w_0 be as above. Then we have the following.*

(i) $\mathcal{V}' = \mathcal{V} \cdot x_0^{-1}$ *and* $\tau'(\mathcal{V}') = \tau(\mathcal{V}) \cdot n_0^{-1}$.
(ii) $\mathcal{I}_{\theta'} = \mathcal{I}_\theta \cdot w_0$.

3.25. **W-action on $\mathcal{I}_{\theta'}$.** As in 3.2 there is also an action of the Weyl group W on $\mathcal{I}_{\theta'}$. Namely if $w \in W$ and $a' \in \mathcal{I}_{\theta'}$, then define an action $w *' a' = w a' \theta'(w)^{-1}$. Since $\theta' = \theta w_0$ and $a' = a w_0$ for some $a \in \mathcal{I}_\theta$, we get

$$\begin{aligned} w *' a' &= w a w_0 \theta' w^{-1} \theta' = w a w_0 w_0^{-1} \theta w^{-1} \theta w_0 \\ &= w a \theta w^{-1} \theta w_0 = (w * a) w_0. \end{aligned}$$

This means that the isomorphism ι_{w_0} is equivariant with respect to the actions of W on $\mathcal{I}$ and $\mathcal{I}_{\theta'}$. This leads to the following result.

Proposition 3.26. *Let $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{I}$, $\mathcal{I}_{\theta'}$, n_0 and w_0 be as above. Then we have the following.*

(i) *The map $\iota_{w_0} : \mathcal{I} \to \mathcal{I}_{\theta'}$ induces an isomorphism between $\mathcal{I}/W$ and $\mathcal{I}_{\theta'}/W$.*
(ii) $\varphi'(V')/W \simeq \varphi(V)/W$.

Remark 3.27. In the case that G contains a θ-stable parabolic k-subgroup we can get the same results for V_k and $\varphi_k(V_k)$.

The following result is useful in the study of twisted involutions and will be used in the sequel.

Lemma 3.28. *Let $a \in \mathcal{I}_\theta$, $a' = a w_0 \in \mathcal{I}_{\theta'}$ and $w \in W$. Then $E(a'\theta', -1) = E(a\theta, -1)$ and $w(E(a\theta, -1)) = E((w * a)\theta, -1)$.*

Proof. The first statement follows from Lemma 3.12 and the second statement is immediate from the definition of the twisted W-action on $\mathcal{I}_\theta$. □

4. (Γ, θ)-index

The computation of $\varphi(V)$ and V will depend on the (Γ, θ)-index of the k-involution θ. In this section we briefly review some results about these from [15].

4.1. Let G be a reductive k-group and $\theta \in \mathrm{Aut}(G)$ a k-involution. Let A be a θ-stable maximal k-split torus, $T \supset A$ a θ-stable maximal k-torus of G, $X = X^*(T)$ and $\Phi = \Phi(T)$. There is a finite Galois extension K/k such that T splits over K. Denote the Galois group $\mathrm{Gal}(K/k)$ of K/k by Γ and let $\Gamma_0 \subset \mathrm{Aut}(X, \Phi)$ be the subgroup corresponding to the action of Γ on (X, Φ). Similarly let $\Theta = \{1, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta \mid T$ and let $\Gamma_\theta = \Gamma_0.\Theta$ the subgroup of $\mathrm{Aut}(X, \Phi)$ generated by Γ_0 and Θ. In the following $\mathcal{E} \subset \mathrm{Aut}(X, \Phi)$ will be one of Γ_0, Θ or Γ_θ.

4.2. For $\sigma \in \mathcal{E}$ and $\chi \in X$ we will also write χ^σ or $\sigma(\chi)$ for the element $\sigma.\chi \in X$. Write $W = W(\Phi)$ for the Weyl group of Φ. For any closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite group generated by the s_α for $\alpha \in \Phi_1$. Now define the following:

$$X_0 = X_0(\mathcal{E}) = \left\{ \chi \in X \;\middle|\; \sum_{\sigma \in \mathcal{E}} \chi^\sigma = 0 \right\} \tag{6}$$

This is a co-torsion free submodule of X, invariant under the action of $\mathcal{E}$. Let $\Phi_0 = \Phi \cap X_0$. This is a closed subsystem of Φ invariant under the action of $\mathcal{E}$. Denote the Weyl group of Φ_0 by W_0 and identify it with $W(\Phi_0)$. Put $W^{\mathcal{E}} = \{w \in W \mid w(X_0) = X_0\}$, $\bar{X}_{\mathcal{E}} = X/X_0(\mathcal{E})$ and let π be the natural projection from X to $\bar{X}_{\mathcal{E}}$. If we take $S = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ and $Y = X^*(S)$, then Y may be identified with $\bar{X}_{\mathcal{E}} = X/X_0$. Let $\bar{\Phi}_{\mathcal{E}} = \pi(\Phi - \Phi_0(\mathcal{E}))$ denote the set of *restricted roots of* Φ *relative to* $\mathcal{E}$.

Remark 4.3. In the case that $\mathcal{E} = \Gamma$, then X_0 is the annihilator of a maximal k-split torus S of T. Similarly in the case that $\mathcal{E} = \Theta$, then X_0 is the annihilator of a θ-split torus S of T. In both these cases $\bar{\Phi}_{\mathcal{E}}$ is the root system of $\Phi(S)$ with Weyl group $\bar{W}_{\mathcal{E}}$.

An order on (X, Φ) related to the action of $\mathcal{E}$ can be defined as follows.

Definition 4.4. A linear order $\succ$ on X which satisfies

$$\text{if } \chi \succ 0 \text{ and } \chi \notin X_0, \text{ then } \chi^\sigma \succ 0 \text{ for all } \sigma \in \mathcal{E} \tag{7}$$

is called a $\mathcal{E}$*-linear order*. A fundamental system of Φ with respect to a $\mathcal{E}$-linear order is called a $\mathcal{E}$*-fundamental system of* Φ.

A $\mathcal{E}$-linear order on X induces linear orders on $Y = X/X_0$ and X_0, and conversely, given linear orders on X_0 and on Y, these uniquely determine a $\mathcal{E}$-linear order on X, which induces the given linear orders (i.e., if $\chi \notin X_0$, then define $\chi \succ 0$ if and only if $\pi(\chi) \succ 0$).

4.5. Fix a $\mathcal{E}$-linear order $\succ$ on X, let Δ be a $\mathcal{E}$-fundamental system of Φ and let Δ_0 be a fundamental system of Φ_0 with respect to the induced order on X_0. Define $\bar{\Delta}_{\mathcal{E}} = \pi(\Delta - \Delta_0)$. This is called a *restricted fundamental system* of Φ relative to S.

There is also a natural (Weyl) group associated with the set of restricted roots, which is related to $W^{\mathcal{E}}/W_0$. Since W_0 is a normal subgroup of $W^{\mathcal{E}}$, every $w \in W^{\mathcal{E}}$ induces an automorphism of $\bar{X}_{\mathcal{E}} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ $(\chi \in X)$. Define $\bar{W}_{\mathcal{E}} = \{\pi(w) \mid w \in W^{\mathcal{E}}\}$. We call this the *restricted Weyl group*, with respect to the action of $\mathcal{E}$ on X. It is not necessarily a Weyl group in the sense of Bourbaki [7, Ch.VI,no.1]. However in the case that S is a maximal k-split, θ-split or (θ, k)-split torus, then $\bar{\Phi}_{\mathcal{E}}$ is actually a root system with Weyl group $\bar{W}_{\mathcal{E}}$.

4.6. There is a natural action of $\mathcal{E}$ on the set of $\mathcal{E}$-fundamental systems of Φ. this set. If Δ is a $\mathcal{E}$-fundamental system of Φ, and $\sigma \in \mathcal{E}$, then the $\mathcal{E}$-fundamental system $\Delta^\sigma = \{\alpha^\sigma \mid \alpha \in \Delta\}$ gives the same restricted basis as Δ, i.e. $\bar{\Delta}^\sigma = \bar{\Delta}$. Let $w_\sigma \in W_0$ be the unique element $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma \Delta$ and define a new action of $\mathcal{E}$ on (X, Φ) as follows:

$$\chi^{[\sigma]} = w_\sigma^{-1} \chi^\sigma, \quad \chi \in X, \quad \sigma \in \mathcal{E}. \tag{8}$$

It is easily verified that $\chi \to \chi^{[\sigma]}$ is an automorphism of the triple (X, Φ, Δ) and that $\chi^{[\sigma][\tau]} = \chi^{[\sigma\tau]}$ for all $\sigma,\ \tau \in \mathcal{E},\ \chi \in X$. This action of $\mathcal{E}$ on Ψ is essentially determined by Δ, Δ_0 and $[\sigma]$. Following Tits [42] the quadruple $(X,\ \Delta,\ \Delta_0,\ [\sigma])$ is called an *index of* $\mathcal{E}$ or an $\mathcal{E}$-index. We will also use the name $\mathcal{E}$-diagram, following the notation in Satake [36, 2.4].

4.7. As in [42] we make a diagrammatic representation of the index of $\mathcal{E}$ by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\mathcal{E})$ and indicating the action of $[\sigma]$ on Δ by arrows.

In the cases of $\mathcal{E} = \Theta$ and $\mathcal{E} = \Gamma$ we get the well known θ-index and Γ-index, which are essential in the respective classifications. For k-involutions we do not use the Γ_θ-index, but we combine it with the Γ-index and θ-index to add additional information. These indices are defined as follows.

4.8. **(Γ, θ)-order.** Assume $A_0 \subset A$ is a maximal (θ, k)-split torus of G, S a maximal θ-split torus of G such that $A_0 \subset S \subset T$. By [22, 4.5] tori A_0, A, S and T exist. A linear order on $X = X^*(T)$ which is simultaneously a Γ-, θ- and Γ_θ-order is called a (Γ, θ)-*order*. A fundamental system of Φ with respect to a (Γ, θ)-order is called a (Γ, θ)-*fundamental system of* Φ.

An Γ_θ-order on (X, Φ) is a (Γ, θ)-order if and only of the following condition is satisfied:

$$\text{If } \Phi_1 \subset \Phi_0(\Gamma_\theta) \text{ irreducible component then } \Phi_1 \subset \Phi_0(\theta) \text{ or } \Phi_1 \subset \Phi_0(\Gamma).$$

Remarks 4.9. (1) A (Γ, θ)-order, as above, is completely determined by the sextuple

$$(X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*).$$

We will call this sextuple an *index of* (Γ, θ) or an (Γ, θ)-index. This terminology follows again Tits [42]. We will also use the name (Γ, θ)-diagram, following the notation in Satake [36, 2.4].
(2) We can make a diagrammatic representation of the (Γ, θ)-index by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\Gamma, \theta)$ and giving the vertices of $\Delta_0(\Gamma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\Gamma) \cap \Delta_0(\theta)$ a label k or θ if $\alpha \in \Delta_0(\Gamma) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ or $\alpha \in \Delta_0(\theta) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ respectively. The actions of $[\sigma]$ and θ^* are indicated by arrows.
(3) The above index of (Γ, θ) determines the indices of both Γ and θ and vice versa.

We conclude this section with some additional results for the action of the involution on (X, Φ).

4.10. ***θ*-orders on quasi *k*-split tori.** For θ-stable maximal quasi k-split tori we will need both the action of θ and $-\theta$. The action of θ is as defined in 4.4. The action of $-\theta$ can be defined in a similar manner. We will discuss this action in the following. We will use the same notation as in 4.2. In particular let A be a θ-stable maximal quasi k-split torus of G and write $X = X^*(A)$ and $\Phi = \Phi(A)$. Similar as in 4.2 let

$$X_0(\theta) = \{\chi \in X \mid \theta(\chi) = \chi\}, \qquad \Phi_0(\theta) = \Phi \cap X_0(\theta)$$
$$X_0(-\theta) = \{\chi \in X \mid \theta(\chi) = -\chi\}, \quad \Phi_0(-\theta) = \Phi \cap X_0(-\theta).$$

Let π be the natural projection from X to $X/X_0(\theta)$ and π^- the natural projection from X to $X/X_0(-\theta)$. As in [12] we define a θ-order (resp. $-\theta$-order) on Φ by choosing orders on $X_0(\theta)$ and $X/X_0(\theta)$ (resp $X_0(-\theta)$ and $X/X_0(-\theta)$). To be more precise:

Definition 4.11. Let $\succ$ be a linear order on X. The order $\succ$ is called a θ^+-order if it has the following property:

$$\text{(9)} \qquad \text{if } \chi \in X,\ \chi \succ 0,\ \text{and } \chi \notin X_0(\theta),\ \text{then } \theta(\chi) \prec 0.$$

The order $\succ$ is called a θ^--order if it has the following property:

$$\text{(10)} \qquad \text{if } \chi \in X,\ \chi \succ 0,\ \text{and } \chi \notin X_0(\theta),\ \text{then } \theta(\chi) \succ 0.$$

Similar as in [11], a θ^+-order on X will also be called a θ-order on X. Note that a θ^--order on X is a θ-order on X for the involution $-\theta$ of (X, Φ).

A basis Δ of Φ with respect to a θ^+-order (resp. θ^--order) on X will be called a θ^+-basis (resp. θ^--basis) of Φ. If Δ is a basis of Φ with respect to a θ^+-order on X, then we write $\Delta_0(\theta) = \Delta \cap \Phi_0(\theta)$ and $\bar{\Delta}_\theta = \pi(\Delta - \Delta_0(\theta))$. Similarly if Δ is a basis of Φ with respect to a θ^--order on X, then we write $\Delta_0(-\theta) = \Delta \cap \Phi_0(-\theta)$ and $\bar{\Delta}_{-\theta} = \pi(\Delta - \Delta_0(-\theta))$. Clearly $\Delta_0(\theta)$ (resp. $\Delta_0(-\theta)$) is a basis of $\Phi_0(\theta)$ (resp. $\Phi_0(-\theta)$). A similar property holds for $\bar{\Delta}_\theta$ and $\bar{\Delta}_{-\theta}$ (see [11, 2.4]).

4.12. **A characterization of θ on a θ-basis of Φ.** Let Δ_1 be a θ-basis of Φ. As in [11, 2.8] we can write $\theta = -\operatorname{id}\theta_1^* w_0(\theta)$, where $w_0(\theta) \in W_0(\theta)$ is the longest element of $W_0(\theta)$ with respect to $\Delta_0(\theta)$, and $\theta_1^* \in \operatorname{Aut}(X, \Phi, \Delta_1, \Delta_0(\theta)) = \{\phi \in \operatorname{Aut}(X, \Phi) \mid \phi(\Delta_1) = \Delta_1 \text{ and } \phi(\Delta_0(\theta)) = \Delta_0(\theta)\}$ with $(\theta_1^*)^2 = \operatorname{id}$. For more details see [11, sect. 2]. This is called a characterization of θ on its $(+1)$-eigenspace (because $W_0(\theta)$ is the Weyl group of $\Phi_0(\theta)$).

Similarly we get a characterization of θ on a θ^--basis of Φ as follows. Let Δ_2 be a θ^--basis of Φ. Then $\theta = \theta_2^* \cdot w_0(-\theta)$, where $w_0(-\theta) \in W_0(-\theta)$ is the longest element of $W_0(-\theta)$ with respect to $\Delta_0(-\theta)$, and $\theta_2^* \in \operatorname{Aut}(X, \Phi, \Delta_2, \Delta_0(-\theta)) = \{\phi \in \operatorname{Aut}(X, \Phi) \mid \phi(\Delta_2) = \Delta_2 \text{ and } \phi(\Delta_0(-\theta)) = \Delta_0(-\theta)\}$ with $(\theta_2^*)^2 = \operatorname{id}$. This is called a characterization of θ on its (-1)-eigenspace.

5. (θ, k)-singular involutions

In this section we introduce two classes of involutions in W which will be used in the characterization of the W-orbits in $\varphi(V)$ and $\varphi_k(V_k)$.

5.1. We use the notation of section 2. In particular let G be a connected reductive k-group, θ an involution of G defined over k and H a k-open subgroup of G_θ.

For quasi k-split tori we defined singular roots with respect to the involution (see 3.8). For k-split tori we have to combine this with the k-structure of the group itself. Before we can define this all we need a bit more notation. Let A be a θ-stable maximal k-split torus. For $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A_1 \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. If α is either real or imaginary, then G_α is θ-stable. We first define θ-singular roots:

Definition 5.2. Let A be as above. A root $\alpha \in \Phi(A)$ with $\theta(\alpha) = \pm\alpha$ is called *θ-singular* (resp. *θ-compact*) if $\overline{G}_\alpha \not\subset H$ (resp. $\overline{G}_\alpha \subset H$).

The θ-singular roots can be divided in those which are singular with respect to the k-structure and those which are not. These are defined now as follows.

Definition 5.3. A real θ-singular root $\alpha \in \Phi(A)$ is called *(θ, k)-singular* (resp. *θ-singular anisotropic*) if $\overline{G}_\alpha \cap H$ is isotropic (resp. $\overline{G}_\alpha \cap H$ is anisotropic). Similarly a imaginary θ-singular root is called *(θ, k)-singular* (resp. *θ-singular anisotropic*) if $\overline{G}_\alpha$ has a non-trivial (θ, k)-split torus (resp. $\overline{G}_\alpha$ has no non-trivial (θ, k)-split tori).

Similar as for θ-stable maximal tori we have the following result (see [12]).

Proposition 5.4. *Let A be a θ-stable maximal k-split torus of G. Then we have the following.*

(1) *A^+ is a maximal k-split torus of H if and only if $\Phi(A)$ has no (θ, k)-singular real roots.*
(2) *A^- is a maximal (θ, k)-split torus of G if and only if $\Phi(A)$ has no (θ, k)-singular imaginary roots.*

Proof. We will prove (1). The proof of (2) is similar.

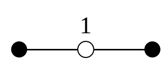
Assume first that A^+ is a maximal k-split torus of H. If $\alpha \in \Phi(A)$ is a (θ, k)-singular real root, then $\overline{G}_\alpha \cap H$ contains a k-split torus, say S. Since α is real, $A^+ \subset \ker(\alpha) \subset Z(G_\alpha)$. So $S.A^+$ is a k-split torus of $G_\alpha \cap H$, what contradicts the maximality of A^+.

Conversely assume $\Phi(A)$ has no (θ, k)-singular real roots. By passing to $Z_G(A^+)$, we can assume that $A^+ = \{e\}$. Since $\Phi(A)$ has no (θ, k)-singular real roots, it follows that $G_\alpha \cap H$ is anisotropic for all $\alpha \in \Phi(A)$. But since A is maximal k-split, we get that H is anisotropic and hence does not contain any k-split tori. □

Remark 5.5. In [13, 3.8] it was shown that every real root of a θ-stable maximal k-split torus A is θ-singular. If A_θ^- is maximal (θ, k)-split then all θ-singular are real and the set of θ-singular roots in $\Phi(A)$ is completely determined by the (Γ, θ)-index of the k-involution θ.

Unfortunately not all real roots are (θ, k)-singular. Whether a θ-singular root is (θ, k)-singular or not depends on other properties of the k-involution, like the quadratic elements. We illustrate this in the following example.

Example 5.6. Let $k = \mathbb{R}$. Assume $G_\mathbb{R} = SU^*(4, \mathbb{R})$ and θ is of type AII. Then the (Γ, θ)-index of (G, θ) is:



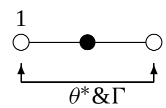
There are two k-involutions related to this (Γ, θ)-index. Let θ_1 be the standard involution of type $A_3^1(II)$ and $\theta_2 = \theta_1 \operatorname{Int}(\epsilon_1)$ the involution of type $A_3^1(II)(\epsilon_1)$. We use here the same notation as in [11]. See also [15].

In both these cases G has a θ-split maximal k-split torus A and $\Phi(A) = \Phi(A^-)$ is of type A_1. In the first case the Lie algebra of $(G_{\theta_1})_k$ is $\mathfrak{sp}(2)$ and $(G_{\theta_1})_k^0$ is compact. So all roots of $\Phi(A)$ are θ-singular, but not (θ, k)-singular. In the second case the Lie algebra of $(G_{\theta_2})_k$ is $\mathfrak{sp}(1, 1)$ and $(G_{\theta_2})_k^0$ is isotropic. So all roots of $\Phi(A)$ are (θ, k)-singular.

Remarks 5.7. (1) From [13, 3.8] it follows that every real root is θ-singular.

(2) In general the root system $\Phi(A)$ of a θ-stable maximal quasi k-split torus A is reduced. It would be quite natural to expect that if $\lambda, 2\lambda \in \Phi(A)$ and λ is θ-singular, then also 2λ is θ-singular. This is true for real roots, however for imaginary θ-singular roots this is in general not true, as can be seen from the following example.

Example 5.8. Assume (Γ, θ)-index of (G, θ) is:



Then G has a maximal k-split torus A, which is also θ-split. Let $T \supset A$ be a θ-stable maximal torus of G. The root system $\Phi(A)$ is of type BC_1 and the long root α is contained in $\Phi(A) \cap \Phi(T)$. This root is θ-singular, since H contains a maximal torus T_1 of G. Consequently T_1 contains a θ-stable maximal quasi k-split torus A_1 and the long root $\beta \in \Phi(A_1) \cap \Phi(T_1)$ is θ-singular. On the other hand, since $\Phi(T_1)$ contains θ-compact roots, there exists $w \in W(T_1)$ such that $w(\beta)$ is θ-compact and consequently $w(\beta)$ is not θ-singular. However, $\frac{1}{2}w(\beta)$ is still θ-singular, since $G_{\frac{1}{2}w(\beta)} = G$. Note that $\overline{G}_{w(\beta)}$ is of type A_1. In this example G has a no θ-stable minimal parabolic k-subgroup, but G has a θ-stable minimal quasi parabolic k-subgroup.

5.9. Now we have defined θ-singular and (θ, k)-singular roots we can generalize these definitions to include involutions as well. For the remainder of this section let A_0 be a θ-stable maximal k-split torus with A_0^- a maximal (θ, k)-split torus of G. Let A be a θ-stable maximal quasi k-split torus with $A^- \subset A_0^-$ and $A^+ \supset A_0^+$. For $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A_1 \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. If α is either real or imaginary, then G_α is θ-stable. Similarly if $w \in W(A)$ satisfies $w^2 = e$ and $w\theta = \theta w$, then we set $G_w = Z_G(A_w^+)$ and $\overline{G}_w = [G_w, G_w]$. Let n be a preimage of w in $N_G(A)$. Then $n \in Z_G(A_w^+)$ and $A_w^- \cap Z(G_w)$ is finite. As a consequence, A_w^- is a θ-stable maximal k-split torus of $[G_w, G_w]$.

Recall that a k-involution σ of a connected reductive k-group M is called (σ, k)-split if there exists a τ-split maximal k-split torus of M.

In 5.2 and 5.3 we defined θ-singular and (θ, k)-singular roots. We can lift these definitions to involutions in the Weyl group by defining the following.

Definition 5.10. Let A be a θ-stable maximal quasi k-split torus with $A^- \subset A_0^-$ and $A^+ \supset A_0^+$ and $w \in W(A)$. Then w is called θ-*singular* if

(1) $w^2 = e$.
(2) $\theta w = w\theta$.
(3) The involution $\theta \,|\, [G_w, G_w]$ is (θ, k)-split.

(4) $[G_w, G_w] \cap H$ contains a maximal quasi k-split torus of $[G_w, G_w]$.

An involution $w \in W(A)$ is called (θ, k)*-singular* if it satisfies (1), (2), (3) and if

(5) $[G_w, G_w] \cap H$ contains a maximal k-split torus of $[G_w, G_w]$.

A root $\alpha \in \Phi(A)$ is called θ-singular (resp. (θ, k)-singular) if the corresponding reflection $s_\alpha \in W(A)$ is θ-singular (resp. (θ, k)-singular).

Remark 5.11. Note that these definitions of θ-singular (resp. (θ, k)-singular) roots coincide with those defined in 5.2 and 5.3.

Lemma 5.12. *Let A be a θ-stable maximal k-split torus with A^- a maximal (θ, k)-split torus of G and $w \in W(A)$ a (θ, k)-singular (resp. θ-singular) involution. Then we have the following conditions:*

(i) $A_w^- = (A_w^-)^-$ *is a θ-split maximal k-split torus of* $[G_w, G_w]$.
(ii) $A_w^+ = A^+(A^-)_w^+$.

Proof. We show the assertion for the case that w is (θ, k)-singular. The result for the case that w is θ-singular follows with a similar argument.

(i). From $\theta w = w\theta$ it follows that $A^- = (A^-)_w^+(A^-)_w^-$. Since A^- is a maximal (θ, k)-split torus of G, $(A^-)_w^-$ is also a maximal θ-split torus of $[G_w, G_w]$. But since $\theta|[G_w, G_w]$ is (θ, k)-split, it follows that $(A^-)_w^-$ is a maximal k-split torus of $[G_w, G_w]$. Hence $(A^-)_w^- = A_w^-$.

(ii) is immediate from (i) and the observation that $A_w^+ = (A_w^+)^+(A_w^+)^-$. □

The θ-singular involutions in $W(A_0)$ can be characterized as follows.

Theorem 5.13. *Let A_0 be a θ-stable maximal k-split torus of G with A_0^- a maximal (θ, k)-split torus of G and $w \in W(A_0)$, $w^2 = e$. Then the following are equivalent:*

(i) *w is θ-singular*
(ii) $(A_0)_w^- \subset A_0^-$

Proof. $(ii) \Rightarrow (i)$. From $(A_0)_w^- \subset A_0^-$ it follows that w and θ commute. Since $(A_0)_w^-$ is a θ-split maximal k-split torus of $[G_w, G_w]$ it suffices to show that $[G_w, G_w] \cap H$ contains a maximal quasi k-split torus of $[G_w, G_w]$. But this follows from [13, 8.13].

$(i) \Rightarrow (ii)$ follows from Lemma 5.12. □

Corollary 5.14. *Let A_0 be a θ-stable maximal k-split torus of G with A_0^- a maximal (θ, k)-split torus of G. Then we have the following.*

(i) *Every involution in $W(A_0) \cap W(A_0^-)$ is θ-singular.*
(ii) *$\alpha \in \Phi(A_0)$ is θ-singular if and only if α is a real root.*

This result is immediate from Theorem 5.13.

Remark 5.15. Let A be a θ-stable maximal k-split torus of G. Using a simple induction one can show that an involution $w \in W(A)$ is θ-singular if and only if there exists a set of strongly orthogonal θ-singular roots $\{\alpha_1, \dots, \alpha_r\}$ of of $\Phi(A)$, such that $w = s_{\alpha_1} \dots s_{\alpha_r}$.

Similarly $w \in W(A)$ is (θ, k)-singular if and only if there exists a set of strongly orthogonal (θ, k)-singular roots $\{\alpha_1, \dots, \alpha_r\}$ of of $\Phi(A)$, such that $w = s_{\alpha_1} \dots s_{\alpha_r}$.

6. Twisted W-orbits in $\mathscr{I}$

In this section we establish the correspondence between W-twisted isomorphy classes of the involutions $w^0_\Pi \in \mathscr{I}$ and the W-conjugacy classes of the involutions $w^0_\Pi \in \mathscr{I}$.

6.1. The map $\phi\gamma^{-1} : \mathscr{A}^\theta/H \to \varphi(V)/W \subset \mathscr{I}/W$, where γ is as in 2.9 and ϕ as in 3.4, enables us to use results proved for conjugacy classes of θ-stable quasi k-split tori in [13, 14] to characterize $\varphi(V)$. In the following we use this correspondence to prove that every W-orbit in $\varphi(V)$ contains a θ-singular involution.

6.2. We give first another description of the involutions w^0_Π in the characterization of the twisted involutions as in Proposition 3.13.

Let A_0 be a θ-stable maximal k-split torus of G with A_0^- a maximal (θ, k)-split torus and A_1 a θ-stable maximal quasi k-split torus of G. There exists $g \in G$ such that $A_1 = gA_0g^{-1}$. Let $w = \tau(g)Z_G(A_0) \in W(A_0)$. Fix a basis Δ_0 of $\Phi_0 = \Phi_0(-\theta) = \{\alpha \in \Phi(A_0) \mid \theta(\alpha) = -\alpha\}$. We can extend Δ_0 to a θ^--basis Δ of $\Phi(A_0)$. Let w_0 be the opposition involution of $W(\Phi_0)$ with respect to Δ_0. Then by 4.12 it follows that $\theta = \theta^* w_0$. So $\theta' = \theta w_0 = \theta^*$ and hence $\theta'(\Delta) = \Delta$. Write $w' = ww_0 = s_{\alpha_1} \dots s_{\alpha_k} w^0_\Pi \theta'(s_{\alpha_k}) \dots \theta'(s_{\alpha_1})$ as in Proposition 3.13. Here Π is a subset of Δ and $\alpha_1, \dots, \alpha_k \in \Delta$. We can reduce to w^0_Π by removing the twisting factor $s_1 \dots s_k$ as follows. Choose $n_1, \dots, n_k \in N_G(A)$ with images $s_1, \dots, s_k$ in $W(A)$ respectively. Set $u = gn_1 \dots n_k$, $m = \tau(u) = u^{-1}\theta(u)$, $w_1 = \varphi(u) \in W(A_1)$. Then $w_1 w_0 = w^0_\Pi$. So assume $w' = ww_0 = w^0_\Pi$.

Proposition 6.3. *Let g, w, w' and w_0 be as above. Then there exists $w_1 \in W(A_0)$ such that $w_1 w^0_\Pi \theta'(w_1)^{-1} = w^0_{\Pi_1}$, with $\Pi_1 \subset \Delta_0$ and $w^0_{\Pi_1}$ the longest involution of Φ_{Π_1} with respect to Π_1.*

Proof. By [13, 5.8] there exists $h \in H^0$ such that $hA_1^-h^{-1} \subset A_0^-$ and $hA_1^+h^{-1} \subset A_0^+$. Let $A_2 = hA_1h^{-1}$ and let $g_1 \in Z_G(A_2^- A_0^+)$ such that $g_1 A_0 g_1^{-1} = A_2$. Then $h^{-1}g_1$ and g both map A_0 to A_1, namely $h^{-1}g_1A_0g_1^{-1}h = gA_0g^{-1} = A_1$. So there exists $n \in N_G(A_0)$ such that $h^{-1}g_1 = gn$. Let $w_1 \in W(A_0)$ be the Weyl group element corresponding to n and let $w_2 = \tau(h^{-1}g_1)Z_G(A_0) = \tau(g_1)Z_G(A_0) \in W(A_0)$. Then $w_2 = w_1^{-1}w\theta(w_1)$ and $(A_0)^-_{w_2} \subset A_0^-$. Since $\theta' = w_0^{-1}\theta = \theta w_0$

and $w' = w^0_\Pi$, we get

$$\begin{aligned} w_2 &= w_1^{-1} w\theta(w_1) = w_1^{-1} w w_0 \theta' w_1 \theta = w_1^{-1} w' \theta' w_1 \theta' w_0^{-1} \\ &= w_1^{-1} w^0_\Pi \theta'(w_1) w_0^{-1}. \end{aligned} \tag{11}$$

By [12, 2.14] there exists $w_3 \in W(\Phi_0)$ such that $w_3 w_2 w_3^{-1} w_0 = w^0_{\Pi_1}$ with $\Pi_1 \subset \Delta_0$. Together with (11) this gives:

$$\begin{aligned} w^0_{\Pi_1} &= w_3 w_2 w_3^{-1} w_0 = w_3 w_1^{-1} w^0_\Pi \theta' w_1 \theta' w_0^{-1} w_3^{-1} w_0 \\ &= w_3 w_1^{-1} w^0_\Pi \theta' w_1 w_3^{-1} \theta w_0 = w_3 w_1^{-1} w^0_\Pi \theta'(w_1 w_3^{-1}). \end{aligned}$$

This proves the result. □

Corollary 6.4. *Let A_0 be as above. Then we have the following.*

1. *Every W-orbit in $\varphi(V)$ contains a θ-singular involution.*
2. *Every W-orbit in $\varphi(V)$ has a representative $w^0_\Pi \in W(A_0) \cap W(A_0^-)$.*

Proof. These results are immediate from Proposition 6.3, using the isomorphism between $\mathcal{I}_\theta / W$ and $\mathcal{I}_{\theta'} / W$ as in 3.26. □

For $\varphi_k(V_k)$ we will prove in Theorem 8.5 a similar result using the (θ, k)-singular involutions.

6.5. Since $w^0_\Pi \in W(A_0) \cap W(A_0^-)$ we can characterize $\varphi(V)/W$ by looking at $W(A_0^-)$-conjugacy classes of the involution w^0_Π. For this it will be useful to have another characterization of $W(A_0^-)$ by linking it to the group $W(A_0, H)$. Let $X = X^*(A_0)$, $X_0(\theta)$ and $\Phi_0(\theta)$ be as in 4.2. Write

$$W_1(\theta) = \{w \in W(A_0) \mid w(X_0(\theta)) \subset X_0(\theta)\} \tag{12}$$

and $W_0(\theta) = W_0(A_0, \theta) = W(\Phi_0(\theta))$. Then by [36, 2.1.3] we have

$$W(A_0^-) \simeq W_1(\theta)/W_0(\theta). \tag{13}$$

The group $W(A_0, H)$ corresponds with $W_1(\theta)$ due to the following result.

Proposition 6.6. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ-stable maximal k-split torus of G with A_0^- a maximal (θ, k)-split torus of G. Then we have the following.*

(*i*) *Any $w \in W(A_0^-)$ has a representative in $(H^0 Z_G(A_0))_k \cap N_G(A_0^-)$.*
(*ii*) *$N_G(A_0^-) = N_{H^0}(A_0^-) Z_G(A_0^-)$.*

Proof. (*i*) Let $n \in N_{G_k}(A_0^-)$ and P a minimal θ-split parabolic k-subgroup of G. Then $P_1 = nPn^{-1}$ is also a minimal θ-split parabolic k-subgroup of G containing A_0. By [22, 4.9] there exists $x \in (H^0 P)_k$ such that $xPx^{-1} = P_1$. Let P_0 be a minimal parabolic k-subgroup of P containing A_0. By [22, 4.8] $H^0 P_0 = H^0 P$. On the other hand $(H^0 P)_k = (H^0 Z_G(A_0))_k U_k$, where $U = R_u(P_0)$ (see [22, 10.2]). It follows that $x = hzu$ with $h \in H^0$, $z \in Z_G(A_0)$ and $u \in U_k$. If we take $g = hz \in (H^0 Z_G(A_0))_k$, then $gPg^{-1} = P_1$ and $gA_0^- g^{-1}$ is (θ, k)-split. Moreover $gA_0^- g^{-1} \subset P_1 \cap \theta(P_1) = Z_G(A_0^-)$, so $gA_0^- g^{-1} A_0^-$ is a (θ, k)-split torus of G. Since A_0^- is maximal (θ, k)-split it follows that $gA_0^- g^{-1} = A_0^-$, what proves (i). Finally (ii) is immediate from (i). □

Remark 6.7. Ideally one would like $W(A_0^-)$ to have representatives in H_k, but as the proof of this result indicates, this will unfortunately not always be true. However in many cases, including the standard pairs for $k = \mathbb{R}$ (for a definition see [11]), one can show that $W(A_0^-)$ has representatives in H_k. It is an interesting open question to give necessary and sufficient conditions such that $W(A_0^-)$ has representatives in H_k.

Corollary 6.8. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ-stable maximal k-split torus of G with A_0^- a maximal (θ, k)-split torus of G. Then $W_0(A_0, H) = W_1(\theta)$.*

Proof. Clearly $W_0(A_0, H) \subset W_1(\theta)$. As for the other inclusion, it suffices by Proposition 6.6 to show that $W_0(\theta) = W_0(A_0, \theta) \subset W_0(A_0, H)$. But this follows from the fact that every root $\alpha \in \Phi_0(A_0, \theta)$ is compact and therefor $G_\alpha \subset H$. □

To prove the correspondence between $W_1(\theta)$-conjugacy classes of involutions in $W(A_0) \cap W(A_0^-)$ and the W-conjugacy classes we need the following result from [19]. Write $E = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ and for $\sigma \in \mathrm{Aut}(\Phi)$ denote the eigenspace of σ for the eigenvalue ξ, by $E(\sigma, \xi)$.

Proposition 6.9. [19, 9.7] *Let θ be an involution of Φ such that $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$. If $w_1, w_2 \in W$ are involutions with $E(w_i, -1) \subset E(\theta, -1)$ $(i = 1, 2)$, then the following are equivalent:*

(i) *w_1 and w_2 are conjugate under W.*
(ii) *w_1 and w_2 are conjugate under $W_1(\theta)$.*
(iii) *$w_1\theta$ and $w_2\theta$ are conjugate under W.*
(iv) *$w_1\theta$ and $w_2\theta$ are conjugate under $W_1(\theta)$.*

Remark 6.10. Note that the involutions w_Π^0 satisfy the condition $E(w_\Pi^0, -1) \subset E(\theta, -1)$. So this result reduces the characterization of the $W_1(\theta)$-conjugacy classes of these involutions to W-conjugacy classes. A characterization of the latter can be found in [12].

For the θ-singular involutions in a twisted W-orbit we can show now the following:

Proposition 6.11. *Let $w_1, w_2 \in \varphi(V)$ be θ-singular involutions. Then w_1 and w_2 are in the same twisted W-orbit if and only if w_1 and w_2 are W-conjugate.*

Proof. Assume first that w_1 and w_2 are W-conjugate. Since A_0^- is a maximal (θ, k)-split torus, we have $(A_0)_{w_i}^- \subset A_0^-$ $(i = 1, 2)$ and hence $w_1, w_2 \in W(A_0) \cap$

$W(A_0^-)$. By Proposition 6.9 w_1 and w_2 are conjugate under $W_1(A_0)$. Let $w \in W_1(A_0)$ such that $ww_1w^{-1} = w_2$. Since $W_1(A_0) = W(A_0, H)$ (see Corollary 6.8) it follows that $\theta(w) = w$ and hence $ww_1\theta(w^{-1}) = ww_1w^{-1} = w_2$.

For the converse statement assume $w \in W$ such that $ww_1\theta(w^{-1}) = w_2$. Then by Lemma 3.28 we have $w((A_0^-)^+_{w_1}) = (A_0^-)^+_{w_2}$ and with a similar argument we also have $w((A_0^-)^-_{w_1}A_0^+) = (A_0^-)^-_{w_2}A_0^+$. So $ww_1\theta w^{-1} = w_2\theta$. Then by Proposition 6.9 w_1 and w_2 are W-conjugate, what proves the result. □

7. Computing $\varphi(V)$ and V

In this section we give an algorithm to compute $\varphi(V)$ and V. This algorithm is essentially a modification of the one used for computing orbits or Borel subgroups acting on symmetric varieties as in [19].

7.1. **Characterization of w_0.** To compute $\varphi(V)$ or $\varphi'(V')$ we first need a characterization of the involution w_0 as in 3.10. For this we give a characterization of the open orbit which will lead to a description of w_0. We use the same notation as in sections 2 and 3. In particular let A be a θ-stable maximal k-split torus of G, $P \supset A$ a minimal parabolic k-subgroup, $\Phi = \Phi(A)$, $\Phi^+ = \Phi(P, A)$ the set of positive roots of Φ related to P, Δ the corresponding basis of Φ, w_0 the element in the Weyl group $W = N_G(A)/Z_G(A)$ with $w_0(\Phi^+) = \theta(\Phi^+)$ and $\theta' = \theta w_0$. If $\Pi \subset \Delta$, then we write Φ_Π for the subsystem of $\Phi(G, A)$ consisting of integral combinations of Π and we write P_Π for the standard parabolic k-subgroup of G containing P with $\Phi(P_\Pi, A) = \Phi_\Pi \cup \Phi^+$. The following result from [22, Proposition 9.2] characterizes the open orbit.

Proposition 7.2. *Let $v \in V$, $n = x(v)\theta(x(v))^{-1}$, w the image of n in W and $w' = ww_0$. Let ζ be the involution of G corresponding to $w'\theta' = w\theta$ (i.e. ζ is given by $\zeta(x) = n\theta(x)n^{-1}$, $x \in G$). The following conditions are equivalent:*

- (i) *$P * n$ is open in $Q = \{x\theta(x)^{-1} | x \in G\}$.*
- (ii) *Let $\Pi = I(w) \cap \Delta$. Then $C''(w) \cap \Delta = \emptyset$ and ζ is trivial on G_{Φ_Π}.*
- (iii) *$w' = w^0_\Pi w^0_\Delta$ and ζ is trivial on G_{Φ_Π}.*
- (iv) *$x(v)^{-1}P_\Pi x(v)$ is a minimal θ-split parabolic k-subgroup of G.*
- (v) *There exists a minimal θ-split parabolic k-subgroup of G containing $x(v)^{-1}Px(v)$.*

Combining this result with Corollary 3.22 we get the following characterization of w_0 and θ'.

Corollary 7.3. *Let A be a θ-stable maximal k-split torus of G such that A^- is a maximal (θ, k)-split torus and $P \supset A$ a minimal parabolic k-subgroup of G such that $PH \subset G$ is open. Let $w_0 \in W$ satisfy $\theta(\Phi^+) = w_0(\Phi^+)$, and take $v_0 \in V$ such that if $x_0 = x(v_0)$ and $n_0 = x_0\theta(x_0)^{-1} \in N_G(A)$ then n_0 induces w_0 in W;*

let ζ be the involution of G given by $\zeta(x) = n_0^{-1}\theta(x)n_0$ for $x \in G$. Then we have the following.

(i) $w_0 = w^0_{\Delta_0(\theta)} w^0_\Delta$, *where* $\Delta_0(\theta) = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\}$ *is as in 4.10.*
(ii) $\zeta|\Phi = w_0^{-1}\theta = \theta w_0 = \theta'$.

Remark 7.4. The element $w^0_{\Delta_0(\theta)} \in W$ follows from the (Γ, θ)-index of the k-involution θ. For k the real numbers, $\mathfrak{p}$-adic numbers, finite field or numbers field the (Γ, θ)-indices of k-involutions are given in [15] (see also [16]). For k algebraically closed the (Γ, θ)-indices are given in [11]. The element $w^0_\Delta \in W$ follows from the classification of involutions in [12]. Combining these two classifications and we get a list of the elements $w_0 \in W$. From the classification of (Γ, θ)-indices in [15] we also get a list of the involutions θ' for each of the irreducible (Γ, θ)-indices. Note that in each case $\theta' = \mathrm{id}$ or a diagram automorphism.

7.5. **Image and fibers of φ.** Similar as in [19] the classification of the image and fibers of $\varphi : V \to \mathcal{I}$ (or $\varphi' : V' \to \mathcal{I}_{\theta'}$) can be reduced to a problem related to the involutions w^0_Π as in the characterization of the twisted involutions in Proposition 3.13. The results about these involutions as proved in [19, §9] carry over to the present situation. Let Δ be a basis of Φ, $w_0 \in W = W(T)$ be such that $\theta(\Phi^+) = w_0(\Phi^+)$, $\theta' = \theta w_0 = w_0^{-1}\theta$ and let $\mathcal{I}_{\theta'}$ be as in 3.23. Write

$$\Lambda_\Delta = \{\Pi \subset \Delta \mid \theta'(\Pi) = \Pi \text{ and } w^0_\Pi \theta'(\alpha) = -\alpha,\ \forall \alpha \in \Phi_\Pi\},$$

$$\mathcal{I}_\Delta = \{w^0_\Pi \mid \Pi \in \Lambda_\Delta\}. \tag{14}$$

The set $\mathcal{I}_\Delta$ contains a set of representatives of $\mathcal{I}_{\theta'}/W$ and also of $\varphi'(V')/W$. Since by Proposition 3.26 $\varphi'(V')/W \simeq \varphi(V)/W$ we have the following result.

Lemma 7.6. *Let W act on $\mathcal{I}$, $\mathcal{I}_{\theta'}$, V and V' as in 3.23 and 3.25. Let Δ be a basis of Φ and let $\mathcal{I}_\Delta$ be as in* (14). *Then we have the following:*

(i) *Each orbit in $\mathcal{I}_{\theta'}/W$ and $\varphi'(V')/W$ has a representative in $\mathcal{I}_\Delta$;*
(ii) *Each orbit in $\mathcal{I}/W$ and $\varphi(V)/W$ has a representative in $\mathcal{I}_\Delta w_0^{-1}$.*

7.7. **Classification of $\varphi(V)/W$ and $\varphi'(V')/W$.** We need to classify the involutions in $\mathcal{I}_\Delta$ which represent the different classes in $\mathcal{I}_{\theta'}/W$ resp. $\varphi'(V')/W$. By Propositions 6.9 and 6.11 it suffices to look at the W-conjugacy classes of the involutions w^0_Π. A classification of conjugacy classes of involutions in the Weyl group is given in [12]. Similar as in [19] let

$$\mathcal{I}^0_\Delta = \{w^0_{\Pi_1}, \cdots, w^0_{\Pi_k}\} \subset \mathcal{I}_\Delta$$

be a set of representatives of $\mathcal{I}_{\theta'}/W$ and let

$$\Lambda^0_\Delta = \{\Pi_1, \cdots, \Pi_k\} \subset \Lambda_\Delta$$

be the corresponding subset of Λ_Δ. Similarly let

$$\mathcal{I}_\Delta(V') = \mathcal{I}_\Delta^0 \cap \varphi'(V')$$

and

$$\mathcal{I}_\Delta(V) = \mathcal{I}_\Delta^0 \cdot w_0^{-1} \cap \varphi(V)$$

Then $\mathcal{I}_\Delta(V')$ is a set of representatives of $\varphi'(V')/W$ and similarly $\mathcal{I}_\Delta(V)$ is a set of representatives of $\varphi(V)/W$. The sets $\mathcal{I}_\Delta(V')$ and $\mathcal{I}_\Delta(V')$ are related as follows:

$$\mathcal{I}_\Delta(V') = \mathcal{I}_\Delta(V) \cdot w_0^{-1}$$

7.8. Next we show that the orbits in $\varphi'(V')$ (resp. $\varphi(V)$) under the twisted action of W correspond with the conjugation classes of the involutions $w_\Pi^0 \in \mathcal{I}_\Delta(V')$ (resp $\mathcal{I}_\Delta(V)$). For this we need to choose first a suitable maximal torus to characterize the sets $\mathcal{I}_\Delta(V')$ and $\mathcal{I}_\Delta(V)$. In the following let A be a θ-stable maximal k-split torus with A^- a maximal (θ,k)-split torus of G. Let $\Delta_0 = \Delta_0(-\theta)$ be a basis of $\Phi_0(-\theta)$ and extend this to a θ^--basis Δ of Φ. Then by 4.12 we have $\theta' = \theta w_{\Delta_0}^0$. As in [12, 7.4] we call an involution w_Π^0 a Δ_0-*standard involution* if $\Pi \subset \Delta_0$. We can choose now Δ_0-standard involutions as representatives for both $\mathcal{I}_\Delta(V)$ and $\mathcal{I}_\Delta(V')$ (see also [12, §7]). Using Proposition 6.3 and a similar argument as in [19] we get the following result:

Proposition 7.9. (i) *Every W-orbit in $\varphi(V)$ contains a Δ_0-standard involution.*
(ii) *Every W-orbit in $\varphi'(V') = \varphi(V) \cdot w_{\Delta_0}^0$ contains a Δ_0-standard involution.*

Remarks 7.10. (1). It follows from the above results that in the case that A^- is a maximal (θ,k)-split torus of G, we can represent the elements of $\mathcal{I}_\Delta(V)$ and $\mathcal{I}_\Delta(V')$ as involutions $w_{\Pi_1}^0$ or $w_{\Pi_2}^0 w_{\Delta_0}^0$. For $\mathcal{I}_\Delta(V')$ we will only use the first characterization, but for $\mathcal{I}_\Delta(V)$ we will use both characterizations depending on whether $\theta' = \mathrm{id}$ or a diagram automorphism.

(2). If $\mathcal{I}_\Delta(V') = \{w_{\Pi_1}^0, \cdots, w_{\Pi_k}^0\}$, where $w_{\Pi_1}^0, \cdots, w_{\Pi_k}^0$ are Δ_0-standard involutions, then $\mathcal{I}_\Delta(V) = \{w_{\Pi_1}^0 w_0, \cdots, w_{\Pi_k}^0 w_0\}$ and $w_{\Pi_1}^0 w_0, \cdots, w_{\Pi_k}^0 w_0$ are involutions. In most cases we will use this characterization of $\mathcal{I}_\Delta(V')$ and $\mathcal{I}_\Delta(V)$.

(3). Using the above results, the classification of (Γ,θ)-indices of k-involutions in [15] and the classification of involutions in the Weyl group in [12] we get a list of the possible subsets $\Pi \in \Lambda_\Delta^0$, which correspond to involutions which are not W-conjugate.

With only very few exceptions, the isomorphism class of an involution $w_\Pi^0 \in \mathcal{I}_\Delta(V)$ is determined by the type of the root system Φ_Π spanned by Π. Only in the case that (G,θ) is (θ,k)-split (i.e. G contains a θ-split maximal k-split torus) can it happen that two involutions $w_{\Pi_1}^0$ and $w_{\Pi_2}^0$ are not W-conjugate, while the root systems Φ_{Π_1} and Φ_{Π_2} $(\Pi_1, \Pi_2 \subset \Delta)$ are of the same type. In these cases it is easy to find two subsets $\Pi_1, \Pi_2 \subset \Delta$, such that the root systems Φ_{Π_1} and Φ_{Π_2} are of the same type and $w^0_{\Pi_1}$ and $w^0_{\Pi_2}$ not W-conjugate. For more details, see [12, sect. 7].

7.11. **Image of φ.** To compute $\varphi'(V')$ (or $\varphi(V)$) from the subsets $\Pi \in \Lambda^0_\Delta$ as above we can use the same method as in [11]. If $w^0_\Pi \in \mathcal{I}_\Delta$, then we also write $W(\Pi)$ for the stabilizer subgroup $W_{w^0_\Pi} = \{w \in W \mid w * w^0_\Pi = w^0_\Pi\}$. The orbits $W *' w^0_\Pi \subset \mathcal{I}_{\theta'}$ and $W * w^0_\Pi w_0^{-1} \subset \mathcal{I}$ can be characterized as follows:

Lemma 7.12. [19, 9.12] *Let θ, θ', $\mathcal{I}$ and $\mathcal{I}_{\theta'}$ be as above. If $w^0_\Pi \in \mathcal{I}_\Delta$, $W(\Pi)$ as above and $w_1, \cdots, w_n$ minimal coset representatives of $W/W(\Pi)$, then we have the following.*

(i) $W *' w^0_\Pi = \{w_1 *' w^0_\Pi, \cdots, w_n *' w^0_\Pi\}$.
(ii) $W * w^0_\Pi w_0^{-1} = \{w_1 * w^0_\Pi w_0^{-1}, \cdots, w_n * w^0_\Pi w_0^{-1}\}$.

7.13. It remains to determine the subsets $\Pi \in \Lambda^0_\Delta$ and compute the corresponding subgroups $W(\Pi)$ for $\Pi \in \Lambda^0_\Delta$. We first recall the following. A pair (G, θ) is called (θ, k)-*split* if there exists a θ-split maximal k-split torus of G. The pair (G, θ) is called *quasi* (θ, k)-*split* if there exists a θ-split minimal parabolic k-subgroup of G. Note that if (G, θ) is (θ, k)-split and A is a θ-stable maximal k-split torus with A^-_θ a maximal (θ, k)-split torus of G, then $\theta|\Phi(A) = -\,\mathrm{id}$. This means that if $s \in \mathcal{I}_\theta$ and $w \in W$, then $w * s = ws\theta(w^{-1}) = wsw^{-1}$. So $\mathcal{I}_\theta$ consists of the set of involutions in $W(A)$ and the $W(A)$-orbits in $\mathcal{I}_\theta$ are precisely the $W(A)$-conjugacy classes of involutions in $W(A)$. We summarize this in the following result.

Lemma 7.14. *Let (G, θ) be (θ, k)-split, P a θ-split minimal parabolic k-subgroup of G, $A \subset P$ a θ-split maximal k-split torus of G and $\mathcal{I}$ the set of twisted involutions in $W(A)$. Then $\mathcal{I}$ consists of the set of involutions in $W(A)$.*

Similar as in [11] we can now show that to compute the subgroups $W(\Pi)$ of W as above, it suffices to consider $\mathcal{I}_{\mathrm{id}}$ or $\mathcal{I}_{-\,\mathrm{id}}$. The W-orbits in all other cases can be identified with orbits for these two cases. This can be seen as follows.

7.15. Let A be a maximal k-split torus of G, $\Phi = \Phi(A)$, $W = W(A)$, $\theta \in \mathrm{Aut}(G, A)$ an involution with A^-_θ a maximal (θ, k)-split torus and Δ a θ^--basis of Φ. Write $\theta = \theta^* w^0_{\Delta(-\theta)}$ as in 4.12 and let $\mathcal{I}^0 \subset W$ denote the set of involutions in W. Assume Φ is irreducible. Then we have three cases.

(1) **$-\,$id $\in W$.** Then Φ is of type B_n, C_n, D_{2n}, E_7, E_8, F_4 or G_2. In this case $\theta' = \theta^* = \mathrm{id}$, $\mathcal{I}_{\theta'} = \mathcal{I}^0 = \mathcal{I}_{-\,\mathrm{id}}$ and $\mathcal{I}_\theta = \mathcal{I}^0 w^0_{\Delta(-\theta)}$. So we can compute either $\varphi(V)$ or $\varphi'(V')$. If $w^0_\Pi \in \mathcal{I}_\Delta(V')$, then $W(\Pi)$ is the commutator subgroup for the involution w^0_Π which can easily be computed using LiE or other symbolic manipulation programs.

(2) **$-\mathrm{id} \notin W$ and $\theta' = \mathrm{id}$.** In this case Φ is of type A_n, D_{2n+1} or E_6 and θ is an inner automorphism of G. Since $\theta' = \theta^* = \theta w^0_{\Delta(-\theta)} = \mathrm{id}$ it follows that $\mathcal{I}_{\theta'} = \mathcal{I}^0 = \mathcal{I}_\theta w^0_{\Delta(-\theta)}$. In this case it is easier to compute $\mathcal{I}_{\theta'}$ and $\varphi'(V')$ instead of $\varphi(V)$. The computation of the groups w^0_Π is similar as in (1).

(3) **$-\mathrm{id} \notin W$ and $\theta' \neq \mathrm{id}$.** In this case Φ is again of type A_n, D_{2n+1} or E_6, but θ is now an outer automorphism of G. In this case it is easier to switch to $-\mathrm{id}$ instead of θ'. Since $\theta = \theta' w^0_{\Delta(-\theta)}$ we get $\theta w^0_{\Delta(-\theta)} w^0_\Delta = \theta' w^0_\Delta = -\mathrm{id}$. So $\mathcal{I}_\theta = \mathcal{I}_{-\mathrm{id}} w^0_{\Delta(-\theta)} w^0_\Delta = \mathcal{I}^0 w^0_{\Delta(-\theta)} w^0_\Delta$ and $\mathcal{I}^0$ follows as in (2). In this case it is easier to compute $\mathcal{I}_\theta$ and $\varphi(V)$ instead of $\mathcal{I}_{\theta'}$ and $\varphi'(V')$.

Remark 7.16. The involution $\theta|A$ can also be described by a θ-index. This θ-index of$\theta|A$ follows from the corresponding (Γ, θ)-index. A list of these θ-indices is given in [13]. Both θ^* and $\Delta(-\theta)$ are immediate from these θ-indices.

Using a similar argument as in [19, 9.15] it follows from the above observations that by switching from a θ^--order to a θ^+-order one can reduce the characterization of $\varphi(V)$ from twisted involutions to involutions in W. We summarize this in the following result.

Lemma 7.17. *Let A be a θ-stable maximal k-split torus, $W = W(A)$, $\Phi = \Phi(A)$ and assume Φ is irreducible. Let Δ_1 be a θ-basis of Φ, Δ_2 a θ^--basis of Φ, $w_0(\theta)$ the longest element of $W_0(\theta)$ with respect to $\Delta_0(\theta) \subset \Delta_1$ and $w_0(-\theta)$ the longest element of $W_0(-\theta)$ with respect to $\Delta_0(-\theta) \subset \Delta_2$. If $\mathcal{I}_1 = \mathcal{I}_\theta \cdot w_0(\theta) \subset W$ and $\mathcal{I}_2 = \mathcal{I}_\theta \cdot w_0(-\theta) \subset W$, then $\mathcal{I}_1$ or $\mathcal{I}_2$ consists of the set of involutions in W.*

Remarks 7.18. (1) The same argument as in [19] can be used compute the fibers of φ. Again φ is injective if and only if there is a unique closed orbit of P on G/H and φ is surjective if and only if (G, θ) is quasi (θ, k)-split.
(2) The computation of the orbits in V can be reduced again to the case that G is semisimple and simply connected. For more details see [19, 9.21].

7.19. **An algorithm to classify V.** Combining the results in this paper we get now an algorithm to classify the orbits in V. Each step of this algorithm can be implemented in LiE or a number of other programs. Assume that G is semisimple and simply connected and assume that the pair (G, θ) is irreducible. In the following let A be a θ-stable maximal k-split torus such that A^- is a maximal (θ, k)-split torus of G, $W = W(A)$, $\Phi = \Phi(A)$ and Δ a θ^--basis of Φ. The algorithm is as follows.

(1) The first step is to compute the set $\mathcal{I}_{\mathrm{id}} = \mathcal{I}_{-\mathrm{id}}$ for all the irreducible root systems. These were already determined in the computation of the orbits of a Borel subgroup on G/H in [19].

The next step is to determine θ' and w_0 and to classify $\varphi(V)$ or $\varphi'(V')$ by identifying the corresponding W-orbits in $\mathcal{I}_{\mathrm{id}}$ or $\mathcal{I}_{-\mathrm{id}}$. Recall that by 7.3 the involution w_0 is completely determined by the subset $\Delta_0(\theta) = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\}$.

(2) Get a list of the pairs $\{\Delta_0(\theta), \theta'\}$ from the classification of (Γ, θ)-indices of k-involutions of G in [15] (see also [16]) and the classification of involutions of W in [12, §7]. If $\theta' = \mathrm{id}$, then compute $\varphi'(V')$ and if $\theta' \neq \mathrm{id}$ (i.e. a diagram automorphism), then compute $\varphi(V)$ by identifying $\varphi(V)$ as a subset of $\mathcal{I}_{-\mathrm{id}} w^0_{\Delta_0(-\theta)} w^0_\Delta$.

(3) Find $\mathcal{I}_\Delta(V') \subset \mathcal{I}^0_\Delta$ or $\mathcal{I}_\Delta(V) \subset \mathcal{I}^0_\Delta w^0_{\Delta_0(-\theta)} w^0_\Delta$. For $\mathcal{I}_\Delta(V')$ this easily follows from Proposition 7.9 and the classification in [12, Table IV]. For $\mathcal{I}_\Delta(V)$ note that we can choose $\Delta_0(-\theta)$-standard involutions as representatives for the W-orbits in $\varphi(V)$. Since these involutions commute with $w^0_{\Delta_0(-\theta)} w^0_\Delta$ a list of these also follows from Proposition 7.9 and the classification of involutions of W in [12, Table IV].

(4) Determine $\varphi'(V') \subset \mathcal{I}_{\theta'}$ or $\varphi(V) \subset \mathcal{I}_{-\mathrm{id}} w^0_{\Delta_0(-\theta)} w^0_\Delta$, by finding the W-orbits of the involutions in $\mathcal{I}_\Delta(V')$ or $\mathcal{I}_\Delta(V)$. For this we use (1) and (2).

After this one can finally find V as follows:

(5) Determine $|\varphi'^{-1}(w^0_\Pi)|$ for each involution w^0_Π in $\mathcal{I}_\Delta(V')$ or $\mathcal{I}_\Delta(V)$. Then, using the same argument as in [19, 9.17] we find the W-orbits in V and V'.

Remarks 7.20. (1). The results about the Bruhat order on the set of twisted involutions in [19] and [34] carry over to this situation and can also be used to characterize the Bruhat order on V. For k a $\mathfrak{p}$-adic or real field one can also look at the Bruhat order on V_k. This situation is however much more complicated and will need a lot of additional work. The main complication is the fact that there is no longer a unique open orbit. We intend to deal with this in a future paper.

(2). The above algorithm can be implemented on a computer for each symmetric k-variety G_k/H_k, as in [15]. The eventual goal is to be able to compute the orbits of all finite cases and the infinite families up to a certain dimension. We should note that some of the finite cases are extremely large and it will be hard to compute these. The highest dimension of the infinite cases that can be handled will depend on several factors. First of all it depends of course on the processor used, but more importantly, it depends on the efficiency of both the algorithm and its implementation. To optimize the algorithm one will have to use some specific properties of each of the symmetric k-varieties G_k/H_k.

(3). To implement the above algorithm we can use most of the code written for the computation of the orbits of a Borel subgroup acting on a symmetric variety as in [19]. Nevertheless there will be a lot of additional work, mainly due to the fact that there are many more (Γ, θ)-indices then in the case of orbits of a Borel subgroup acting on a symmetric variety as in [19]. There are 137 different types of absolutely irreducible (Γ, θ)-indices, while there are only 24 absolutely irreducible θ-indices related to symmetric varieties.

The amount of work needed to implement the algorithm can be reduced considerably by using one of the available symbolic manipulation programs for which a lot of Weyl group algorithms already have been implemented. For example one could use the excellent package of Stembridge (see [41]), who has implemented several Weyl group algorithms in Maple. Even more suited is the package LiE (see [27]). Using this package the calculation of $W(\Pi)$ and the calculation of the minimal coset representatives in $W/W(\Pi)$ are easily implemented. Another reason that LiE is a good choice is that the source code (written in C) is available, so one can also optimize its algorithms to suit the above calculation of $\mathcal{I}_{\theta'}$ and V. An implementation in LiE should be able to handle all the finite dimensional cases. If a more efficient implementation of the algorithm is needed, then one will have to write an independent program.

8. Computing $\varphi_k(V_k)$

In this section we give an algorithm to compute $\varphi_k(V_k)$. This algorithm necessarily depends on the classification of k-involutions. To get a classification of these one needs to classify both the (Γ, θ)-index and the quadratic elements (see [15] for more on this). The (Γ, θ)-indices have been classified for k the real numbers, $\mathfrak{p}$-adic numbers, finite field, numbers field or algebraically closed. However for most of these fields a classification of the quadratic elements is still lacking. For $k = \mathbb{R}$ and $k = \bar{k}$ a classification of the k-involutions can be found in [11]. In these cases the algorithm in this section can compute $\varphi_k(V_k)$.

8.1. We will use the same notation as in the previous sections. In particular let A be a maximal k-split torus of G, $\Phi = \Phi(A)$, $W = W(A)$, $\theta \in \mathrm{Aut}(G, A)$ an involution with A^- a maximal (θ, k)-split torus. Let $\Delta_0 = \Delta_0(-\theta)$ be a basis of $\Phi_0(-\theta)$ and extend this to a θ^--basis Δ of Φ. Then by 4.12 we have $\theta' = \theta w^0_{\Delta_0}$.

8.2. For the characterization of $\varphi(V)$ we used the map $\phi\gamma^{-1} : \mathcal{A}^\theta/H \to \varphi(V)/W \subset \mathcal{I}/W$. To compute $\varphi_k(V_k)$ we need to look at the map $\phi_k\gamma_k^{-1} : \mathcal{A}^\theta_k/H_k \to \varphi_k(V_k)/W$, where γ_k is as in 2.9 and ϕ_k as in 3.4.

Recall from 2.10 that we have a natural embedding $\zeta : \mathcal{A}^\theta_k/H_k \to \mathcal{A}^\theta/H$ sending the H_k-conjugacy class of a θ-stable maximal k-split torus onto its H-conjugacy class and we have that $\varphi_k(V_k) \subset \varphi(V) \subset \mathcal{I}$ is W-stable. From [13, 4.13] it follows now that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{A}^\theta_k/H_k & \xrightarrow{\phi_k\gamma_k^{-1}} & \varphi_k(V_k)/W \\ \Big\downarrow \zeta & & \Big\downarrow \mathrm{id} \\ \mathcal{A}^\theta_0/H & \xrightarrow{\phi\gamma_0^{-1}} & \varphi(V)/W \end{array}$$

This means that in order to find representatives for the W-orbits in $\varphi_k(V_k)$ we can look at H-conjugacy classes of θ-stable maximal k-split tori instead of H_k-conjugacy classes. This will enable us to characterize the Δ_0-standard involutions in $\varphi_k(V_k)/W$ by using the classification of θ-stable maximal k-split tori in [12, 13, 14].

8.3. Let A be a θ-stable maximal k-split torus of G with A^- a maximal (θ, k)-split torus and let S be a θ-stable maximal k-split torus of G such that S^+ a maximal k-split torus of H, $A^+ \subset S^+$ and $A^- \supset S^-$. By [13, 4.10] such a pair (A, S) always exists. Let $g_1 \in Z_G(A^+S^-)$ such that $g_1Sg_1^{-1} = A$, $n_1 = g_1\theta(g_1)^{-1} \in N_G(A)$ and $w_1 \in W(A)$ corresponding Weyl group element. One easily checks that w_1 is an involution and by 5.3 w_1 is (θ, k)-singular. It is easy to see that we can choose w_1 to be Δ_0-standard. Let $\Pi_1 \subset \Delta_0(-\theta)$ such that $w_1 = w^0_{\Pi_1}$. This is called the maximal (θ, k)-singular involution. Since all θ-stable maximal k-split tori G containing a maximal k-split torus of H are conjugate under H we get the following result:

Lemma 8.4. *All maximal (θ, k)-singular involutions are W-conjugate.*

Combining the results in [13, §4] with the results of section 7 we get the following result.

Theorem 8.5. *Let A, S, $w^0_{\Pi_1}$ be as above and let A_1 be a θ-stable maximal k-split torus of G. Then we have the following.*

1. *There exists $h \in H$ such that $A_2 = hA_1h^{-1}$ satisfies $A^+ \subset A_1^+ \subset S^+$ and $A^- \supset A_1^- \supset S^-$.*
2. *There exists $g \in Z_G(A^+A_2^-)$ such that $gSg^{-1} = A$.*
3. *Let $n = g\theta(g)^{-1} \in N_G(A)$ and $w \in W(A)$ corresponding Weyl group element. Then w is a (θ, k)-singular involution.*
4. *If $w^0_\Pi \in \varphi_k(V_k)$ is a Δ_0-standard involution, then w^0_Π is (θ, k)-singular and there exists $s \in W$ such that $s(\Pi) \subset \Pi_1$.*
5. *Every W-orbit in $\varphi_k(V_k)$ has a representative w^0_Π with $\Pi \subset \Pi_1$.*

Remark 8.6. It follows from the above result that in order to compute representatives for $\varphi_k(V_k)/W$ it suffices to determine the maximal (θ, k)-singular involutions, which we can choose Δ_0-standard and representatives for all other W-orbits in $\varphi_k(V_k)$ are determined by the subsets of Π_1 which correspond to involutions in Λ^0_Δ. So to compute $\varphi_k(V_k)$ we need to extend the algorithm for $\varphi(V)$ in 7.19 with the following additional step:

(6) For each k-involution determine the subset $\Pi \subset \Delta_0$ which represents the maximal (θ, k)-singular involution. The involutions $w^0_\Pi \in \Lambda^0_\Delta \cap \varphi(V)$ with $\Pi \subset \Pi_1$ are representatives for the W-orbits in $\varphi_k(V_k)$.

Remarks 8.7. (1). The maximal (θ, k)-singular involutions depend on the k-involution and not on the (Γ, θ)-index. In order to get a list of these for a specific field k one needs first a classification of the k-involutions for that field. Sofar that classification is only known for $k = \bar{k}$ and $k = \mathbb{R}$ (see [11]).
(2). For $k = \mathbb{R}$ a list of the maximal (θ, k)-singular involutions is given in [14]. So combined with the algorithm in 7.19 we can compute $\varphi_k(V_k)$ for $k = \mathbb{R}$.
(3). The computation of the orbit closures in V_k is much more complicated than for V. For a further discussion of this see [18].

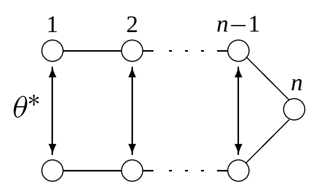
8.8. In [14] we showed that for $k = \mathbb{R}$ there exists a one to one correspondence between (θ, k)-singular involutions in A and H_k-conjugacy classes of θ-stable maximal k-split tori. This leads to the following result:

Theorem 8.9. *Assume $k = \mathbb{R}$ and let A, V_k, φ_k, etc. be as above. The map $\phi_k \gamma_k^{-1} : \mathcal{A}_k^\theta / H_k \to \varphi_k(V_k)/W \subset \mathcal{I}/W$ is a bijection.*

Remark 8.10. It follows from this result that similar as in the case of $k = \bar{k}$ in [19] the fibers of $\varphi_k : V_k \to \mathcal{I}$ for $k = \mathbb{R}$ differ a Weyl group element. So these can be computed in a similar way as in [19] for $k = \bar{k}$. We note however that a more efficient description of these is still needed. We hope to address this question in a future paper.

We conclude with giving an example, which illustrates the above algorithm.

Example 8.11. Let $G = \mathrm{SL}_{2n}(k)$ and assume $k = \mathbb{R}$. Let $\theta(g) = L({}^t g^{-1})L^{-1}$, where $L = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Then $H \simeq \mathrm{SP}_{2n}$. Using the notation in [15] (see also [11] or [16]) the pair (G, θ) is of type $A_{2n-1}^{n,2n-1}(III_b, I, \epsilon_0)$. The corresponding (Γ, θ)-index is:



Let B be the Borel subgroup of upper triangular matrices, T the group of diagonal matrices and $A = \{\mathrm{diag}(a_1, \dots, a_n, a_1, \dots, a_n) \mid a_1, \dots, a_n \in k^*\}$. Clearly $A \subset T$, T is maximal k-split and A is both maximal θ-split and (θ, k)-split. The orbit BH is open in G. Let $\Phi = \Phi(T)$ be the root system of T with respect to G, Φ^+ the set of positive roots of Φ related to B, Δ the corresponding basis of Φ, $W = W(T)$ the Weyl group of T and $X = X^*(T)$ the group of characters of T. Let $w_0 \in W$ such that $\theta(\Phi^+) = w_0(\Phi^+)$, $\theta' = \theta w_0$, $v_0 \in V$ and $n_0 = x(v_0)\theta(x(v_0))^{-1} \in N_G(T)$ such that n_0 induces w_0 in W and let ζ be the involution of G given by $\zeta(x) = n_0^{-1}\theta(x)n_0$, $x \in G$. By Corollary 7.3 $w_0 = w^0_{\Delta_0(\theta)} w^0_\Delta$, where $\Delta_0(\theta) = \{\alpha \in \Delta \mid$

$\theta(\alpha) = \alpha\}$. Since (G, θ) is quasi (θ, k)-split, we have $\Delta_0(\theta) = \emptyset$ and $w_0 = w^0_\Delta$. The set $\mathcal{I}$ consists of the set of involutions in W. Let $\mathcal{I}_{\theta'} = \mathcal{I} \cdot w_0$ as in Lemma 3.24.

From [12, Table II] it follows that the involutions w^0_Π are all of type $A_1 + \ldots + A_1$, (r times) and there is only one conjugacy class for each type.

Since (G, θ) is quasi (θ, k)-split it follows that the map $\varphi : V \to \mathcal{I}$ is surjective. So in particular $\varphi(V) = \mathcal{I}$. However from [14, Table 2] it follows that the longest (θ, k)-split involution is the id, so $\varphi_k(V_k) = \{\mathrm{id}\}$. The computation of $\mathcal{I}$ is as in [19].

An implementation of the above algorithm in LiE is illustrated in Table 1. We use there the following notation. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and write $s_i = s_{\alpha_i}$ for the reflection defined by $\alpha_i \in \Delta$. In the table we list the minimal coset representatives of $W/W(\Pi)$ for each of the involutions $w^0_\Pi \in \mathcal{I}_\Delta \simeq \mathcal{I}_\Delta(V)$.

Table 1. Calculation of $\varphi(V) = \mathcal{I}$

	w^0_Π	$W/W(\Pi)$
$n = 2$	id	$\{\mathrm{id}\}$
	$w_0 = s_1$	$\{\mathrm{id}\}$
$n = 3$	id	$\{\mathrm{id}, s_1, s_2\}$
	$w_0 = s_2s_1s_2$	$\{\mathrm{id}\}$
$n = 4$	id	$\{\mathrm{id}, s_1, s_2s_1\}$
	s_2	$\{\mathrm{id}, s_1, s_3, s_1s_3, s_2s_1, s_2s_3\}$
	$w_0 = s_1s_2s_1s_3s_2s_1$	$\{\mathrm{id}\}$

References

[1] S. Abeasis, *On a remarkable class of subvarieties of a symmetric variety*, Adv. in Math. **71** (1988), 113–129.

[2] E. van den Ban, *The principal series for a reductive symmetric space I. H-fixed distribution vectors*, Ann. Sci. Ec. Norm. Sup. **21** (1988), 359–412.

[3] van den Ban, E. and Schlichtkrull, H., *The most continuous part of the Plancherel decomposition for a reductive symmetric space I*, To appear.

[4] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate texts in mathematics, Springer Verlag, New York, 1991.

[5] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–152.

[6] ______, *Compléments a l'article "groupes réductifs"*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 253–276.

[7] N. Bourbaki, *Groups et algébres de Lie, Chapitres 4,5,6*, Éléments de Mathématique, Hermann, Paris, 1968.

[8] J.-L. Brylinski and P. Delorme, *Vecteurs distributions H-invariants pour les séries principales géneralisées d'espaces symétriques réductifs et prolongement méromorphe d'integrales d'Eisenstein*, Invent. Math. **109** (1992), 619–664.
[9] M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Annals of Math. **111** (1980), 253–311.
[10] I. Grojnowski, *Character sheaves on symmetric spaces*, Ph.D. thesis, Massachusetts Institute of Technology, June 1992.
[11] A. G. Helminck, *Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces*, Adv. in Math. **71** (1988), 21–91.
[12] ______, *Tori invariant under an involutorial automorphism I*, Adv. in Math. **85** (1991), 1–38.
[13] ______, *Tori invariant under an involutorial automorphism II*, (1993), Adv. in Math., To appear.
[14] ______, *Tori invariant under an involutorial automorphism III*, (1994), To appear.
[15] ______, *On the classification of symmetric k-varieties I*, To appear.
[16] ______, *Symmetric k-varieties*, Algebraic Groups and Their Generalizations: Classical Methods (Providence, RI), vol. 56, Proc. Sympos. Pure Math., no. Part 1, Amer.Math. Soc, 1994, pp. 233–279.
[17] ______, *On the orbit closures of parabolic k-subgroups acting on symmetric varieties I*, (1994), To appear.
[18] ______, *On the orbit closures of parabolic k-subgroups acting on symmetric varieties II, real reductive groups*, To appear.
[19] ______, *Computing B-orbits on G/H*, J. Symbolic Computation **21** (1996), 169–209.
[20] A. G. Helminck and G. F. Helminck, *H_k-fixed distributionvectors for representations related to $\mathfrak{p}$-adic symmetric varieties*, To appear.
[21] A. G. Helminck and G. Schwarz, *Orbits and invariants associated with a pair of involutions of a reductive group*, In preparation (1996).
[22] A. G. Helminck and S. P. Wang, *On rationality properties of involutions of reductive groups*, Adv. in Math. **99** (1993), 26–96.
[23] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer-Verlag, New York, 1975.
[24] H. Jacquet, K. Lai, and S. Rallis, *A trace formula for symmetric spaces*, Duke Math. J. **70** (1993).
[25] G. Lusztig, *Symmetric spaces over a finite field*, The Grothendieck Festschrift Vol. III (Boston, MA), Progr. Math., vol. 88, Birkhäuser, 1990, pp. 57–81.
[26] G. Lusztig and D.A. Vogan, *Singularities of closures of K-orbits on flag manifolds*, Invent. Math. **71** (1983), 365–379.
[27] M. A. A. van Leeuwen, A. M. Cohen and B. Lisser, *A package for Lie group computations*, CAN (Computer Algebra Nederland), Amsterdam, 1992.
[28] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331–357.
[29] T. Oshima and T. Matsuki, *A description of discrete series for semisimple symmetric spaces*, Adv. Stud. in Pure Math., vol. 4, Academic Press, Orlando, FL, 1984, pp. 331–390.
[30] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators in an affine symmetric space*, Invent. Math. **57** (1980), 1–81.
[31] C. Procesi and C. de Concini, *Complete symmetric varieties I*, Lecture notes in Math., vol. 996, Springer-Verlag, New York/Berlin, 1983, (II, III preprints, University of Rome), pp. 1–44.

[32] C. Rader and S. Rallis, *Spherical characters on $\mathfrak{p}$-adic symmetric spaces*, American Journal of Mathematics **118** (1996), no. 1, 91–178.
[33] R.W. Richardson, *Orbits, invariants and representations associated to involutions of reductive groups*, Invent. Math. **66** (1982), 287–312.
[34] R.W. Richardson and T.A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), 389–436.
[35] W. Rossmann, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979), 157–180.
[36] I. Satake, *Classification theory of semisimple algebraic groups*, Lecture Notes in Pure and Appl. Math., vol. 3, Dekker, Berlin, 1971.
[37] P. Slodowy, *Habilitationsschrift*, 1984, Univ. of Bonn.
[38] T. A. Springer, *Linear algebraic groups*, Progr. Math., vol. 9, Birkhäuser, Boston/Basel/Stuttgart, 1981.
[39] ______, *Some results on algebraic groups with involutions*, Algebraic groups and related topics, Adv. Stud. in Pure Math., vol. 6, Academic Press, Orlando, FL, 1984, pp. 525–543.
[40] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., vol. 80, Amer. Math. Soc., Providence, RI, 1968.
[41] J. R. Stembridge, *A Maple package for root systems and finite Coxeter groups*, 1992.
[42] J. Tits, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups (Providence, RI), Proc. Sympos. Pure Math., vol. IX, Amer.Math. Soc., 1966, pp. 33–62.
[43] Y. L. Tong and S. P. Wang, *Geometric realization of discrete series for semisimple symmetric space*, Invent. Math. **96** (1989), 425–458.
[44] D. A. Vogan, *Irreducible characters of semi-simple Lie groups III*, Invent. Math. **71** (1983), 381–417.
[45] J. A. Wolf, *Finiteness of orbit structure for real flag manifolds*, Geom. Dedicata **3** (1974), 377–384.

E-mail address: `loek@math.ncsu.edu`