

# COMPUTING $B$ -ORBITS ON $G/H$

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**Abstract.** The orbits of a Borel subgroup acting on a symmetric variety  $G/H$  occur in several areas of mathematics. For example, these orbits and their closures are essential in the study of Harish Chandra modules (see [Vog83]). There are several descriptions of these orbits, but in practice it is actually very difficult and cumbersome to compute the orbits and their closures. Since the characterizations of these orbits are very combinatorial in nature, this work could conceivably be done by a computer.

In this paper we prove a number of additional properties of these orbits and combine these with properties of the various descriptions of these orbits to obtain an efficient algorithm. This algorithm can be implemented on a computer by using existing symbolic manipulation programs or by writing an independent program.

## Introduction

In the last two decades computer algebra has had a major impact on many areas of mathematics. Best known are its accomplishments in number theory, algebraic geometry and group theory. In the last few years several people have also started to devise and implement algorithms related to Lie theory. The most noteworthy examples of this are the package LiE written by CAN (see [MLL92]) and the packages *Coxeter* and *Weyl* by J. Stembridge, which are written in Maple (see [Ste92]). In the LiE package most of the basic combinatorial aspects of Lie theory have been implemented, following the excellent description and tables in [Bou68]. There remain many, more complex aspects of Lie theory for which it would be useful to have a computer implementation of the structure. In this paper we lay the foundation for a computer algebra package for computations related to symmetric varieties. These varieties are defined as the spherical homogeneous spaces  $G/H$  with  $G$  a reductive algebraic group and  $H$  the fixed point group of an involution. They occur in many problems in representation theory, geometry, and number theory. (See for example [OM84], [PdC83], [TW89], [Vog83]). Perhaps the best known application is in the representation theory of Lie groups. There the symmetric varieties are of fundamental importance in many problems, ranging from the representation theory on symmetric spaces to the characterization of the characters of the irreducible representations of a semisimple Lie group.

Most of the fine structure of the symmetric varieties can be described by the orbits of a Borel subgroup  $B$  acting on the symmetric variety  $G/H$ . The closure of an orbit is an algebraic variety, which in most cases has quite complicated singularities. In practice it is quite difficult and cumbersome to actually compute these

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orbits and their closures. Fortunately most of the geometry of these orbits and their closures can be described combinatorially and therefore most of this work could be done by a computer. For this we will need a description which is both precise and efficient enough to be implemented on a computer. Since the number of orbits becomes quite large very quickly (approximately  $|W|^{\frac{1}{2}}$ , where  $W$  is the Weyl group of  $G$ ), it will be important to find an algorithm that is as efficient as possible. This means that the algorithm has to use as much as possible of the rich combinatorial structure, instead of blindly computing the orbits.

There are several descriptions of these orbits and their closures (see [Spr84], [Mat79, Mat83], [HW93], [Vog83] and [RS90]). For example we will often use the orbits of  $H$  acting on the flag variety  $G/B$  instead of the  $B$ -orbits on  $G/H$ . In principle one can derive a method to compute the orbits from each of the different characterizations of the orbits, but not all of these methods are precise enough to be implemented on a computer.

A small additional problem in describing an algorithm is that the results about these orbits are contained in a number of different papers, in which different approaches and notations are used. Translating the results from one setting to another becomes sometimes technically quite complicated. So part of what needs to be done is to describe and collect the relevant combinatorial structure of these orbits in one setting. In this paper we will mainly follow [HW93] and give a description of the orbit structure starting from an arbitrary orbit instead of the traditional closed orbit as in [Spr84] and [RS90]. These results enable us to compute the orbits and their closures starting from the unique open orbit, instead of a closed orbit. Since most of the fine structure of a symmetric variety  $G/H$  is associated with the open orbit, this is in general a more natural starting point for computing the orbit decomposition. Many other results, like the combinatorial description of the orbit closures in [RS90] which are needed for the computation of the closures of these orbits, will also be translated to this setting. We will also derive a number of additional results about these orbits which are useful for the actual computation. Finally, aspects from these various characterizations will be combined in an efficient algorithm, which computes not only the orbits, but also their closures. This algorithm can be implemented on a computer using existing symbolic manipulation programs or by writing an independent program. Probably the most efficient implementation would be to write an extension to the program LiE. An example of this is given in 8.33. Note that most of the remaining structure of a symmetric variety, like the restricted root system, will also follow from the above structure.

These results provide a first step towards a solution of a more general problem: namely, to classify the orbits of minimal parabolic  $\mathbb{R}$ -subgroups acting on semisimple symmetric spaces. These orbits are of importance in the characterization of the discrete series representations of the semisimple symmetric space  $G_{\mathbb{R}}/H_{\mathbb{R}}$ . There are similar characterizations of these orbits, but there are many more semisimple symmetric spaces to consider and also some additional technical problems. For more details, see [HW93], [Mat79, Mat83], [Hel91, Hel97, Hel98] and [Hel].

A brief outline of this paper is as follows. In section 1 we introduce notation and review a few generalities about symmetric varieties. In section 2 we discuss the various characterizations of the orbits of a Borel subgroup on  $G/H$ . These orbits have been studied by many people. A first characterization for algebraically closed fields of characteristic not 2 was given in [Spr84]. For  $k = \mathbb{R}$  or  $\mathbb{C}$  a slightly different characterization was given in [Mat79]. Additional results about these orbits appear in [RS90], [Mat83] and [Vog83]. Here we will mainly follow [HW93], where a generalization of this orbit decomposition to non-algebraically closed fields was discussed. One way to characterize the orbits is by  $H$ -orbits of pairs  $(B, T)$ , where  $T$  is a  $\theta$ -stable maximal torus and  $B \supset T$  a Borel subgroup. This is the easiest characterization to describe and we will use this one in the introduction. In section 2 we also discuss the action of the Weyl group  $W$  on these orbits and the connection of these orbits with twisted involutions in the Weyl group. In particular we analyze the natural map  $\varphi : V \rightarrow \mathcal{I} \subset W$ , where  $V$  is the set of orbits and  $\mathcal{I}$  the set of twisted involutions in the Weyl group  $W$ . We will classify the set of orbits  $V$  by computing the image and fibers of  $\varphi$ . Most of these results are from [HW93], [RS90] and [Hel].

In section 3 we discuss the twisted involutions in the Weyl group and give a detailed characterization. In [Spr84] a characterization of the twisted involutions was given in the case that there exists a basis  $\Delta$  of the root system  $\Phi(T)$  such that  $\theta(\Delta) = \Delta$ . This condition means that the corresponding orbit of  $B$  in  $G/H$  is closed. We generalize this characterization to an arbitrary basis  $\Delta$  of  $\Phi(T)$ . For this characterization we pass to an involution  $\theta'$  of  $\Phi(T)$ , which stabilizes the fixed basis  $\Delta$ . This involution is also essential in the description of the orbit closures.

In section 4 we discuss the relation between the  $B$ -orbits on the symmetric varieties associated with the involutions  $\theta$  and  $\theta'$  as above. This is of importance if one wants to classify the orbits starting from a pair  $(B, T)$  other than a standard pair (i.e. one in which  $\theta(B) = B$ ).

The next section discusses the orbit closures and the (Bruhat) order on these orbits, induced by the closure relations. A combinatorial description of this geometrically defined order on  $V$  was given in [RS90]. Their description of the order was given with respect to a standard pair  $(B, T)$ . In section 5 we show how the results can be generalized to an arbitrary pair  $(B, T)$ . Our formulation of the combinatorial Bruhat order differs slightly from the one in [RS90] and is more geared toward the computation of the orbit closures. For example we generalize the notion of “Bruhat descendants” from the Weyl group to  $V$  and the set of twisted involutions in the Weyl group.

The relation between Borel subgroups containing a fixed maximal torus  $T$  and orders on  $\Phi(T)$  is discussed in section 6. This includes a characterization of the orbits of minimal and maximal dimension in a  $W$ -orbit in  $V$ . In section 7 we discuss a number of properties of the open orbit.

Finally in section 8 we present the algorithm to compute the orbits and their closures. This includes a discussion of possible computational complications and also an example.

## 1. Preliminaries

In this section we introduce notation and recall a few results from [HW93], [Hel91] and [Hel98]. All algebraic groups and algebraic varieties are taken over an algebraically closed field  $k$  of characteristic  $\neq 2$  and all algebraic groups considered are linear algebraic groups. Our basic reference for reductive groups will be the papers [BT65, BT72] and also the books [Hum75] and [Spr81]. We shall follow their notations and terminology.

**1.1.** Given an algebraic group  $G$ , the identity component is denoted by  $G^0$ . We use  $L(G)$  (or  $\mathfrak{g}$ , the corresponding lower case German letter) for the Lie algebra of  $G$ . If  $H$  is a subset of  $G$ ,  $N_G(H)$  (resp.  $Z_G(H)$ ) is the normalizer (resp. centralizer) of  $H$  in  $G$ . We write  $Z(G)$  for the center of  $G$ . The commutator subgroup of  $G$  is denoted by  $[G, G]$ .

If  $G$  is a reductive algebraic group,  $H$  a closed subgroup of  $G$  and  $A$  a subtorus of  $H$  then we denote by  $X^*(A)$  (resp.  $X_*(A)$ ) the group of characters of  $A$  (resp. one-parameter subgroups of  $A$ ) and by  $\Phi(H, A)$  the set of roots of  $A$  in  $H$ . Let  $W(H, A) = N_H(A)/Z_H(A)$  denote the Weyl group of  $H$  relative to  $A$ . If  $\alpha \in \Phi(H, A)$ , then let  $U_\alpha$  denote the unipotent subgroup of  $H$  corresponding to  $\alpha$ . If  $A$  is a maximal torus, then  $U_\alpha$  is one-dimensional. Given a quasi-closed subset  $\psi$  of  $\Phi(G, A)$ , the groups  $G_\psi$  and  $G_\psi^*$  are defined in [BT65, 3.8]. If  $G_\psi^*$  is unipotent,  $\psi$  is said to be unipotent and often one writes  $U_\psi$  for  $G_\psi^*$ .

Throughout the paper  $G$  will denote a connected reductive algebraic group.

**1.2. Involutions of  $G$ .** Let  $G$  be a connected algebraic group,  $\theta$  an automorphism of  $G$  of order two and  $G_\theta = \{g \in G \mid \theta(g) = g\}$  the set of fixed points of  $\theta$ . This is a subgroup of  $G$  which is reductive if  $G$  is reductive. If  $G$  is semisimple and simply connected, then  $G_\theta$  is connected, but in general  $G_\theta$  is not necessarily connected. The automorphism  $\theta$  will also be called an involution of  $G$ .

If  $G$  is reductive and  $H$  an open subgroup of  $G_\theta$ , then we call the variety  $G/H$  a *symmetric variety*. Symmetric varieties are spherical.

*Notation 1.3.* Given  $g, x \in G$ , the *twisted action* associated to  $\theta$  is given by  $(g, x) \mapsto g * x = gx\theta(g)^{-1}$ . Let  $Q$  and  $Q'$  denote the subsets of  $G$  defined by

$$Q = \{g^{-1}\theta(g) \mid g \in G\},$$

$$Q' = \{g \in G \mid \theta(g) = g^{-1}\}.$$

The set  $Q$  is contained in  $Q'$ . Both  $Q$  and  $Q'$  are invariant under the twisted action associated to  $\theta$ . There are only finitely many twisted  $G$ -orbits in  $Q'$  and each such orbit is closed [?, see]sect. 9]ri2. In particular,  $Q$  is a connected closed subvariety of  $G$ .

Define a morphism  $\tau : G \rightarrow G$  by

$$(1) \quad \tau(x) = x\theta(x^{-1}), \quad (x \in G).$$

Then  $\tau(G) = Q$  is a closed subvariety of  $G$  and  $\tau(x) = \tau(y)$  if and only if  $x^{-1}y \in G_\theta$ . The morphism  $\tau$  induces an isomorphism of the coset space  $G/G_\theta$  onto  $Q$

[?, see]2.2]sp1. This map will be essential in the study of involutions of reductive algebraic groups and their symmetric varieties.

**1.4.  $\theta$ -stable tori.** Let  $G$  be a connected reductive algebraic group and  $\theta$  an involution of  $G$ . In this subsection we discuss  $\theta$ -stable tori and the restricted root system associated with a symmetric variety. First we give some notation.

**1.5.** Let  $T$  be a  $\theta$ -stable torus of  $G$ . (Recall that according to a result of Steinberg [?, see]7.5]st, there exists a  $\theta$ -stable torus  $T$  of  $G$ .) Write  $T_\theta^+ = (T \cap G_\theta)^0$  and  $T_\theta^- = \{x \in T \mid \theta(x) = x^{-1}\}^0$ . Then it is easy to verify that the product map

$$\mu : T_\theta^+ \times T_\theta^- \rightarrow T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular  $T = T_\theta^+ T_\theta^-$  and  $T_\theta^+ \cap T_\theta^-$  is a finite group. (In fact it is an elementary abelian 2-group.) The automorphisms of  $\Phi(G, T)$  and  $W(G, T)$  induced by  $\theta$  will also be denoted by  $\theta$ .

The natural tori associated with the symmetric variety  $G/H$  are defined as follows:

**Definition 1.6.** A torus  $A$  of  $G$  is called  $\theta$ -split if  $\theta(a) = a^{-1}$  for every  $a \in A$ .

The set of roots of a maximal  $\theta$ -split torus gives a restricted root system, which is the natural root system of a symmetric variety (see [Ric82b]). If  $k = \mathbb{C}$ , then  $G/H$  is the complexification of a Riemannian symmetric space. Therefore much of the fine structure of the symmetric varieties is similar to that of the Riemannian symmetric spaces.

## 2. Orbits of Borel subgroups on symmetric varieties

In this section we review some results about the orbits of Borel subgroups acting on the symmetric varieties. Most of these results are due to Springer (see [Spr84]). Over the real numbers these orbits were also studied by Matsuki (see [Mat79, Mat83]). We follow the description given in [HW93], which is basically a generalization of [Spr84] and [Mat79] to non-algebraically closed fields. However, besides such a generalization, the results in [HW93] yield also some additional information about these orbits which is not contained in [Spr84] and [Mat79].

**2.1.** Let  $G$  be a reductive algebraic group,  $\theta$  an involution of  $G$ ,  $G_\theta = \{g \in G \mid \theta(g) = g\}$  the set of fixed points of  $\theta$  and  $H$  an open subgroup of  $G_\theta$ . As in (1) let  $\tau : G \rightarrow G$  be the morphism defined by  $\tau(x) = x\theta(x)^{-1}$ ,  $x \in G$ .

Let  $B$  be a Borel subgroup of  $G$ ,  $T$  a  $\theta$ -stable maximal torus of  $B$ ,  $N = N_G(T)$  and  $W = W(T) = N_G(T)/T$  the corresponding Weyl group. The key lemma in the study of the orbits of Borel subgroups on symmetric varieties is the following result [?, see]Lemma 2.6]h-w.

**Lemma 2.2.** *Every Borel subgroup  $B$  of  $G$  contains a  $\theta$ -stable maximal torus of  $B$ , unique up to conjugation by an element of  $H \cap R_u(B)$ .*

Using this result we can reduce consideration of the orbit decomposition to a question about  $\theta$ -stable maximal tori and their Weyl groups.

**Proposition 2.3.** *If  $g \in G$  satisfies  $\theta(g) = g^{-1}$ , then there exists  $x \in U$  such that  $xg\theta(x)^{-1} \in N_G(T)$ .*

**2.4.** The  $B$  orbits on  $G/H$  can be described now as follows. Set  $\mathcal{V} = \{x \in G \mid \tau(x) \in N\}$ . The group  $T \times H$  acts on  $\mathcal{V}$  by  $(x, z)y = xyz^{-1}$ , for  $(x, z) \in T \times H$  and  $y \in \mathcal{V}$ . Let  $V$  be the set of  $(T \times H)$ -orbits on  $\mathcal{V}$ . If  $v \in V$ , we let  $x(v) \in \mathcal{V}$  be a representative of the orbit  $v$  in  $V$ . The set  $V$  characterizes the double cosets (see [HW93]).

**Proposition 2.5.** *The inclusion map  $\mathcal{V} \rightarrow G$  induces a bijection from the set  $V$  of  $(T \times H)$ -orbits on  $\mathcal{V}$  onto the set of  $(B \times H)$ -orbits on  $G$ . In particular  $G$  is the disjoint union of the double cosets  $Bx(v)H$ ,  $v \in V$ .*

*Remark 2.6.* For algebraically closed fields this result is due to [Spr84]. For  $k = \mathbb{R}$  these orbits were characterized in a slightly different way (see [Mat79] and [Ros79]). It is not hard to derive this characterization of the double cosets from Proposition 2.5. The result is as follows.

**Corollary 2.7.** *Let  $\{T_i \mid i \in I\}$  be representatives of the  $H$ -conjugacy classes of  $\theta$ -stable maximal tori in  $G$ . Then*

$$H \backslash G/B \cong \bigcup_{i \in I} W(H, T_i) \backslash W(G, T_i)$$

*Remark 2.8.* The finiteness of  $V$  was proved in [Spr84]. The finiteness of the orbit decomposition was also discussed in [Wol74], [Ros79], [Mat79] and [HW93].

**2.9.** From the above results it follows that we can also characterize the orbits as follows. Let  $\mathcal{B}$  denote the set of all pairs  $(B, T)$ , where  $T$  is a  $\theta$ -stable maximal torus of  $G$  and  $B \supset T$  a Borel subgroup of  $G$ . The group  $H$  acts on  $\mathcal{B}$  by conjugation and we can identify the set of orbits  $V$  with  $\mathcal{B}/H$ , the set of  $H$ -conjugacy classes of pairs  $(B, T)$  in  $\mathcal{B}$ .

A pair  $(B, T)$  in  $\mathcal{B}$  is called a *standard pair* if  $\theta(B) = B$ . In that case it easily follows that the orbit  $BH$  is closed and  $T_\theta^+$  is a maximal torus of  $H$ . For more details, see [Spr84] or [HW93].

**2.10.** From Proposition 2.5 it follows that for studying the  $B$ -orbits on  $G/H$  it suffices to consider the set  $\mathcal{V}$ . Using the twisted action of  $G$  on  $G$ , as in 1.3, one can also characterize the orbit decomposition as follows. Consider the restriction map  $\tau|_{\mathcal{V}} : \mathcal{V} \rightarrow G$ . Then  $\tau(\mathcal{V}) = N_G(T) \cap \tau(G)$ , where  $\tau(G) = G * \text{id}$ , the twisted orbit of  $\text{id} \in G$ . The set  $N_G(T) \cap \tau(G)$  is stable under the twisted action of  $T$  and we have the following result:

**Corollary 2.11.** *There is a bijection, induced by the map  $\tau|_{\mathcal{V}}$ , between the set  $V$  of  $(T \times H)$ -orbits on  $\mathcal{V}$  and the set of twisted  $T$ -orbits on  $N_G(T) \cap \tau(G)$ .*

This characterization is often very useful for calculating the fibers of the twisted involutions in the natural characterization of  $V$  in the Weyl group. For more details see 2.15 and 8.21.

**2.12. Action of  $W$  on  $V$ .** There is a natural (left) action of the Weyl group  $W = W(T)$  on  $V$ , which is defined as follows.

Let  $v \in V$  and let  $x = x(v)$ . If  $n \in N$ , then  $nx \in \mathcal{V}$  and its image in  $V$  depends only on the image of  $n$  in  $W$ . This defines a left action of  $W$  on  $V$ , denoted by  $(w, v) \rightarrow w \cdot v$  ( $w \in W, v \in V$ ). The  $W$  orbits on  $V$  are now in correspondence with conjugacy classes of maximal tori. This can be seen as follows.

**2.13.** Let  $\mathcal{T}$  denote the set of maximal tori of  $G$  and let  $\mathcal{T}^\theta$  be the fixed point set of  $\theta$ , i.e., the set of  $\theta$ -stable maximal tori. Let  $T$  and  $\mathcal{V}$  be as above. If  $x \in \mathcal{V}$ , then  $xTx^{-1}$  is a  $\theta$ -stable maximal torus and conversely any  $\theta$ -stable maximal torus in  $\mathcal{T}^\theta$  can be written as  $xTx^{-1}$  for some  $x \in \mathcal{V}$ . The group  $H$  acts on  $\mathcal{T}^\theta$  by conjugation. If  $v \in V$ , then  $x(v)^{-1}Tx(v) \in \mathcal{T}^\theta$ . This determines a map from  $V$  to the orbit set  $\mathcal{T}^\theta/H$ . Since this map is constant on  $W$ -orbits, we also get a map of orbit sets

$$(2) \quad \gamma : V/W \rightarrow \mathcal{T}^\theta/H.$$

We have now the following characterization of the  $W$  orbits in  $V$ .

**Proposition 2.14.** *Let  $G, \mathcal{T}^\theta$  and  $\gamma$  be as above. Then  $\gamma : V/W \rightarrow \mathcal{T}^\theta/H$  is bijective.*

For a proof of this result, see [HW93] and [Spr84].

There is a natural map from  $V$  into  $W$ , induced by the map  $\tau|_{\mathcal{V}} : \mathcal{V} \rightarrow N_G(T)$ . In the next subsection we analyze this map in more detail.

**2.15. Twisted involutions.** Recall that an element  $a \in W(T)$  is a *twisted involution* if  $\theta(a) = a^{-1}$  (see [Spr84] or [HW93]). Let  $\mathcal{I} = \mathcal{I}_\theta = \mathcal{I}(W(T), \theta)$  be the set of twisted involutions in  $W(T)$ .

If  $v \in V$ , then  $\tau(x(v))T \in W(T)$  is a twisted involution and is independent of the choice of the representative  $x(v) \in \mathcal{V}$  of  $v$ . This defines a map  $\varphi : V \rightarrow \mathcal{I}$ , given by  $\varphi(v) = \tau(x(v))T$ , which is essential in the study of the orbit closures of the  $B$ -orbits in  $G/H$ .

**2.16.** The Weyl group  $W$  acts also on  $\mathcal{I}$ . This action comes from the *twisted action* of  $W$  on (the set)  $W$ , which is defined as follows: if  $w, w_1 \in W$ , then  $w * w_1 = ww_1\theta(w)^{-1}$ . If  $w_1 \in W$ , then  $W * w_1 = \{w * w_1 \mid w \in W\}$  is the *twisted  $W$ -orbit* of  $w_1$ . Now  $\mathcal{I}$  is stable under the twisted action, so that we get a twisted action of  $W$  on  $\mathcal{I}$ .

The map  $\varphi : V \rightarrow \mathcal{I}$  is equivariant with respect to the action of  $W$  on  $V$  and the twisted action of  $W$  on  $\mathcal{I}$ :

**Lemma 2.17.** *Let  $w \in W$  and  $v \in V$ . Then  $\varphi(w \cdot v) = w * \varphi(v)$ .*

Since  $\varphi : V \rightarrow \mathcal{I}$  is  $W$ -equivariant, we get a map  $\phi : V/W \rightarrow \mathcal{I}/W$ . Note that the image of  $\varphi$  is a union of twisted  $W$ -orbits. One easily proves now the following properties of the maps  $\varphi$  and  $\phi$  (see [HW93] and [RS90]).

**Proposition 2.18.** *Let  $G, \varphi$  and  $\phi$  be as above. Then we have the following.*

- (i) *Let  $v_1, v_2 \in V$ . If  $\varphi(v_1) = \varphi(v_2)$ , then  $v_1 = w \cdot v_2$  for some  $w \in W$ .*

- (ii)  $\phi : V/W \rightarrow \mathfrak{L}/W$  is injective.
- (iii) There is a bijection from  $\varphi(V)/W$  onto  $\mathcal{T}^\theta/H$ .

Before we can give a description of the orbit closures, we need first a description of the twisted involutions in the Weyl group. This will be discussed in the next section.

### 3. Twisted Involutions and Singular Roots

In this section we give a description of the set of twisted involutions  $\mathfrak{L}$ . In the case that  $\theta(\Phi^+) = \Phi^+$  a description of the twisted involutions was given in [Spr84]. One can easily generalize this description and give a similar description of the twisted involutions when  $\theta(\Phi^+) \neq \Phi^+$ . This follows essentially from the results in [HW93]. In the following we will discuss this characterization for the case that  $\theta(\Phi^+) \neq \Phi^+$ . First we need to review a few facts about singular roots.

**3.1.** Let  $B$  be a Borel subgroup of  $G$ ,  $T \subset B$  a  $\theta$ -stable maximal torus of  $B$ ,  $\Phi = \Phi(T)$  the root system of  $T$  with respect to  $G$ ,  $\Phi^+$  the set of positive roots of  $\Phi$  related to  $B$ ,  $\Delta$  the corresponding basis of  $\Phi$ ,  $W = W(T)$  the Weyl group of  $T$  and  $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$ . The Weyl group  $W$  is generated by  $\Sigma$ . In the sequel, the length function  $l$  and Bruhat order  $\leq$  on  $W$  are defined relative to  $\Sigma$ . Let  $E = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . If  $\sigma \in \text{Aut}(\Phi)$ , then we denote the eigenspace of  $\sigma$  for the eigenvalue  $\xi$  by  $E(\sigma, \xi)$ . For a subset  $\Pi$  of  $\Delta$  denote the subset of  $\Phi$  consisting of integral combinations of  $\Pi$  by  $\Phi_\Pi$ . Then  $\Phi_\Pi$  is a subsystem of  $\Phi$  with Weyl group  $W_\Pi$ . Let  $w_\Pi^0$  denote the longest element of  $W_\Pi$  with respect to  $\Pi$ .

The involution  $\theta$  of  $G$  induces an automorphism of  $W$ , also denoted by  $\theta$ , which is given by

$$\theta(w) = \theta \circ w \circ \theta, \quad w \in W.$$

If  $s_\alpha$  is the reflection defined by  $\alpha$ , then  $\theta(s_\alpha) = s_{\theta(\alpha)}$ ,  $\alpha \in \Phi$ .

**3.2.** The roots of  $\Phi$  can be divided into three subsets, according to the action of  $\theta$ , as follows.

- (a)  $\theta(\alpha) \neq \pm\alpha$ . Then  $\alpha$  is called *complex* (relative to  $\theta$ ).
- (b)  $\theta(\alpha) = -\alpha$ . Then  $\alpha$  is called *real* (relative to  $\theta$ ).
- (c)  $\theta(\alpha) = \alpha$ . Then  $\alpha$  is called *imaginary* (relative to  $\theta$ ).

For  $\alpha \in \Phi$  let  $T_\alpha = \{a \in T \mid s_\alpha(a) = a\}^0$ ,  $G_\alpha = Z_G(T_\alpha)$  and  $\overline{G}_\alpha = [G_\alpha, G_\alpha]$ . If  $\alpha$  is either real or imaginary, then  $G_\alpha$  is  $\theta$ -stable. The root  $\alpha$  is called  *$\theta$ -singular* if  $\overline{G}_\alpha \not\subset H$ .

A root  $\alpha \in \Phi$  with  $\theta(\alpha) = \alpha$  is called *compact imaginary* if  $\overline{G}_\alpha \subset H$ . A root  $\alpha \in \Phi$  with  $\theta(\alpha) = \alpha$  and  $\overline{G}_\alpha \not\subset H$  is called *imaginary  $\theta$ -singular*.

For real roots we have the following result (see [Hel88] or [Spr84]):

**Lemma 3.3.** *All roots  $\alpha \in \Phi$  with  $\theta(\alpha) = -\alpha$  are  $\theta$ -singular.*

*Remark 3.4.* In 3.2 we considered roots under the action of an involution  $\theta$ . In the case that  $(G, \theta)$  is defined over a field  $k$  which is not algebraically closed, one needs to consider also the action of the Galois group of  $\overline{k}/k$ , besides the action of

$\theta$ . This situation occurs frequently in the study of affine symmetric spaces or more generally in the study of symmetric  $k$ -varieties. For a more detailed discussion of the different singular roots which occur in this situation, see [Hel97].

The above definitions of real, complex and imaginary roots carry over to the Weyl group and the set  $V$ . We define first real, complex and imaginary elements for the Weyl group, which then will imply similar definitions for  $V$ .

**Definition 3.5.** Let  $\mathcal{I} = \mathcal{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$  denote the set of twisted involutions.

Given  $w \in \mathcal{I}_\theta$ , an element  $\alpha \in \Phi$  is called *complex* (resp. *real*, *imaginary*) relative to  $w$  if  $w\theta\alpha \neq \pm\alpha$  (resp.  $w\theta\alpha = -\alpha$ ,  $w\theta\alpha = \alpha$ ). We use the following notation:

$$\begin{aligned} C'(w, \theta) &= \{\alpha \in \Phi^+ \mid -\alpha \neq w\theta\alpha < 0\}, & R(w, \theta) &= \{\alpha \in \Phi^+ \mid -\alpha = w\theta\alpha\}, \\ C''(w, \theta) &= \{\alpha \in \Phi^+ \mid \alpha \neq w\theta\alpha > 0\}, & I(w, \theta) &= \{\alpha \in \Phi^+ \mid \alpha = w\theta\alpha\}. \end{aligned}$$

We will omit  $\theta$  from this notation if there is no ambiguity as to which involution we consider.

**3.6.** These definitions carry over to  $V$  now as follows. Fix  $v \in V$ , let  $x = x(v)$  and let  $T_1 = x^{-1}Tx$ . Then  $T_1$  is a  $\theta$ -stable maximal torus. The inner automorphism  $\text{Int}(x^{-1})$  defines an isomorphism  $\omega_v$  from  $\Phi(T)$  onto the root system  $\Phi_1 = \Phi(T_1)$ . We say that  $\alpha$  is real, complex or imaginary for  $v$  if  $\alpha_1 = \omega_v(\alpha)$  is real, complex or imaginary in the sense of 3.2 for  $T_1$ . If  $\alpha$  is real, complex or imaginary for  $v$ , then we also say that the reflection  $s_\alpha \in W$  is real, complex or imaginary for  $v$ .

The notions of real, complex and imaginary roots for  $v$  and for  $w = \varphi(v)$  are the same:

**Lemma 3.7.** *Let  $v \in V$ ,  $w = \varphi(v)$  and  $\alpha \in \Phi$ . Then  $\alpha$  is real, complex or imaginary for  $v$  if and only if  $\alpha$  is real, complex or imaginary for  $w$ .*

*Proof.* Let  $x = x(v)$ ,  $T_1 = x^{-1}Tx$  and  $\omega_v$  be as above. Then  $\tau(x) = x\theta(x)^{-1}$  is a representative for  $w = \varphi(v)$  in  $N_G(T)$ . Let  $\alpha \in \Phi(T)$  and  $\alpha_1 = \omega_v(\alpha) \in \Phi(T_1)$ . Then  $\omega_v^{-1}\theta(\alpha_1) = \omega_v^{-1}\theta(\omega_v(\alpha)) = \omega_v^{-1}\theta(\omega_v)(\theta(\alpha)) = \text{Int}(x\theta(x)^{-1})(\theta(\alpha)) = w\theta(\alpha)$ . The result follows.  $\square$

We will use the same notation for  $v$  as for  $w = \varphi(v)$ . In particular we will write  $C'(v)$ ,  $C''(v)$ ,  $R(v)$ ,  $I(v)$  for  $C'(w)$ ,  $C''(w)$ ,  $R(w)$ ,  $I(w)$  respectively. It follows from Lemma 3.7 that this notation does not depend on  $v \in V$ .

**3.8. Springer characterization of twisted involutions.** The discussion on twisted involutions in [Spr84] depends on the fact that  $\theta(\Phi^+) = \Phi^+$ . This is equivalent to the condition that  $\theta(B) = B$ . This in its turn is equivalent to the condition that the orbit  $BH$  is closed (see [HW93]). From the results in [HW93], one can derive a generalization of the characterization of twisted involutions in [Spr84] to cover  $B$  which need not be  $\theta$ -stable. Basically all one needs to do is pass to another involution which leaves  $\Phi^+$  invariant. Let  $w_0 \in W$  such that

$$(3) \quad \theta(\Phi^+) = w_0(\Phi^+).$$

and let  $\theta' = \theta w_0$ . Then  $w_0$  and  $\theta'$  satisfy the following conditions:

**Proposition 3.9.** *Let  $\Phi$ ,  $\Phi^+$ ,  $\theta$ ,  $w_0$  and  $\theta'$  be as above. Then we have the following properties:*

- (i)  $w_0 \in \mathfrak{I}_\theta$ .
- (ii)  $\theta'(\Phi^+) = \Phi^+$ .
- (iii)  $\theta'$  is an involution of  $\Phi$ .
- (iv)  $\mathfrak{I}_{\theta'} = \mathfrak{I}_\theta \cdot w_0$ .

*Proof.* (i). Since  $\theta(\Phi^+) = w_0(\Phi^+)$  it follows that  $w_0^{-1}\theta(\Phi^+) = \Phi^+$  and consequently  $\theta(w_0^{-1})(\Phi^+) = \theta w_0^{-1}\theta(\Phi^+) = \theta(\Phi^+) = w_0(\Phi^+)$ . So  $\theta(w_0^{-1}) = w_0$ , which proves (i).

(ii) follows from the fact that  $\theta$  is an involution and  $\theta(\Phi^+) = w_0(\Phi^+)$ . Namely then  $\theta'(\Phi^+) = \theta w_0(\Phi^+) = \Phi^+$ . So  $\theta'$  equals the id or a diagram automorphism. All diagram automorphisms are involutorial, except when  $\Phi$  is of type  $D_4$ . That  $\theta'$  is in fact involutorial follows as follows:

$$(\theta')^2 = \theta w_0 \theta w_0 = \theta(w_0)w_0 = w_0^{-1}w_0 = \text{id}.$$

This proves (iii).

(iv). Let  $w \in \mathfrak{I}_{\theta'}$  and consider  $w w_0^{-1}$ . Note that from (i) it follows that  $\theta w_0 = w_0^{-1}\theta$ . But then

$$\theta(w w_0^{-1}) = \theta w w_0^{-1}\theta = \theta w \theta w_0 = w_0 w_0^{-1}\theta w \theta w_0 = w_0 \theta' w \theta' = w_0 w^{-1}.$$

So  $\mathfrak{I}_{\theta'} \subset \mathfrak{I}_\theta \cdot w_0$ . A similar argument shows the opposite inclusion, which proves the result.  $\square$

*Remark 3.10.* By choosing a suitable minimal parabolic  $k$ -subgroup,  $w_0$  becomes an involution (see also Lemma 6.4).

For the real, complex and imaginary roots we can show the following:

**Lemma 3.11.** *If  $w \in \mathfrak{I}_\theta$  and  $w' = w w_0$ , then  $w'\theta' = w\theta$ . In particular we have*

$$\begin{aligned} I(w, \theta) &= I(w', \theta'), & R(w, \theta) &= R(w', \theta'), \\ C'(w, \theta) &= C'(w', \theta'), & C''(w, \theta) &= C''(w', \theta'). \end{aligned}$$

*Proof.* The first statement follows from Proposition 3.9. Namely,

$$w'\theta' = w w_0 \theta w_0 = w w_0 \theta w_0 \theta \theta = w w_0 \theta(w_0)\theta = w w_0 w_0^{-1}\theta = w\theta.$$

The remaining statements follow from this, since if  $\alpha \in \Phi^+$ , then  $w'\theta'(\alpha) = w\theta(\alpha)$ .  $\square$

Instead of working with  $\theta$  and  $w$  we can now work with  $\theta'$  and  $w'$ . Combining the above with the description of twisted involutions in [Spr84], we get now the following characterization of the twisted involutions.

**Proposition 3.12.** [HW93, Proposition 7.9] *If  $w \in \mathfrak{I}_\theta$  and  $w' = w w_0 \in \mathfrak{I}_{\theta'}$ , then there exist  $s_1, \dots, s_h \in \Sigma$  and a  $\theta'$ -stable subset  $\Pi$  of  $\Delta$  satisfying the following conditions.*

- (i)  $w' = s_1 \dots s_h w_\Pi^0 \theta'(s_h) \dots \theta'(s_1)$  and  $l(w') = 2h + l(w_\Pi^0)$ .

(ii)  $w_{\Pi}^0 \theta' \alpha = -\alpha$ ,  $\alpha \in \Phi_{\Pi}$  (i.e.  $\Phi_{\Pi}^+ \subset R(w_{\Pi}^0, \theta')$ ).

Moreover if  $w' = t_1 \dots t_m w_{\Lambda}^0 \theta'(t_m) \dots \theta'(t_1)$ , where  $t_1, \dots, t_m \in \Sigma$  and  $\Lambda$  a  $\theta'$ -stable subset of  $\Delta$  satisfying conditions (i) and (ii), then  $m = h$ ,  $s_1 \dots s_h \Pi = t_1 \dots t_h \Lambda$  and

$$s_1 \dots s_h \theta'(s_h) \dots \theta'(s_1) = t_1 \dots t_h \theta'(t_h) \dots \theta'(t_1).$$

The above results suggest the the following definition.

**Definition 3.13.** Given  $w \in \mathcal{L}_{\theta}$ , a decomposition

$$ww_0 = uw_{\Pi}^0 \theta'(u^{-1})$$

with  $u \in W$  and  $\Pi \subset \Delta$  is called a *Springer decomposition* of  $w$  if

- (i)  $l(ww_0) = 2l(u) + l(w_{\Pi}^0)$ ,
- (ii)  $w_{\Pi}^0 \theta' |_{\Pi} = -1$ .

In the remainder of this subsection we give a criterion to classify the Springer decompositions of twisted involutions. These results are from [HW93, sect. 7].

**Definition 3.14.** A subset  $\psi$  of  $\Phi$  is parabolic if  $\psi$  is closed and  $\psi \cup -\psi = \Phi$ . Given any subset  $\psi$  of  $\Phi$ , let  $\psi_s$  denote the set  $\psi \cap -\psi$  and  $\psi_u$  the complement of  $\psi_s$  in  $\psi$ .

The minimal  $\theta$ -stable parabolic subsets of  $\Phi$  can be characterized as follows (see [HW93]).

**Lemma 3.15.** *Let  $\psi$  be a  $\theta$ -stable parabolic subset of  $\Phi$ . Then  $\psi$  is a minimal  $\theta$ -stable parabolic subset of  $\Phi$  if and only if  $\psi_s = \{\alpha \in \Phi | \theta\alpha = -\alpha\}$ .*

**Definition 3.16.** A parabolic subset  $\psi$  of  $\Phi$  is called  $(\theta, \Phi^+)$ -special if,

$$\psi \supset \Phi^+ \cap \theta(\Phi^+).$$

**3.17.** Let  $\psi$  be a  $(\theta, \Phi^+)$ -special minimal  $\theta$ -stable parabolic subset of  $\Phi$ . Set

$$(4) \quad \psi^+ = (\psi_s \cap \Phi^+) \cup \psi_u.$$

Then  $\psi^+$  is a system of positive roots of  $\Phi$ . Consider the sets

$$\Omega_0 = \Phi^+ \cap \psi_s = \{\alpha \in \Phi^+ | \theta\alpha = -\alpha\},$$

$$\Omega_+ = \Phi^+ \cap \psi_u,$$

$$\Omega_- = \Phi^+ \cap -\psi_u.$$

It follows readily that we have the decompositions

$$\Phi^+ = \Omega_0 \cup \Omega_+ \cup \Omega_-,$$

$$\psi^+ = \Omega_0 \cup \Omega_+ \cup -\Omega_-.$$

Since  $\psi$  is  $(\theta, \Phi^+)$ -special,  $\psi_u \supset \Phi^+ \cap \theta(\Phi^+)$  and so

$$(5) \quad \Omega_+ \supset \Phi^+ \cap \theta(\Phi^+).$$

Now let  $\Omega'_+$  denote the complement of  $\Phi^+ \cap \theta(\Phi^+)$  in  $\Omega_+$ . From (5), we have that

$$(6) \quad \theta(\Omega'_+), \quad \theta(\Omega_-) \subset -\Phi^+.$$

**Lemma 3.18.** *Let  $\psi$  be a  $(\theta, \Phi^+)$ -special minimal  $\theta$ -stable parabolic subset of  $\Phi$ ,  $\psi^+$  be as in (4) and  $w \in W$  such that  $w(\Phi^+) = \psi^+$ . Let  $\Pi'$  be the set of simple roots of  $\Omega_0$  and  $\Pi$  the subset of  $\Delta$  with  $\Pi' = w\Pi$ . Then we have the following conditions.*

- (i)  $w_0 = ww_{\Pi}^0\theta'(w^{-1})$ .
- (ii)  $l(w_0) = 2l(w) + l(w_{\Pi}^0)$ .
- (iii)  $w_{\Pi}^0\theta'|\Pi = -1$ .
- (iv)  $\theta|\Pi' = -1$ .

**Lemma 3.19.** *Let  $w \in W$  and  $\Pi \subset \Delta$ . Suppose that*

$$w_0 = ww_{\Pi}^0\theta'(w^{-1})$$

*such that  $l(w_0) = 2l(w) + l(w_{\Pi}^0)$  and  $w_{\Pi}^0\theta'|\Pi = -1$ . Then there is a unique  $(\theta, \Phi^+)$ -special minimal  $\theta$ -stable parabolic subset  $\psi$  of  $\Phi$  such that*

$$w(\Phi^+) = (\psi_s \cap \Phi^+) \cup \psi_u.$$

The following result characterizes the Springer decompositions of the twisted involutions [?, see]Proposition 7.24]h-w.

**Proposition 3.20.** *Let  $w \in \mathfrak{I}_{\theta}$  and  $\xi = w\theta$ . Then we have the following conditions.*

- (i)  $\xi$  is an involution leaving  $\Phi$  invariant.
- (ii) *Given any  $(\xi, \Phi^+)$ -special minimal  $\xi$ -stable parabolic subset  $\psi$  of  $\Phi$ , let  $u \in W$  be the element such that*

$$u(\Phi^+) = (\psi_s \cap \Phi^+) \cup \psi_u.$$

*Then there exists  $\Pi \subset \Delta$  such that*

- (a)  $u(\Phi_{\Pi}^+) = \psi_s \cap \Phi^+$ ,
- (b)  $ww_0 = uw_{\Pi}^0\theta'(u^{-1})$  is a Springer decomposition.
- (iii) *There is a one to one correspondence, given in (ii), between the set of  $(\xi, \Phi^+)$ -special minimal  $\xi$ -stable parabolic subsets of  $\Phi$  and the set of Springer decompositions of  $w$ .*

#### 4. Relation between $\theta$ and $\theta'$

In section 3 we saw that if we characterize the orbits in terms of twisted involutions with respect to a pair  $(B, T)$  which is not standard, then one needs to use the involution  $\theta'$  of  $(X^*(T), \Phi(T))$  instead of  $\theta$  to get a combinatorial description of the related twisted involutions. This involution will also be of importance for the description of the orbit closures (see section 5). So naturally there arise the questions of whether this involution  $\theta'$  can be lifted to an involution of  $G$ , and if so, whether that involution is conjugate to  $\theta$ . In this section we show when  $\theta'$  can be lifted to  $G$  and what the relation is between the double cosets for  $\theta$  and  $\theta'$ . First we address the question of lifting, for which we look at a slightly more general problem.

**4.1.** Let  $B$  be a Borel subgroup of  $G$ ,  $T \subset B$  a  $\theta$ -stable maximal torus,  $W = W(T)$ ,  $\Phi = \Phi(T)$ , and  $\mathfrak{I} = \mathfrak{I}_\theta$  the set of twisted involutions in  $W$ . Let  $w \in \mathfrak{I}$  and  $\xi = w\theta$ . Then  $\xi$  is an involution of  $\Phi$ , which is of importance in the Springer decomposition of  $w$  (see Proposition 3.20). In the following we show when this involution  $\xi$  can be lifted to an involution of  $G$  and when this involution is conjugate to  $\theta$ .

The first question can be solved as follows.

**Lemma 4.2.** *Let  $w \in W$  and  $n \in N_G(T)$  a representative. Then  $\text{Int}(n)\theta$  is an involution of  $G$  if and only if  $\theta(n) = n^{-1}z$ , with  $z \in Z(G)$ .*

*Proof.* Since  $\text{Int}(n)\theta\text{Int}(n)\theta = \text{Int}(n\theta(n))$  it follows that  $\text{Int}(n)\theta$  is an involution if and only if  $n\theta(n) \in Z(G)$ .  $\square$

To answer the second question we need more notation. Let  $\mathcal{V}$ ,  $V$  and  $\varphi : V \rightarrow \mathfrak{I}$  be as in 2.15. If  $w \in \varphi(V)$ , take  $v \in V$  with  $w = \varphi(v)$ , let  $x = x(v) \in \mathcal{V}$  be a representative and set  $n = x\theta(x)^{-1} \in N_G(T)$ ; then  $n$  represents  $w$ , and  $\xi = \text{Int}(n)\theta = \text{Int}(x)\theta\text{Int}(x^{-1})$ . So the involution  $\xi$  is conjugate to  $\theta$ . This observation leads to the following result:

**Lemma 4.3.** *Let  $\mathcal{V}$ ,  $V$ ,  $\varphi : V \rightarrow \mathfrak{I}$  be as above. Let  $w \in \mathfrak{I}$ ,  $n \in N_G(T)$  a representative of  $w$  and  $\xi = \text{Int}(n)\theta$ . Assume that  $\theta(n) = n^{-1}z$ , with  $z \in Z(G)$ . Then  $\xi$  is conjugate to  $\theta$  if and only if  $n \in \tau(\mathcal{V})Z(G)$ .*

Combining the above results we get:

**Lemma 4.4.** *Let  $\mathcal{V}$ ,  $V$ ,  $\varphi : V \rightarrow \mathfrak{I}$  be as above and let  $w \in \mathfrak{I}$ . Then the following are equivalent.*

- (1) *There exists a representative  $n \in N_G(T)$  for  $w$ , such that  $\xi = \text{Int}(n)\theta$  is an involution of  $G$  conjugate to  $\theta$ .*
- (2)  *$w \in \varphi(V) \subset \mathfrak{I}$ .*

**4.5.** Let  $B$  and  $T$  be as above. As in (3), let  $w_0 \in W = W(T)$  be such that  $\theta(\Phi^+) = w_0(\Phi^+)$  and let  $\theta' = \theta w_0 = w_0^{-1}\theta$ . We can solve now the question of when the involution  $\theta'$  can be lifted to a conjugate of  $\theta$ . By Lemma 4.4 it suffices to show that  $w_0 \in \varphi(V)$ . This is equivalent to the following:

**Proposition 4.6.** *Let  $\mathcal{V}$ ,  $V$ ,  $\varphi : V \rightarrow \mathfrak{I}$  and  $w_0$  be as above. Then  $w_0 \in \varphi(V)$  if and only if  $G$  has a  $\theta$ -stable Borel subgroup.*

*Proof.* Assume first  $w_0 \in \varphi(V)$ . Let  $v_0 \in V$  be such that  $\varphi(v_0) = w_0$  and let  $x_0 = x(v_0) \in \mathcal{V}$  be a representative of  $v_0$ . Then  $\tau(x_0)$  is a representative of  $w_0$  in  $N_G(T)$ . Since  $\theta(\Phi^+) = w_0(\Phi^+)$ , we have  $\theta(B) = \tau(x_0)B\tau(x_0^{-1})$ . But then  $B_1 = \theta(x_0^{-1})B\theta(x_0)$  is a  $\theta$ -stable Borel subgroup of  $G$ .

Conversely, assume  $B_0 \subset G$  is a  $\theta$ -stable Borel subgroup. By Proposition 2.5 there exists  $x \in \mathcal{V}$  such that  $B_0 = xBx^{-1}$ . Since  $\theta(B_0) = B_0$  it follows that

$$\theta(B) = \theta(x)^{-1}xBx^{-1}\theta(x) = \tau(\theta(x)^{-1})B\tau(\theta(x))$$

Now  $\tau(\theta(x)^{-1}) \in N_G(T)$ . Let  $w \in W$  be the corresponding Weyl group element. Then  $\theta(\Phi^+) = w(\Phi^+)$ , so  $w = w_0 \in \varphi(V)$ .  $\square$

*Remark 4.7.* Since  $G$  is defined over an algebraically closed field, the existence of a  $\theta$ -stable Borel subgroup is guaranteed by a result of Steinberg (see [Ste68]). For non-algebraically closed fields one can prove a similar result, using minimal parabolic  $k$ -subgroups instead of Borel subgroups. However in this case  $\theta$ -stable minimal parabolic  $k$ -subgroups of  $G$  do not necessarily exist. See [HW93, sect. 3] for a discussion of  $\theta$ -stable parabolic  $k$ -subgroups.

If  $\xi \in \text{Aut}(G)$  with  $\xi(T) = T$ , then by abuse of notation we will write  $\xi|\Phi$  for the action of  $\xi$  on  $\Phi$ . Summarizing the above results we get now the following result.

**Corollary 4.8.** *Let  $w_0, \theta'$  be as above. There exists a representative  $n \in N_G(T)$  of  $w_0$ , such that  $\xi = \text{Int}(n)\theta$  is an involution of  $G$  conjugate to  $\theta$  satisfying  $\xi|\Phi = \theta'$ .*

**4.9.** Since the involutions  $\theta$  and  $\theta'$  are conjugate, the orbit decompositions of the corresponding symmetric varieties under the action of a Borel subgroup are similar. In the remainder of this section we discuss the relation between these double coset decompositions.

Let  $B, T, \mathcal{V}, V, \mathfrak{I}$  and  $\varphi : V \rightarrow \mathfrak{I}$  be as above. Write  $\Phi = \Phi(T)$  and  $W = W(T)$ . Take  $w_0 \in W$  such that  $\theta(\Phi^+) = w_0(\Phi^+)$ ,  $n_0 = x_0\theta(x_0)^{-1} \in N_G(T) \cap \tau(G)$  a representative of  $w_0^{-1}$ ,  $\theta' = \text{Int}(n_0)\theta$  and  $H' = x_0Hx_0^{-1}$ . Then  $H'$  is a closed reductive subgroup of  $G$  satisfying

$$G_{\theta'}^0 \subset H' \subset G_{\theta'}.$$

Denote the actions of  $\theta$  and  $\theta'$  on  $\Phi$  also by  $\theta$  and  $\theta'$ . Then  $\theta' = \theta w_0 = w_0^{-1}\theta$ .

As for  $\theta$  let  $\tau' : G \rightarrow G$  be the map defined by  $\tau'(x) = x\theta'(x)^{-1}$ ,  $\mathcal{V}' = \{x \in G \mid \tau'(x) \in N_G(T)\}$ ,  $V'$  the set of  $(T \times H')$ -orbits in  $\mathcal{V}'$ ,  $\mathfrak{I}_{\theta'}$  the set of twisted involutions of  $W$  with respect to  $\theta'$  and  $\varphi' : V' \rightarrow \mathfrak{I}_{\theta'}$  as in 2.15.

From 3.8 we get the following relations between the sets  $\mathcal{V}, \mathcal{V}', \mathfrak{I}$  and  $\mathfrak{I}_{\theta'}$ .

**Lemma 4.10.** *Let  $\mathcal{V}, \mathcal{V}', \mathfrak{I}, \mathfrak{I}_{\theta'}, x_0, n_0$  and  $w_0$  be as above. Then we have the following.*

- (i)  $\mathcal{V}' = \mathcal{V} \cdot x_0^{-1}$  and  $\tau'(\mathcal{V}') = \tau(\mathcal{V}) \cdot n_0^{-1}$ .
- (ii)  $\mathfrak{I}_{\theta'} = \mathfrak{I}_{\theta} \cdot w_0$ .

**4.11.** As in 2.16 there is also an action of the Weyl group  $W$  on  $\mathfrak{I}_{\theta'}$ . Namely if  $w \in W$  and  $a' \in \mathfrak{I}_{\theta'}$ , then define an action  $w *' a' = wa'\theta'(w)^{-1}$ . Since  $\theta' = \theta w_0$  and  $a' = aw_0$  for some  $a \in \mathfrak{I}_{\theta}$ , we get

$$(7) \quad w *' a' = waw_0\theta'w^{-1}\theta' = waw_0w_0^{-1}\theta w^{-1}\theta w_0 = wa\theta w^{-1}\theta w_0 = (w * a)w_0.$$

This means that right translation by  $w_0$  gives an isomorphism  $\iota_{w_0} : \mathfrak{I} \rightarrow \mathfrak{I}_{\theta'}$ , which is equivariant with respect to the actions of  $W$  on  $\mathfrak{I}$  and  $\mathfrak{I}_{\theta'}$ .

On the other hand, the map  $g \rightarrow gx_0^{-1}$  from  $\mathcal{V}$  to  $\mathcal{V}'$  induces a map  $\delta : V \rightarrow V'$ . Then  $\varphi' \circ \delta = \iota_{w_0} \circ \varphi$ . We summarize this in the following diagram:

$$\begin{array}{ccc}
 V & \xrightarrow{\delta} & V' \\
 \downarrow \varphi & & \downarrow \varphi' \\
 \mathfrak{I}_\theta & \xrightarrow{\iota_{w_0}} & \mathfrak{I}_{\theta'}
 \end{array}$$

The above analysis of the actions of  $W$  on  $\mathfrak{I}$  and  $\mathfrak{I}_{\theta'}$  leads to the following result.

**Proposition 4.12.** *Let  $\mathcal{V}$ ,  $\mathcal{V}'$ ,  $\mathfrak{I}$ ,  $\mathfrak{I}_{\theta'}$ ,  $n_0$  and  $w_0$  be as above. Then we have the following.*

- (i) *The map  $\iota_{w_0} : \mathfrak{I} \rightarrow \mathfrak{I}_{\theta'}$  induces an isomorphism between  $\mathfrak{I}/W$  and  $\mathfrak{I}_{\theta'}/W$ .*
- (ii)  *$V'/W \simeq V/W \simeq \varphi'(V')/W \simeq \varphi(V)/W$ .*

*Proof.* Since  $\iota_{w_0} : \mathfrak{I} \rightarrow \mathfrak{I}_{\theta'}$  is equivariant with respect to the actions  $*$  resp.  $*'$  of  $W$  on  $\mathfrak{I}$  resp.  $\mathfrak{I}_{\theta'}$ , the first result is clear.

(ii) is immediate from the above analysis and Proposition 2.18.  $\square$

*Remark 4.13.* The above results describe the relation between the orbits for  $\theta$  and  $\theta'$ . In many explicit calculations it is easier to switch to the open orbit, instead of the closed orbit and the above results describe the relation between the cosets. See also section 8.

## 5. Orbit closures and a Bruhat type order

In this section we discuss a number of results about the orbit closures and the Bruhat type order induced by the closure relations. First some more notation.

**5.1.** Let  $B$  be a fixed Borel subgroup of  $G$  and  $T$  a  $\theta$ -stable maximal torus of  $B$ . Let  $\Phi = \Phi(G, T)$  denote the set of roots of  $T$  in  $G$  and  $\Phi^+ = \Phi(B, T)$  with basis  $\Delta$ . Let  $s_\alpha$  denote the reflection defined by  $\alpha \in \Phi$  and write  $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$  for the set of simple reflections in  $W$ . Then  $(W, \Sigma)$  is a Coxeter group. For  $\alpha \in \Phi$ , let  $\mathfrak{g}_\alpha$  be the root subspace of the Lie algebra  $\mathfrak{g} = L(G)$  of  $G$  corresponding to  $\alpha$ . We have the decomposition  $\mathfrak{g} = L(G) = L(T) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ . Given  $\alpha \in \Delta$ , let  $P_\alpha = P_{s_\alpha}$  denote the standard parabolic subgroup of  $G$  containing  $B$  such that  $\Phi(P_\alpha, T) = (\mathbb{Z}\alpha \cap \Phi) \cup \Phi^+$ . It is easy to see that

$$(8) \quad \dim(P_\alpha) = \dim(B) + \dim\left(\bigoplus_{\gamma \in \mathbb{Z}\alpha \cap \Phi^+} \mathfrak{g}_\gamma\right) = \dim(B) + 1.$$

**5.2.** Let  $Q = \{g\theta(g)^{-1} \mid g \in G\}$  be as in 1.3. The set  $Q$  is stable under the twisted action of  $G$  defined by  $g * x = gx\theta(g)^{-1}$ ,  $g \in G$ ,  $x \in Q$ . Let  $v \in V$ ,  $x = x(v)$ ,  $\mathcal{O}_v = BxH$  and  $n = x\theta(x)^{-1}$ . In the following we will describe the closure of the orbit  $B * n$  in  $Q$ . Since  $cl(\mathcal{O}_v) = \tau^{-1}(cl(B * n))$  this will also give a description of the orbits  $\mathcal{O}_v = BxH$  in  $G$ .

Let  $w$  be the image of  $n$  in  $W$  and  $w' = ww_0$ . We write  $w'$  as in Proposition 3.12,

$$w' = s_1 \dots s_h w_\Pi^0 \theta'(s_h) \dots \theta'(s_1),$$

with  $l(w') = 2h + l(w_\Pi^0)$ . Choose  $n_1, \dots, n_h \in N_G(T)$  with images  $s_1, \dots, s_h$  in  $W$  respectively. Set  $s_i = s_{\alpha_i}$  and let  $P_i = P_{\alpha_i}$ ,  $1 \leq i \leq h$  be as in 5.1. If we write  $u = n_h^{-1} \dots n_1^{-1}x$  and  $m = u\theta(u)^{-1}$ , then we have the following description of the orbit closures [?, see]Proposition 9.5]h-w.

**Proposition 5.3.** *Let  $B, v, x, n, w, w', u, m, \Pi$  be as above. Then we have the following.*

- (i)  $cl(B * n) = P_1 * \dots * P_h * P_\Pi * m$ .
- (ii)  $cl(\mathcal{O}_v) = cl(BxH) = \tau^{-1}(cl(B * n)) = P_1 \dots P_h P_\Pi \mathcal{O}_u$ .
- (iii)  $\dim(B * n) = \sum_{i=1}^h (\dim(P_i) - \dim(B)) + \dim(P_\Pi * m) = h + \dim(P_\Pi * m)$ .

This result gives a fairly detailed description of the orbit closures. It suggests that the closure of an orbit  $\mathcal{O}_v$  is to a large extent determined by the corresponding twisted involution  $\varphi(v)$ . To actually compute the orbits contained in the closure of an orbit one typically uses the Bruhat order defined by the closure relations. We discuss this order in the next subsection.

**5.4. Bruhat order on  $V$ .** Let  $v \in V$ ,  $x = x(v)$  and let  $\mathcal{O}_v = BxH$  denote the corresponding double coset. This is a smooth subvariety and the closure  $cl(\mathcal{O}_v)$  is a union of double cosets. The *Bruhat order on  $V$*  is the order  $\leq$  on  $V$  defined by the closure relations on the double cosets  $\mathcal{O}_v = Bx(v)H$ . Thus  $v_1 \leq v$  if and only if  $\mathcal{O}_{v_1} \subset cl(\mathcal{O}_v)$ .

We can represent the closure relations on the set of double cosets  $Bx(v)H$  by a diagram, using the above Bruhat order. This will be called the diagram of  $V$ .

The Bruhat order on  $V'$  is defined similar as for  $V$  and will be denoted by  $\leq'$ . Using the map  $\delta : V \rightarrow V'$  as in 4.11, we get the following relation between the Bruhat order on  $V$  and  $V'$ . Let  $v_1, v_2 \in V$ , then

$$(9) \quad v_1 \leq v_2 \quad \text{if and only if} \quad \delta(v_1) \leq' \delta(v_2).$$

In the following we will only characterize the Bruhat order for  $V$ . The diagram of the orbits for  $V'$  is the same.

*Remark 5.5.* In the case of the Bruhat decomposition of the group  $G$  (i.e.  $B$ -orbits on  $G/B$ ), Chevalley gave a combinatorial description of this geometrically defined order. Here the Bruhat order on the  $B \times B$ -orbits on  $G$  corresponds to the combinatorially defined Bruhat order on the Weyl group. In the case of  $B$ -orbits on the symmetric variety  $G/H$ , Richardson and Springer recently gave a similar combinatorial description of the Bruhat order on  $V$  (see [RS90]). They used the map  $\varphi : V \rightarrow \mathcal{I}$ , as in 2.15, to map elements of  $V$  to twisted involutions in the Weyl group. An additional complication in this case is that this map is often not one to one. Note that the Bruhat order on  $V$  also induces an order on  $\mathcal{I}$ . A combinatorial description of this order on  $\mathcal{I}$  can be found in [RS90]. In this paper Richardson and Springer also prove that this combinatorial order on  $\mathcal{I}$  and the Bruhat order on  $V$  are compatible (i.e.  $v_1 \leq v_2$  if and only if  $\varphi(v_1) \leq \varphi(v_2)$ ).

A small problem is that the combinatorial description of the Bruhat order on  $V$  and  $\mathcal{I}$  in [RS90] is given with respect to a standard pair  $(B, T)$ . However using

the results about twisted involutions in section 3 and 4 and the above description of the orbit closures, one can generalize these results to an arbitrary pair  $(B, T)$ . We will discuss in the remainder of this section how the results in [RS90] carry over to this more general setting. Note that, if the pair  $(B, T)$  is not a standard pair, then the combinatorics of the orbit closures are actually related to  $\mathcal{L}_{\theta'}$  and not to  $\mathcal{L}$ . So instead of  $\varphi : V \rightarrow \mathcal{L}$  we need to consider  $\iota_{w_0} \circ \varphi : V \rightarrow \mathcal{L}_{\theta'}$ , where  $\iota_{w_0} : \mathcal{L} \rightarrow \mathcal{L}_{\theta'}$  is as in 4.11. We will define two opposite orders on  $V$  and  $\mathcal{L}_{\theta'}$ : one related to a closed orbit, the other related to the open orbit. The latter will be useful for the computation of the orbit closures, when one uses a characterization of the orbits related to the open orbit. For more details see Corollary 5.26 and 8.30. Our formulation of the combinatorial Bruhat orders on  $V$  and  $\mathcal{L}_{\theta'}$  differs slightly from the one in [RS90] and is more geared towards the computation of the orbit closures.

Finally we will also review a number of properties of the combinatorial Bruhat orders on  $V$  and  $\mathcal{L}$  which will be needed for the actual computation of the orbit closures.

**5.6.** To define the combinatorial Bruhat order on  $V$ , the first thing we need to do is to analyze the description of the orbit closure in Proposition 5.3 in more detail. The set  $P_{\Pi}\mathcal{O}_u$  as in Proposition 5.3 can also be described by a sequence  $P_{s_1} \dots P_{s_k}$  acting on a closed orbit, where  $(s_1, \dots, s_k)$  is a sequence in  $\Sigma$ . To prove this, it suffices to analyze the rank one situation. A detailed description of this is given in [RS90] and [Hel]. Roughly the results are as follows. Let  $v \in V$ ,  $x = x(v)$  and  $\mathcal{O}_v = BxH$  the corresponding double coset. If  $s \in \Sigma$ , then  $P_s\mathcal{O}_v = P_sxH$  is a union of one, two or three  $(B \times H)$ -orbits.

The rank one analysis leads to the following result, which enables us to define an action of the Coxeter group  $(W, \Sigma)$  on  $V$ .

**Lemma 5.7.** *Let  $v \in V$  and  $s \in \Sigma$ . Then we have the following.*

- (i)  $P_s \text{cl}(\mathcal{O}_v)$  is closed.
- (ii)  $P_s\mathcal{O}_v$  contains a unique dense  $(B \times H)$ -orbit.

The first statement follows from [Ste74, Lemma 2] and the second statement can be found in [RS90, sect. 4].

**5.8. Admissible sequences in  $V$ .** Related to a sequence  $\mathbf{s} = (s_1, \dots, s_k)$  in  $\Sigma$  we can define now a sequence  $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$  in  $V$  as follows. Let  $\mathcal{O}_{v_0}$  be a closed orbit and for  $i \in [1, k]$  let  $\mathcal{O}_{v_i}$  be the unique dense orbit in  $P_{s_i}\mathcal{O}_{v_{i-1}}$ . We will call  $\mathbf{s}$  an *admissible sequence for  $v \in V$*  if there exists a closed orbit  $\mathcal{O}_{v_0}$ , such that the sequence  $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$  in  $V$  satisfies

$$(10) \quad \dim P_{s_{i+1}} \dots P_{s_1} \mathcal{O}_{v_0} > \dim P_{s_i} \dots P_{s_1} \mathcal{O}_{v_0} \quad \text{for } i = 1, \dots, k-1,$$

and  $v = v_k$ . In this case we will also call the pair  $(\mathcal{O}_{v_0}, \mathbf{s})$  an *admissible pair for  $v$* .

The above sequences start at a closed orbit and build up from there. The question arises of whether one could start at the opposite end with the open orbit and build down from there. Unfortunately this is not possible. The reason for this is that  $P_s\mathcal{O}_v$  can consist of three  $(B \times H)$ -orbits. Later on we will show that for twisted involutions in  $\mathcal{L}_{\theta'}$  we can actually go up and down.

Although we cannot move down from the open orbit, it is possible to move up from any orbit to the open orbit by means of a sequence  $\mathbf{s}$  in  $\Sigma$ . The corresponding sequence in  $V$  is defined as follows. Let  $v \in V$ ,  $\mathbf{s} = (s_1, \dots, s_k)$  a sequence in  $\Sigma$  and  $v_{\max} \in V$  the open orbit. Define the sequence  $\mathbf{v}_o(\mathbf{s}) = (v_0, v_1, \dots, v_k)$  in  $V$  as follows. Let  $v_0 = v$  and for  $i \in [1, k]$  let  $\mathcal{O}_{v_i}$  be the unique dense orbit in  $P_{s_i}\mathcal{O}_{v_{i-1}}$ . If  $v_k = v_{\max}$ , we call this an *open-sequence for  $v$  in  $V$* . We will call an open-sequence for  $v$  in  $V$  an *admissible open-sequence for  $v$  in  $V$*  if it satisfies condition (10).

The existence of both an admissible pair and an admissible open-sequence for each  $v \in V$  is stated in the following result.

**Theorem 5.9.** *Let  $v \in V$ . Then we have the following.*

- (i) *There exists a closed orbit  $\mathcal{O}_{v_0}$  and a sequence  $\mathbf{s} = (s_1, \dots, s_k)$  in  $\Sigma$  such that  $cl(\mathcal{O}_v) = P_{s_k} \dots P_{s_1} \mathcal{O}_{v_0}$  and  $\dim \mathcal{O}_v = k + \dim \mathcal{O}_{v_0}$ .*
- (ii) *There exists an admissible open-sequence  $\mathbf{t}$  for  $v$ .*

This result is a refinement of [RS90, Theorem 4.6] to arbitrary pairs  $(B, T)$  and follows easily from Proposition 5.3 and the rank one analysis.

**5.10.** If  $\mathbf{s} = (s_1, \dots, s_k)$  is a sequence in  $\Sigma$ , then we write  $\ell(\mathbf{s}) = k$  for the length of the sequence. The length  $L(v)$  of an element  $v \in V$  can be defined now as follows. Let  $(\mathcal{O}_{v_0}, \mathbf{s})$  be an admissible pair for  $v$ . Define  $L(v) = k = \dim \mathcal{O}_v - \dim \mathcal{O}_{v_0}$ . Since all closed orbits have the same dimension the above definition of length does not depend on the admissible pair for  $v$ .

The relation between the length of an admissible open-sequence for  $v$  and an admissible sequence for  $v$  is now as follows [?, see]Lemma 7.2]ri-sp.

**Lemma 5.11.** *Let  $v \in V$ ,  $v_{\max} \in V$  the open orbit,  $\mathbf{s}$  an admissible sequence for  $v$  and  $\mathbf{t}$  an admissible open-sequence for  $v$ . Then we have the following.*

- (i)  $L(v) = \ell(\mathbf{s})$ .
- (ii)  $\ell(\mathbf{t}) = L(v_{\max}) - L(v) = L(v_{\max}) - \ell(\mathbf{s})$ .

*Remark 5.12.* The above discussion shows how a sequence  $\mathbf{s} = (s_1, \dots, s_k)$  in  $\Sigma$  can be used to define a sequence  $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$  in  $V$ . Using the map  $\varphi : V \rightarrow \mathcal{L}$  the sequence  $\mathbf{s}$  in  $\Sigma$  also defines sequences  $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$  in  $\mathcal{L}$  and  $\mathbf{a}'(\mathbf{s}) = (a'_0, a'_1, \dots, a'_k)$  in  $\mathcal{L}_{\theta'}$ , where  $a_i = \varphi(v_i)$  and  $a'_i = (t_{w_0} \circ \varphi)(v_i)$ . In 5.19 we will give a combinatorial description of these sequences in  $\mathcal{L}$  and  $\mathcal{L}_{\theta'}$ .

The sets of admissible sequences and admissible open-sequences in  $V$  each have natural orders. These orders are defined as follows.

**5.13. Combinatorial Bruhat order on  $V$ .** Let  $x, y \in V$ . Then we write  $x \leq y$  if there exist admissible pairs  $(\mathcal{O}_{v_0}, \mathbf{s} = (s_1, \dots, s_k))$  for  $x$  and  $(\mathcal{O}_{v_0}, \mathbf{t} = (t_1, \dots, t_r))$  for  $y$  with  $k \leq r$  and  $s_i = t_i$  for  $i = 1, \dots, k$ . We write  $x \leq_o y$  if there exist admissible open-sequences  $\mathbf{s} = (s_1, \dots, s_k)$  for  $x$  and  $\mathbf{t} = (t_1, \dots, t_r)$  for  $y$  with  $k \geq r$  and  $s_i = t_i$  for  $i = 1, \dots, r$ .

It is easy to see that  $\leq$  and  $\leq_o$  define partial orders on  $V$ . Moreover one easily shows that these orders are opposite:

**Lemma 5.14.** *Let  $x, y \in V$ . Then  $x \leq y$  if and only if  $y \leq_o x$ .*

The order  $\leq$  on  $V$  is the same as the standard order on  $V$  as defined in [RS90, sect. 5]. It follows now that the order  $\leq$  on  $V$  and the geometrical Bruhat order  $\leq$  are the same [?, see]Theorem 7.11]ri-sp.

**Theorem 5.15.** *Let  $\leq$  be the order on  $V$  as in 5.13,  $\leq$  the Bruhat order on  $V$  and  $x, y \in V$ . Then  $x \leq y$  if and only if  $x \leq y$ .*

In the following we will define orders on the set  $\mathcal{L}_{\theta'}$  of twisted involutions similar to the orders on  $V$  as in 5.13.

**5.16.  $(W, \Sigma)$ -action on  $\mathcal{L}$  and  $\mathcal{L}_{\theta'}$ .** In order to give combinatorial definitions of the admissible sequences in  $\mathcal{L}$  and  $\mathcal{L}_{\theta'}$  as in 5.12, we have to define first actions of  $(W, \Sigma)$  on  $\mathcal{L}$  and  $\mathcal{L}_{\theta'}$ . We begin with the former following [RS90, sect. 3.1].

If  $s \in \Sigma$ , define a map  $\eta(s) : \mathcal{L} \rightarrow \mathcal{L}$  as follows. Let  $a \in \mathcal{L}$ . If  $s * a = a$  then set  $\eta(s)(a) = sa$  and if  $s * a \neq a$  then set  $\eta(s)(a) = s * a$ . The map  $\eta(s)$  is a bijection of  $\mathcal{L}$  of period two which does not have any fixed points. Write  $s \circ a$  for  $\eta(s)(a)$ . This operation extends to an action of  $(W, \Sigma)$  on  $\mathcal{L}$  as follows. Let  $w \in W$  and  $s_1 \dots s_k$  a reduced expression of  $w$  with respect to  $\Delta$ . Then for  $a \in \mathcal{L}$  define  $w \circ a = s_k \circ \dots \circ s_1 \circ a$ . One easily shows that this definition does not depend on the reduced expression for  $w \in W$ .

The action of  $(W, \Sigma)$  on  $\mathcal{L}_{\theta'}$  can be defined similarly. One can also induce this action from the above action of  $(W, \Sigma)$  on  $\mathcal{L}$  using the bijection  $\iota_{w_0} : \mathcal{L} \rightarrow \mathcal{L}_{\theta'}$  as in 4.11. Let  $\eta'(s) = \iota_{w_0} \eta(s) \iota_{w_0}^{-1}$  be the bijection of  $\mathcal{L}_{\theta'}$  induced by  $\eta(s) : \mathcal{L} \rightarrow \mathcal{L}$ . Recall that by Lemma 4.10  $a \in \mathcal{L}$  if and only if  $aw_0 \in \mathcal{L}_{\theta'}$ . So, if  $s * aw_0 = aw_0$  then  $\eta'(s)(aw_0) = saw_0$  and if  $s * aw_0 \neq aw_0$  then  $\eta'(s)(aw_0) = s * (aw_0)$ . We write  $s \circ' aw_0$  for  $\eta'(s)(aw_0)$ .

**5.17. Admissible sequences in  $\mathcal{L}_{\theta'}$ .** As in  $V$ , a sequence  $\mathbf{s} = (s_1, \dots, s_k)$  in  $\Sigma$  induces sequences in  $\mathcal{L}_{\theta'}$  and  $\mathcal{L}$ . These sequences, which we will call  $\Sigma$ -sequences in  $\mathcal{L}_{\theta'}$  or  $\mathcal{L}$ , are defined by induction as follows. The sequence in  $\mathcal{L}_{\theta'}$  is  $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$ , where  $a_0 = 1$  and  $a_i = s_i \circ' a_{i-1}$  for  $i \in [1, k]$ ; the sequence in  $\mathcal{L}$  is defined similarly. We will mainly use sequences in  $\mathcal{L}_{\theta'}$ .

These  $\Sigma$ -sequences start at the identity in  $\mathcal{L}_{\theta'}$  or  $\mathcal{L}$  and build up from there. One could also start with the longest element in the Weyl group with respect to  $\Delta$  and build the sequence in  $\mathcal{L}_{\theta'}$  or  $\mathcal{L}$  down from there. In that case we get the following. Let  $w_{\Delta}^0 \in W$  be the longest element with respect to the basis  $\Delta$ . If  $\mathbf{s} = (s_1, \dots, s_k)$  is a sequence in  $\Sigma$ , then define a sequence  $\mathbf{b}(\mathbf{s}) = (b_0, b_1, \dots, b_k)$  in  $\mathcal{L}_{\theta'}$  by induction as follows. Let  $b_0 = w_{\Delta}^0$  and for  $i \in [1, k]$  let  $b_i = s_i \circ' b_{i-1}$ . Such a sequence will be called a  $w_{\Delta}^0$ -sequence in  $\mathcal{L}_{\theta'}$ . The  $w_{\Delta}^0$ -sequence in  $\mathcal{L}$  is defined similarly.

*Remark 5.18.* Since  $\mathcal{L}_{\theta'} w_{\Delta}^0 = \mathcal{L}_{\theta' w_{\Delta}^0} = \mathcal{L}_{\theta w_0 w_{\Delta}^0}$  it follows that the  $w_{\Delta}^0$ -sequences in  $\mathcal{L}_{\theta'}$  correspond to  $\Sigma$ -sequences in  $\mathcal{L}_{\theta'} w_{\Delta}^0$ .

To define an order on  $\mathcal{L}_{\theta'}$  compatible with this action we need first to define admissible sequences. These are defined as follows. For  $w \in W$  let  $l(w)$  denote the length of  $w$  with respect to the Bruhat order on  $W$ .

**Definition 5.19.** Let  $\mathbf{s} = (s_1, \dots, s_k)$  be a sequence in  $\Sigma$  and let  $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$  (resp.  $\mathbf{b}(\mathbf{s}) = (b_0, b_1, \dots, b_k)$ ) be the  $\Sigma$ -sequence (resp.  $w_\Delta^0$ -sequence) in  $\mathcal{I}_{\theta'}$  induced by  $\mathbf{s}$ . Then  $\mathbf{s}$  is called an *admissible sequence* (or an admissible  $\Sigma$ -sequence) if  $0 = l(a_0) < l(a_1) < \dots < l(a_k)$ . The sequence  $\mathbf{s}$  is called an *admissible  $w_\Delta^0$ -sequence* if  $l(w_\Delta^0) = l(b_0) > l(b_1) > \dots > l(b_k)$ . If  $a \in \mathcal{I}_{\theta'}$ , then the sequence  $\mathbf{s}$  in  $\Sigma$  is called an *admissible  $\Sigma$ -sequence for  $a$*  (resp. an *admissible  $w_\Delta^0$ -sequence for  $a$* ) if  $\mathbf{s}$  is an admissible  $\Sigma$ -sequence (resp. admissible  $w_\Delta^0$ -sequence) and  $a_k = a$  (resp.  $b_k = a$ ).

If  $a \in \mathcal{I}_{\theta'}$  and  $\mathbf{s} = (s_1, \dots, s_k)$  an admissible  $\Sigma$ -sequence for  $a$ , then we define the length of  $a$  as  $\ell(a) = \ell(\mathbf{s}) = k$ .

Using a similar argument to that in [RS90, Lemma 3.4], it follows that every element of  $\mathcal{I}_{\theta'}$  has an admissible  $\Sigma$ -sequence and an admissible  $w_\Delta^0$ -sequence:

**Lemma 5.20.** *For every  $a \in \mathcal{I}_{\theta'}$ , there exist sequences  $\mathbf{s}$  and  $\mathbf{t}$  in  $\Sigma$ , such that  $\mathbf{s}$  is an admissible  $\Sigma$ -sequence for  $a$  and  $\mathbf{t}$  is an admissible  $w_\Delta^0$ -sequence for  $a$ .*

From this result we get the following relation between the length of an admissible  $\Sigma$ -sequence for  $a$  and an admissible  $w_\Delta^0$ -sequence for  $a$  [?, see]Lemma 8.18]ri-sp.

**Lemma 5.21.** *Let  $a \in \mathcal{I}_{\theta'}$ ,  $\mathbf{s}$  be an admissible  $\Sigma$ -sequence for  $a$  and  $\mathbf{t}$  an admissible  $w_\Delta^0$ -sequence for  $a$ . Then  $\ell(\mathbf{t}) = \ell(w_\Delta^0) - \ell(a) = \ell(w_\Delta^0) - \ell(\mathbf{s})$ .*

As in  $V$  the admissible sequences in  $\mathcal{I}_{\theta'}$  induce partial orders on  $\mathcal{I}_{\theta'}$ . In fact the  $\Sigma$ -sequences and  $w_\Delta^0$ -sequences in  $\mathcal{I}_{\theta'}$  lead to opposite orders in  $\mathcal{I}_{\theta'}$ . These orders are defined as follows.

**5.22. Bruhat order on  $\mathcal{I}_{\theta'}$ .** Let  $a, b \in \mathcal{I}_{\theta'}$ . Define the order  $\leq_1$  on  $\mathcal{I}_{\theta'}$  with respect to the admissible  $\Sigma$ -sequences as follows. Write  $a \leq_1 b$  if there exist admissible  $\Sigma$ -sequences  $\mathbf{s} = (s_1, \dots, s_k)$  for  $a$  and  $\mathbf{t} = (t_1, \dots, t_r)$  for  $b$  with  $k \leq r$  and  $s_i = t_i$  for  $i = 1, \dots, k$ .

Similarly, using admissible  $w_\Delta^0$ -sequences we get an order  $\leq_2$  on  $\mathcal{I}_{\theta'}$  as follows. If  $a, b \in \mathcal{I}_{\theta'}$ , then write  $a \leq_2 b$  if there exists admissible  $w_\Delta^0$ -sequences  $\mathbf{s} = (s_1, \dots, s_k)$  for  $a$  and  $\mathbf{t} = (t_1, \dots, t_r)$  for  $b$  with  $k \leq r$  and  $s_i = t_i$  for  $i = 1, \dots, k$ .

The orders  $\leq_1$  and  $\leq_2$  define partial orders on  $\mathcal{I}_{\theta'}$ . The order  $\leq_1$  will also be called the *Bruhat order on  $\mathcal{I}_{\theta'}$* . Using a combinatorial argument one can show that this order on  $\mathcal{I}_{\theta'}$  is compatible the partial order on  $\mathcal{I}_{\theta'}$  induced by the Bruhat order on the Weyl group  $W$ . The orders  $\leq_1$  and  $\leq_2$  define opposite orders on  $\mathcal{I}_{\theta'}$ . We summarize this in the following result [?, see]sect. 8]ri-sp.

**Proposition 5.23.** *Let  $a, b \in \mathcal{I}_{\theta'}$ . Then we have the following.*

- (i) *An admissible sequence for  $a$  can be extended to an admissible sequence for  $w_\Delta^0$ .*
- (ii)  *$w_\Delta^0$  is the longest element of  $\mathcal{I}_{\theta'}$  with respect to  $\leq_1$ .*
- (iii)  *$a \leq_1 b$  if and only if  $b \leq_2 a$ .*

From the above results it easily follows now that the admissible sequences in  $V$  lead to admissible sequences in  $\mathcal{I}_{\theta'}$ .

**Lemma 5.24.** *Let  $v \in V$ . If  $(\mathcal{O}_{v_0}, \mathbf{s} = (s_1, \dots, s_k))$  is an admissible pair for  $v$  and  $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_r)$  is the corresponding sequence in  $V$ , then  $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$ , with  $a_i = \iota_{w_0}\varphi(v_i)$  ( $i = 1, \dots, k$ ), is an admissible sequence for  $a = \iota_{w_0}\varphi(v)$ .*

Combining the above results we obtain now the following relation between the Bruhat order on  $V$  and the Bruhat order on  $\mathcal{L}_{\theta'}$ .

**Theorem 5.25.** *Let  $v \in V$ ,  $a = \iota_{w_0}\varphi(v) \in \mathcal{L}_{\theta'}$  and  $\mathbf{s} = (s_1, \dots, s_k)$  a sequence in  $\Sigma$ . Then  $\mathbf{s}$  is an admissible sequence for  $a$  if and only if there exists a closed orbit  $\mathcal{O}_{v_0}$  such that  $(\mathcal{O}_{v_0}, \mathbf{s})$  is an admissible pair for  $v$ .*

Using the opposite sequences we also get the following result:

**Corollary 5.26.** *Let  $v \in V$ ,  $v_{\max} \in V$  be the open orbit,  $a = \iota_{w_0}\varphi(v)$ ,  $a_{\max} = \iota_{w_0}\varphi(v_{\max}) \in \mathcal{L}_{\theta'}$ , and  $\mathbf{t} = (t_1, \dots, t_r)$  an admissible  $w_{\Delta}^0$ -sequence for  $a_{\max}$ . A sequence  $\mathbf{s} = (s_1, \dots, s_k)$  in  $\Sigma$  is an admissible open-sequence for  $v$  if and only if  $\mathbf{w} = (t_1, \dots, t_r, s_k, \dots, s_1)$  is an admissible  $w_{\Delta}^0$ -sequence for  $a$ .*

**5.27.** The question which remains now is how to compute the closure of an orbit  $x$  in  $V$ . As we have seen the closure of this orbit consists of  $\{y \in V \mid y \leq x\}$  and therefore it becomes a question of the Bruhat orders on  $V$  and  $\mathcal{L}_{\theta'}$ . In the case of the Bruhat order on the Weyl group one uses Bruhat descendants to calculate all  $y$  in  $W$  which are less than or equal to a given element  $x$  with respect to the Bruhat order. Using the above characterizations of the orders on  $V$  and  $\mathcal{L}_{\theta'}$  in terms of admissible sequences, we can use a similar procedure.

**5.28. Bruhat descendants.** Let  $v \in V$  and let  $w = \iota_{w_0}\varphi(v) \in \mathcal{L}_{\theta'}$ . The *Bruhat descendants* of  $v$  are those elements  $x \in V$  for which  $x \leq v$  and  $L(x) = L(v) - 1$ . Similarly the *Bruhat descendants* of  $w$  are those  $y \in \mathcal{L}_{\theta'}$  for which  $y \leq_1 w$  and  $\ell(y) = \ell(w) - 1$ .

A description of the Bruhat descendants of  $v$  or  $w$  follows easily from the following property of the Bruhat order on  $V$  [?, see]5.12]ri-sp:

**Property 5.29.** (Exchange property). Let  $x \in V$  and  $(\mathcal{O}_{v_0}, \mathbf{s} = (s_1, \dots, s_k))$  be an admissible pair for  $x$ . Let  $s \in \Sigma$  and  $y \in V$  be such that  $x = s \circ y$ ,  $L(y) = L(x) - 1$ . Then there exist  $i \in [1, k]$  and an admissible pair  $(\mathcal{O}'_{v_0}, \mathbf{s}' = (s_1, \dots, \hat{s}_i, \dots, s_k))$  for  $y$ , such that the pair  $(\mathcal{O}'_{v_0}, \mathbf{t} = (s_1, \dots, \hat{s}_i, \dots, s_k, s))$  is an admissible pair for  $x$ .

The Bruhat order on  $\mathcal{L}_{\theta'}$  satisfies a similar property.

**5.30.** It follows from this that in order to compute the Bruhat descendants of  $v \in V$  one needs to do two things.

- (1) Find the closed orbits contained in the closure of  $v$ . For this see 8.23.
- (2) Find the Bruhat descendants of  $w$  in  $\mathcal{L}_{\theta'}$ .

From the exchange property it follows that the second problem is basically a matter of computing admissible subsequences. These are defined as follows. Let  $\mathbf{s} = (s_1, \dots, s_k)$  be a sequence in  $\Sigma$ . A sequence  $\mathbf{t} = (t_1, \dots, t_r)$  in  $\Sigma$  is called an

subsequence of  $\mathbf{s}$  if  $r < k$  and for all  $i \in [1, r]$  we have  $t_i = s_{j_i}$ , where  $j_i \in [1, k]$  and  $j_i < j_{i+1}$ . If  $\mathbf{s}$  is an admissible sequence for  $w \in \mathcal{L}_{\theta'}$ , then a subsequence  $\mathbf{t}$  induces a sequence  $\mathbf{b}(\mathbf{t}) = (b_0, b_1, \dots, b_r)$  as in 5.17. The subsequence  $\mathbf{t}$  in  $\Sigma$  is called an *admissible subsequence for  $w$*  if  $\mathbf{b}(\mathbf{t})$  is an admissible  $\Sigma$ -sequence in  $\mathcal{L}_{\theta'}$ .

Now let  $w \in \mathcal{L}_{\theta'}$ . The Bruhat descendants of  $w$  can be obtained as follows. Let  $\mathbf{s}$  be an admissible sequence for  $w$ . Then the set of subsequences obtained from  $\mathbf{s}$  by removing a single reflection such that the resulting subsequence remains admissible, contains exactly one admissible sequence for each Bruhat descendent of  $w$ . From the exchange property it follows that this is independent of the chosen admissible sequence  $\mathbf{s}$  for  $w$ . This notion of Bruhat descendants is similar to the one in the Weyl group.

Now any  $y \in \mathcal{L}_{\theta'}$  with  $y \preceq_1 w$  can be obtained by starting from  $w$  and repeatedly moving from an element to one of its Bruhat descendants. This all leads to the following characterization of the elements contained in the closure of an orbit in  $V$ :

**Proposition 5.31.** *Let  $x, y \in V$ . Then  $y \preceq x$  if and only if there exist an admissible pair  $(\mathcal{O}_{v_0}, \mathbf{s})$  for  $x$  and a subsequence  $\mathbf{t} = (t_1, \dots, t_r)$  of  $\mathbf{s}$  such that  $(\mathcal{O}_{v_0}, \mathbf{t})$  is an admissible pair for  $y$ .*

This result is easily proved by induction.

*Remark 5.32.* The above Bruhat descendants give an inductive procedure to compute the closure of an orbit  $v \in V$ . This computation basically reduces to computing admissible subsequences in  $\Sigma$ .

## 6. $\theta$ -orders on $\Phi$

In this section we discuss some properties of the orbit of  $W$  in  $V$ . In particular we look at the orbits of maximal (resp. minimal) dimension inside a  $W$ -orbit, which are related to a  $\theta$ -order (resp.  $(-\theta)$ -order) on  $\Phi(T)$ . Here  $T$  is the  $\theta$ -stable maximal torus associated with the  $W$ -orbit in  $V$ . So this also gives a description of the open and closed orbits in  $V$ . We review first a few facts about  $\theta$ -orders in root systems.

**6.1.** Let  $B$  be a Borel subgroup of  $G$ ,  $T \subset B$  a  $\theta$ -stable maximal torus of  $B$ ,  $\Phi = \Phi(T)$  the root system of  $T$  with respect to  $G$ ,  $\Phi^+$  the set of positive roots of  $\Phi$  related to  $B$ ,  $\Delta$  the corresponding basis of  $\Phi$ ,  $W = W(T)$  the Weyl group of  $T$  and  $X = X^*(T)$  the group of characters of  $T$ . The involution  $\theta$  of  $G$  induces an involution of  $X$ , which will also be denoted by  $\theta$ .

Let  $X_0(\theta) = \{\chi \in X \mid \theta(\chi) = \chi\}$  and  $\Phi_0(\theta) = \Phi \cap X_0(\theta)$ . Clearly  $X_0(\theta)$  and  $\Phi_0(\theta)$  are  $\theta$ -stable and  $\Phi_0(\theta)$  is a closed subsystem of  $\Phi$ . We denote the Weyl group of  $\Phi_0(\theta)$  by  $W_0(\theta)$  and identify it with the subgroup  $W(\Phi_0(\theta))$  of  $W$ . Let  $W_1(\theta) = \{w \in W \mid w(X_0(\theta)) = X_0(\theta)\}$ ,  $\overline{X}_\theta = X/X_0(\theta)$  and let  $\pi$  be the natural projection from  $X$  to  $\overline{X}_\theta$ . Every  $w \in W_1(\theta)$  induces an automorphism  $\pi(w)$  of  $\overline{X}_\theta$  and  $\pi(w(\chi)) = \pi(w)(\pi(\chi))$  ( $\chi \in X$ ). If  $\overline{W}_\theta = \{\pi(w) \mid w \in W_1(\theta)\}$ , then  $\overline{W}_\theta \cong W_1(\theta)/W_0(\theta)$ . (See [Sat71, 2.1.3]). This is called the *restricted Weyl group* with respect to the action of  $\theta$  on  $X$ . It is not necessarily a Weyl group in the sense of [Bou68, Ch. V, no. 1].

Let  $\overline{\Phi}_\theta = \pi(\Phi - \Phi_0(\theta))$  denote the set of *restricted roots of  $\Phi$  relative to  $\theta$* . As in [Hel88] we define a  $\theta$ -order on  $\Phi$  by choosing orders on  $X_0(\theta)$  and  $\overline{X}_\theta$ . To be more precise:

**Definition 6.2.** Let  $\succ$  be a linear order on  $X$ . The order  $\succ$  is called a  $\theta^+$ -order if it has the following property:

$$(11) \quad \text{if } \chi \in X, \chi \succ 0, \text{ and } \chi \notin X_0(\theta), \text{ then } \theta(\chi) \prec 0.$$

The order  $\succ$  is called a  $\theta^-$ -order if it has the following property:

$$(12) \quad \text{if } \chi \in X, \chi \succ 0, \text{ and } \chi \notin X_0(\theta), \text{ then } \theta(\chi) \succ 0.$$

As in [Hel88], a  $\theta^+$ -order on  $X$  will also be called a  $\theta$ -order on  $X$ . A basis  $\Delta$  of  $\Phi$  with respect to a  $\theta^+$ -order (resp.  $\theta^-$ -order) on  $X$  will be called a  $\theta^+$ -basis (resp.  $\theta^-$ -basis) of  $\Phi$ .

If  $\Delta$  is a basis of  $\Phi$  with respect to a  $\theta^+$ -order on  $X$ , then we write  $\Delta_0(\theta) = \Delta \cap \Phi_0(\theta)$  and  $\overline{\Delta}_\theta = \pi(\Delta - \Delta_0(\theta))$ . Similarly if  $\Delta$  is a basis of  $\Phi$  with respect to a  $\theta^-$ -order on  $X$ , then we write  $\Delta_0(-\theta) = \Delta \cap \Phi_0(-\theta)$  and  $\overline{\Delta}_{-\theta} = \pi(\Delta - \Delta_0(-\theta))$ . Clearly  $\Delta_0(\theta)$  (resp.  $\Delta_0(-\theta)$ ) is a basis of  $\Phi_0(\theta)$  (resp.  $\Phi_0(-\theta)$ ). A similar property holds for  $\overline{\Delta}_\theta$  and  $\overline{\Delta}_{-\theta}$  [?, see]Lemma 2.4]h1.

**6.3. A characterization of  $\theta$  on a  $\theta$ -basis of  $\Phi$ .** Let  $\Delta_1$  be a  $\theta$ -basis of  $\Phi$ . As in [Hel88, 2.8] we can write  $\theta = -\text{id} \theta_1^* w_0(\theta)$ , where  $w_0(\theta) \in W_0(\theta)$  is the longest element of  $W_0(\theta)$  with respect to  $\Delta_0(\theta)$ , and  $\theta_1^* \in \text{Aut}(X, \Phi, \Delta_1, \Delta_0(\theta)) = \{\phi \in \text{Aut}(X, \Phi) \mid \phi(\Delta_1) = \Delta_1 \text{ and } \phi(\Delta_0(\theta)) = \Delta_0(\theta)\}$  with  $(\theta_1^*)^2 = \text{id}$ . For more details see [Hel88, sect. 2]. This is called a characterization of  $\theta$  on its  $(+1)$ -eigenspace (because  $W_0(\theta)$  is the Weyl group of  $\Phi_0(\theta)$ ).

Similarly we get a characterization of  $\theta$  on a  $\theta^-$ -basis of  $\Phi$  as follows. Let  $\Delta_2$  be a  $\theta^-$ -basis of  $\Phi$ . Then  $\theta = \theta_2^* \cdot w_0(-\theta)$ , where  $w_0(-\theta) \in W_0(-\theta)$  is the longest element of  $W_0(-\theta)$  with respect to  $\Delta_0(-\theta)$ , and  $\theta_2^* \in \text{Aut}(X, \Phi, \Delta_2, \Delta_0(-\theta)) = \{\phi \in \text{Aut}(X, \Phi) \mid \phi(\Delta_2) = \Delta_2 \text{ and } \phi(\Delta_0(-\theta)) = \Delta_0(-\theta)\}$  with  $(\theta_2^*)^2 = \text{id}$ . This is called a characterization of  $\theta$  on its  $(-1)$ -eigenspace.

From the characterization of involutions of root systems in [Hel91], it follows that we cannot have that both  $\theta_1^*$  and  $\theta_2^*$  are non-trivial. This also follows from the above characterizations of  $\theta$ .

**Lemma 6.4.** *Let  $\Delta_1, \Delta_2, \theta_1^*$  and  $\theta_2^*$  be as above. If  $\Phi$  is irreducible, then  $\theta_1^* = \text{id}$  or  $\theta_2^* = \text{id}$ .*

*Proof.* If  $\theta$  is an inner automorphism of  $G$ , then  $\theta = w_0(-\theta)$  and  $\theta_2^* = \text{id}$ . So assume  $\theta$  is an outer automorphism of  $G$ . Then  $\theta_2^* \neq \text{id}$  and by [Hel88, 2.9]  $\Phi$  is of type  $A_n, D_{2n+1}$  or  $E_6$ . In each of these cases  $-\text{id} \notin W$ . If  $\theta_1^* \neq \text{id}$ , then  $-\text{id} \cdot \theta_1^* = w_0(\theta)$ . But then  $\theta = -\text{id} \theta_1^* w_0(\theta) = w_0(\theta)^2 = \text{id}$ , which contradicts the fact that  $\theta$  is an outer automorphism of  $G$ . It follows that  $\theta_1^* = \text{id}$ , which proves the result.  $\square$

**6.5.** The above can be related to orbits of maximal or minimal dimension as follows. Let  $\{T_i \mid i \in I\}$  be a set of representatives of the  $H$ -conjugacy classes of

$\theta$ -stable maximal tori. By Corollary 2.7 we can split the orbits into the subsets corresponding to the  $T_i$  ( $i \in I$ ). Let  $BxH$  be an orbit. We may assume that  $B_1 = xBx^{-1}$  contains  $T_i = xTx^{-1}$  for some  $i$ . Let  $\mathcal{H}(T_i) = \{BxH \mid xBx^{-1} \supset T_i\}$  ( $i \in I$ ). Clearly all the orbits in  $\mathcal{H}(T_i)$  have a representative  $n \in N_G(T_i)/N_H(T_i)$ . We have now the following.

**Proposition 6.6.** *Let  $T$  and  $\mathcal{H}(T)$  be as above. Let  $n \in N_G(T)$  and let  $\succ_n$  be the order on  $X$  induced by  $nBn^{-1}$ . Then we have the following conditions.*

- (i)  $\dim(BnH) = \max\{\dim(BgH) \mid g \in N_G(T)\}$  if and only if  $\succ_n$  is a  $\theta^+$ -order on  $\Phi$ .
- (ii)  $\dim(BnH) = \min\{\dim(BgH) \mid g \in N_G(T)\}$  if and only if  $\succ_n$  is a  $\theta^-$ -order on  $\Phi$ .

*Remark 6.7.* The number of orbits of minimal dimension in  $\mathcal{H}(T)$  corresponds to the number of  $\theta^-$ -orders on  $(X, \Phi)$  which are not conjugate under  $W_H(T)$ . Another characterization of these orbits, using the map  $\varphi$ , will be given in Proposition 8.24 and 8.23.

The following example illustrates the above result.

*Example 6.8.* Let  $G = SL_2(k)$  and  $\theta(g) = {}^t g^{-1}$ . Then  $H = SO(2)$  is a maximal torus. There are two  $\theta$ -stable Borel subgroups containing  $T = H$ , corresponding to the two orders on  $\Phi(T)$ . Hence there are two closed orbits and one open orbit.

**6.9.** The orbit sets  $\mathcal{H}(T_i)$  ( $i \in I$ ) correspond to the following subsets of  $V$ . Let  $V_i = \{v \in V \mid BvH \in \mathcal{H}(T_i)\}$  and  $W_i = \varphi(V_i)$  ( $i \in I$ ). From Lemma 2.17 it follows that the sets  $V_i$  are the orbits under the action of  $W$  on  $V$ , as in 2.16.

**Lemma 6.10.** *Let  $V_i, W_i$  be as above and let  $v_1 \in V_i$  and  $w_1 = \varphi(v_1)$ . Then we have the following.*

- (i)  $V_i = \{w \cdot v_1 \mid w \in W\}$
- (ii)  $W_i = \{w * w_1 \mid w \in W\}$

From this result it follows that the orbits of minimal dimension in  $\mathcal{H}(T)$  are related to the involution  $w_{\Pi}^0$  as in Proposition 3.12.

## 7. The open orbit and $\theta$ -orders

Since most of the combinatorial data of the symmetric variety is related to a maximal  $\theta$ -split torus it is often easiest to classify the orbits starting from a pair  $(B, T)$ , where  $T$  contains a maximal  $\theta$ -split torus of  $G$ . In this case the set  $\mathcal{H}(T)$  contains the unique open orbit.

In the following we will give several characterizations of the open orbit. We use the same notation as in sections 2 and 3. In particular let  $T$  be a  $\theta$ -stable maximal torus of  $G$ ,  $B \supset T$  a Borel subgroup,  $\Phi = \Phi(T)$ ,  $\Phi^+ = \Phi(B, T)$  the set of positive roots of  $\Phi$  related to  $B$ ,  $\Delta$  the corresponding basis of  $\Phi$ ,  $w_0$  the element in the Weyl group  $W = N_G(T)/Z_G(T)$  with  $w_0(\Phi^+) = \theta(\Phi^+)$  and  $\theta' = \theta w_0$ . If  $\Pi \subset \Delta$ , then we write  $\Phi_{\Pi}$  for the subsystem of  $\Phi(G, T)$  consisting of integral combinations of  $\Pi$  and we write  $P_{\Pi}$  for the standard parabolic subgroup of  $G$

containing  $B$  with  $\Phi(P_\Pi, T) = \Phi_\Pi \cup \Phi^+$ . The following result from [HW93, Proposition 9.2] characterizes the open orbit. (See also [Spr84].)

**Proposition 7.1.** *Let  $v \in V$ ,  $n = x(v)\theta(x(v))^{-1}$ ,  $w$  the image of  $n$  in  $W$  and  $w' = ww_0$ . Let  $\zeta$  be the involution of  $G$  corresponding to  $w'\theta' = w\theta$  (i.e.  $\zeta$  is given by  $\zeta(x) = n\theta(x)n^{-1}$ ,  $x \in G$ ). The following conditions are equivalent:*

- (i)  $B * n$  is open in  $Q = \{x\theta(x)^{-1} | x \in G\}$ .
- (ii) Let  $\Pi = I(w) \cap \Delta$ . Then  $C''(w) \cap \Delta = \emptyset$  and  $\zeta$  is trivial on  $G_{\Phi_\Pi}$ .
- (iii)  $w' = w_\Pi^0 w_\Delta^0$  and  $\zeta$  is trivial on  $G_{\Phi_\Pi}$ .
- (iv)  $x(v)^{-1}P_{\Pi}x(v)$  is a minimal  $\theta$ -split parabolic subgroup of  $G$ .
- (v) There exists a minimal  $\theta$ -split parabolic subgroup of  $G$  containing  $x(v)^{-1}Bx(v)$ .

Combining this result with Corollary 4.8 we get the following characterization of  $w_0$  and  $\theta'$ .

**Corollary 7.2.** *Let  $A$  be a maximal  $\theta$ -split torus of  $G$ ,  $T \supset A$  a maximal torus of  $G$  and  $B \supset T$  a Borel subgroup of  $G$  such that  $BH \subset G$  is open. Let  $w_0 \in W$  satisfy  $\theta(\Phi^+) = w_0(\Phi^+)$ , and take  $v_0 \in V$  such that if  $x_0 = x(v_0)$  and  $n_0 = x_0\theta(x_0)^{-1} \in N_G(T)$  then  $n_0$  induces  $w_0$  in  $W$ ; let  $\zeta$  be the involution of  $G$  given by  $\zeta(x) = n_0^{-1}\theta(x)n_0$  for  $x \in G$ . Then we have the following.*

- (i)  $w_0 = w_\Pi^0 w_\Delta^0$ , where  $\Pi = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\}$ .
- (ii)  $\zeta|_\Phi = w_0^{-1}\theta = \theta w_0 = \theta'$ .

*Remark 7.3.* The element  $w_\Pi^0 \in W$  follows from the classification of involutorial automorphisms in [Hel88] and the element  $w_\Delta^0 \in W$  follows from the classification of involutions in [Hel91]. So the above result gives an easy characterization of the elements  $w_0 \in W$ .

Using the results on  $\theta$ -orders in section 6, the unique open orbit  $BgH$  in  $G$  can be described also as follows.

**Proposition 7.4.** *Let  $G$ ,  $B$  and  $T$  be as above and for  $n \in N_G(T)$  let  $\succ_n$  be the order on  $X$  induced by  $nBn^{-1}$ . Then  $BnH$  ( $n \in N_G(T)$ ) is open in  $G$  if and only if  $\dim(T_\theta^-)$  is maximal and  $\succ_n$  is a  $\theta^+$ -order on  $\Phi$ .*

## 8. Computing the orbits

In the previous sections we have seen that there are several slightly different characterizations of the orbits of Borel subgroups on  $G/H$ , which lead to different ways to compute the orbits. However in practice, no matter which method one chooses, it is still quite difficult and cumbersome to actually compute the diagram of all the orbits or simply the closure of an orbit. Since the characterizations of this orbit decomposition are very combinatorial in nature, most of this work could be done by a computer. These various characterizations of the orbits lead to different algorithms, some more efficient than others. Since the actual computation of the orbits is so complicated one needs an algorithm which uses as much as possible of the rich combinatorial structure of the orbit decomposition.

In this section we discuss an algorithm to compute the diagram of all the orbits and the orbit closures. This algorithm can be implemented on a computer and heavily uses the combinatorial structure of the orbit decomposition as discussed in the previous chapters. Since the structures of the various symmetric varieties  $G/H$  differ, minor adjustments to the algorithm will be needed for each case. From the results in this paper it will be clear how to fine-tune the algorithm for specific cases.

**8.1.** We use the notation of the previous chapters and fix a pair  $(B, T)$ , where  $T$  is a  $\theta$ -stable maximal torus and  $B$  a Borel subgroup containing  $T$ . Let  $W$  be the Weyl group of  $T$  and  $\mathcal{I}$  the set of twisted involutions of  $W$ . The map  $\varphi : V \rightarrow \mathcal{I}$  as in 2.15 describes most of the combinatorial data involving the orbits in  $V$ . In the following we will discuss how one can classify the image and fibers of  $\varphi$ . This would then also classify the diagram of all the orbits in  $V$ , including the orbit closures. First we concentrate on the computation of the image of  $\varphi$ . For this it is in most cases easier to compute representatives for the image of  $\varphi' : V' \rightarrow \mathcal{I}_{\theta'}$  which then also gives a set of representatives for the image of  $\varphi$  using the map  $\iota_{w_0} : \mathcal{I} \rightarrow \mathcal{I}_{\theta'}$  as in 4.11. For the remaining cases it will be easier to compute  $\varphi(V)$  directly. In the following we will discuss how one can compute the image of  $\varphi$  or  $\varphi'$  systematically. For this we need to prove first a few more properties of the maps  $\varphi$  and  $\varphi'$ .

**8.2.** The classification of the image and fibers of  $\varphi : V \rightarrow \mathcal{I}$  (or  $\varphi' : V' \rightarrow \mathcal{I}_{\theta'}$ ) can be reduced to a problem related to the involutions  $w_{\Pi}^0$  as in the characterization of the twisted involutions in Proposition 3.12. We will first prove some properties of these involutions. Let  $\Delta$  be a basis of  $\Phi$ ,  $w_0 \in W = W(T)$  be such that  $\theta(\Phi^+) = w_0(\Phi^+)$ ,  $\theta' = \theta w_0 = w_0^{-1}\theta$  and let  $\mathcal{I}_{\theta'}$  be as in 3.8. Write

$$(13) \quad \Lambda_{\Delta} = \{\Pi \subset \Delta \mid \theta'(\Pi) = \Pi \text{ and } w_{\Pi}^0 \theta'(\alpha) = -\alpha, \forall \alpha \in \Phi_{\Pi}\},$$

$$(14) \quad \mathcal{I}_{\Delta} = \{w_{\Pi}^0 \mid \Pi \in \Lambda_{\Delta}\}.$$

The set  $\mathcal{I}_{\Delta}$  contains a set of representatives of  $\mathcal{I}_{\theta'}/W$  and also of  $\varphi'(V')/W$ . Since by Proposition 4.12  $\varphi'(V')/W \simeq \varphi(V)/W$  we have the following result.

**Lemma 8.3.** *Let  $W$  act on  $\mathcal{I}$ ,  $\mathcal{I}_{\theta'}$ ,  $V$  and  $V'$  as in 2.12 and 4.11. Let  $\Delta$  be a basis of  $\Phi$  and let  $\mathcal{I}_{\Delta}$  be as in (14). Then we have the following:*

- (i) *Each orbit in  $\mathcal{I}_{\theta'}/W$  and  $\varphi'(V')/W$  has a representative in  $\mathcal{I}_{\Delta}$ ;*
- (ii) *Each orbit in  $\mathcal{I}/W$  and  $\varphi(V)/W$  has a representative in  $\mathcal{I}_{\Delta} w_0^{-1}$ .*

*Proof.* Let  $w \in \mathcal{I}$ . By Proposition 3.12 we can write  $w' = w w_0 = w_1 w_{\Pi}^0 \theta'(w_1^{-1})$ . Then  $w_{\Pi}^0 = w_1^{-1} *' w'$  which proves the first statement.

(ii) is immediate from Proposition 2.18, using the isomorphism  $\iota_{w_0} : \mathcal{I} \rightarrow \mathcal{I}_{\theta'}$  as in 4.11.  $\square$

*Remark 8.4.* One  $W$ -orbit in  $\mathcal{I}_{\theta'}$  can contain several elements of  $\mathcal{I}_{\Delta}$ . For example if  $\Pi_1 = \{\alpha_1\}$  of type  $A_1$ , then any  $\theta'$ -stable subset  $\Pi_2 = \{\alpha_2\} \subset \Delta$ , where  $|\alpha_1| = |\alpha_2|$  and  $\Pi_1$  and  $\Pi_2$  lie in the same connected component of  $\Delta$ , gives an involution  $w_{\Pi_2} = s_{\alpha_2}$  which is  $W$ -conjugate to  $w_{\Pi_1} = s_{\alpha_1}$ .

**8.5. Classification of  $\varphi(V)/W$  and  $\varphi'(V')/W$ .** We need to classify the involutions in  $\mathfrak{I}_\Delta$  which represent the different classes in  $\mathfrak{I}_{\theta'}/W$  resp.  $\varphi'(V')/W$ . We will show later on in this section how we can link this to the classification of conjugacy classes of involutions in the Weyl group which is given in [Hel91]. First we need some more notation. Let

$$\mathfrak{I}_\Delta^0 = \{w_{\Pi_1}^0, \dots, w_{\Pi_k}^0\} \subset \mathfrak{I}_\Delta$$

be a set of representatives of  $\mathfrak{I}_{\theta'}/W$  and let

$$\Lambda_\Delta^0 = \{\Pi_1, \dots, \Pi_k\} \subset \Lambda_\Delta$$

be the corresponding subset of  $\Lambda_\Delta$ . Similarly let

$$\mathfrak{I}_\Delta(V') = \mathfrak{I}_\Delta^0 \cap \varphi'(V')$$

and

$$\mathfrak{I}_\Delta(V) = \mathfrak{I}_\Delta^0 \cdot w_0^{-1} \cap \varphi(V)$$

Then  $\mathfrak{I}_\Delta(V')$  is a set of representatives of  $\varphi'(V')/W$  and similarly  $\mathfrak{I}_\Delta(V)$  is a set of representatives of  $\varphi(V)/W$ . Note that we have the following relation between the sets  $\mathfrak{I}_\Delta(V')$  and  $\mathfrak{I}_\Delta(V)$

$$\mathfrak{I}_\Delta(V') = \mathfrak{I}_\Delta(V) \cdot w_0^{-1}$$

**8.6.** The set  $\mathfrak{I}_\Delta(V)$  gives also a set of representatives of the  $H$ -conjugacy classes of  $\theta$ -stable maximal tori of  $G$ . This can be seen as follows. As in 2.13 let  $\mathcal{T}^\theta/H$  denote the set of  $H$ -conjugacy classes of  $\theta$ -stable maximal tori of  $G$ . Then by Proposition 2.18 we have  $\mathcal{T}^\theta/H \simeq V/W \simeq \varphi(V)/W$ . For  $w_\Pi^0 w_0^{-1} \in \mathfrak{I}_\Delta(V)$  let  $v_\Pi \in \varphi^{-1}(w_\Pi^0 w_0^{-1})$  and let  $T_\Pi = x(v_\Pi)^{-1} T x(v_\Pi)$ . Let

$$\mathcal{T}_\Delta = \{T_\Pi \mid \Pi \in \Lambda_\Delta^0\}.$$

Then  $\mathcal{T}_\Delta$  is a set of representatives for the  $H$ -conjugacy classes in  $\mathcal{T}^\theta$ .

**8.7.** Next we show that the orbits in  $\varphi'(V')$  (resp.  $\varphi(V)$ ) under the twisted action of  $W$  correspond with the conjugation classes of the involutions  $w_\Pi^0 \in \mathfrak{I}_\Delta(V')$  (resp.  $\mathfrak{I}_\Delta(V)$ ). For this we need to choose first a suitable maximal torus to characterize the sets  $\mathfrak{I}_\Delta(V')$  and  $\mathfrak{I}_\Delta(V)$ . In the following let  $T$  be a  $\theta$ -stable maximal torus  $T$  with  $T^-$  a maximal  $\theta$ -split torus of  $G$ . Let  $\Delta_0 = \Delta_0(-\theta)$  be a basis of  $\Phi_0(-\theta)$  and extend this to a  $\theta^-$ -basis  $\Delta$  of  $\Phi$ . Then by 6.3 we have

$$\theta' = \theta w_{\Delta_0}^0.$$

As in [Hel91, 7.4] we call an involution  $w_\Pi^0$  a  $\Delta_0$ -standard involution if  $\Pi \subset \Delta_0$ . Similar as for  $\mathcal{T}_\Delta$  we can choose now  $\Delta_0$ -standard involutions as representatives for both  $\mathfrak{I}_\Delta(V)$  and  $\mathfrak{I}_\Delta(V')$  [?, see also]sect. 7]h2.

**Proposition 8.8.** *Every  $W$ -orbit in  $\varphi(V)$  contains a  $\Delta_0$ -standard involution.*

*Proof.* Let  $s \in \varphi(V)$ ,  $v \in \varphi^{-1}(s)$ ,  $x = x(v)$  and  $T_1 = xTx^{-1}$ . From [Hel91, Lemma 3.3] it follows that we may assume that  $T_1^- \subset T^-$  and  $T_1^+ \supset T^+$ . Let  $g \in Z_G(T_1^- T_1^+)$  such that  $T_1 = gTg^{-1}$ . Then  $g^{-1}\theta(g) \in \mathcal{V}$ . Let  $v' \in V$  be the corresponding orbit in  $V$  and  $s' = \varphi(v')$ . Then  $s'$  is an involution. Since  $\mathcal{T}^\theta/H \simeq V/W$ , both  $v$  and  $v'$  are contained in the same  $W$ -orbit of  $V$ . Let  $w \in W(T)$  such that  $w \cdot v' = v$ . Then also  $w * s' = s$ . On the other hand, from [Hel91, Corollary 7.5] it follows that  $s'$  is conjugate under  $W_0(-\theta)$  to a  $\Delta_0$ -standard involution, which proves the result.  $\square$

We can show a similar result for  $\varphi'(V')$ .

**Corollary 8.9.** *Every  $W$ -orbit in  $\varphi'(V') = \varphi(V) \cdot w_{\Delta_0}^0$  contains a  $\Delta_0$ -standard involution.*

*Proof.* From the above result it follows that  $\varphi(V)$  contains a  $\Delta_0$ -standard involution  $w_\Pi^0$  with  $\Pi \subset \Delta$ . Since  $w_\Pi^0$  and  $w_{\Delta_0}^0$  commute,  $w_\Pi^0 w_{\Delta_0}^0$  is also an involution of  $\Phi_0(-\theta)$ . Then there exists  $s \in W(\Phi_0(-w_{\Delta_0}^0))$  such that  $sw_\Pi^0 w_{\Delta_0}^0 s^{-1}$  is a standard involution. Since for all  $\alpha \in \Phi_0(-w_{\Delta_0}^0)$  we have  $\theta'(\alpha) = \theta w_{\Delta_0}^0(\alpha) = \alpha$ , it follows that  $sw_\Pi^0 w_{\Delta_0}^0 \theta'(s^{-1}) = sw_\Pi^0 w_{\Delta_0}^0 s^{-1}$ , which proves the result.  $\square$

*Remarks 8.10.* (1). It follows from the above results that in the case that  $T^-$  is a maximal  $\theta$ -split torus of  $G$ , we can represent the elements of  $\mathcal{L}_\Delta(V)$  and  $\mathcal{L}_\Delta(V')$  as involutions  $w_{\Pi_1}^0$  or  $w_{\Pi_2}^0 w_{\Delta_0}^0$ . For  $\mathcal{L}_\Delta(V')$  we will only use the first characterization, but for  $\mathcal{L}_\Delta(V)$  we will use both characterizations depending on whether  $\theta' = \text{id}$  or a diagram automorphism.

(2). If  $\mathcal{L}_\Delta(V') = \{w_{\Pi_1}^0, \dots, w_{\Pi_k}^0\}$ , where  $w_{\Pi_1}^0, \dots, w_{\Pi_k}^0$  are  $\Delta_0$ -standard involutions, then  $\mathcal{L}_\Delta(V) = \{w_{\Pi_1}^0 w_0, \dots, w_{\Pi_k}^0 w_0\}$  and  $w_{\Pi_1}^0 w_0, \dots, w_{\Pi_k}^0 w_0$  are involutions. In most cases we will use this characterization of  $\mathcal{L}_\Delta(V')$  and  $\mathcal{L}_\Delta(V)$ .

In [Hel91, sect. 7] only the  $W$ -conjugacy classes of the  $\Delta_0$ -standard involutions are studied. So the question which remains is whether two  $\Delta_0$ -standard involutions which are in the same twisted  $W$ -orbit are also  $W$ -conjugate and vice versa. Before we consider this, we prove a result which is very useful in the classification of the twisted involutions. First we need some more notation. Let  $T$  be a  $\theta$ -stable maximal torus of  $G$ . Write  $E = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ ,  $\Phi = \Phi(T)$ ,  $W = W(T)$  and for  $\sigma \in \text{Aut}(\Phi)$  denote the eigenspace of  $\sigma$  for the eigenvalue  $\xi$ , by  $E(\sigma, \xi)$ . Write  $\Phi(\sigma)$  for  $\Phi_0(-\sigma)$ . We have now the following result:

**Proposition 8.11.** *Let  $\theta$  be an involution of  $\Phi$  such that  $\overline{\Phi}_\theta$  is a root system with Weyl group  $\overline{W}_\theta$ . If  $w_1, w_2 \in W$  are involutions with  $E(w_i, -1) \subset E(\theta, -1)$  ( $i = 1, 2$ ), then the following are equivalent:*

- (i)  $w_1$  and  $w_2$  are conjugate under  $W$ ;
- (ii)  $w_1$  and  $w_2$  are conjugate under  $W_1(\theta)$ ;
- (iii)  $w_1\theta$  and  $w_2\theta$  are conjugate under  $W$ ;
- (iv)  $w_1\theta$  and  $w_2\theta$  are conjugate under  $W_1(\theta)$ .

*Proof.* (i)  $\Leftrightarrow$  (ii) follows from [Hel91, Proposition 2.16].

(ii)  $\Leftrightarrow$  (iv) follows from the fact that  $W_1(\theta)$  maps  $T^-$  onto itself, since  $\overline{W}_\theta \cong W_1(\theta)/W_0(\theta)$ . Since (iv)  $\Rightarrow$  (iii) is clear, it remains to show (iii)  $\Rightarrow$  (iv).

Assume  $w \in W$  such that  $ww_1\theta w^{-1} = w_2\theta$ . If  $\theta \in W$  then the result follows from (i)  $\Rightarrow$  (ii). So assume  $\theta \notin W$ . We will use induction on the rank of  $\Phi$ .

If  $\text{rank } \Phi = 1$ , then  $\theta = -\text{id}$ ,  $E(w_1, -1) = E(w_2, -1) = E$  and the result is clear.

Assume now that the result is proved for root systems of rank  $\leq n$  and assume that  $\Phi$  has rank  $n + 1$ . Since  $ww_1\theta w^{-1} = w_2\theta$ , we have  $w(\Phi(w_1\theta)) = \Phi(w_2\theta)$ . If  $\Phi(w_1\theta) = \emptyset$ , then  $w_1$  and  $w_2$  are opposition involutions of  $W(\theta)$  with respect to bases of  $\Phi(\theta)$  and hence are conjugate under  $W(\theta)$  [?, see]Corollary 2.13]h2. So assume now that  $\Phi(w_1\theta) \neq \emptyset$ . Let  $\alpha \in \Phi(w_1\theta)$  and  $\beta = w(\alpha) \in \Phi(w_2\theta)$ . Since  $E(w_i, -1) \subset E(\theta, -1)$  ( $i = 1, 2$ ), it follows that both  $\alpha, \beta \in \overline{\Phi}_\theta$ . Moreover, since  $\beta = w(\alpha)$ , both  $\alpha$  and  $\beta$  are contained in one irreducible component of  $\Phi$ . Hence by [Hel91, Lemma 2.15] they are contained in one irreducible component of  $\overline{\Phi}_\theta$ . Since  $W(\overline{\Phi}_\theta) \cong W_1(\theta)/W_0(\theta)$ , it follows that there exists  $\tilde{w} \in W_1(\theta)$  such that  $\tilde{w}(\alpha) = \beta$ .

Let  $r_1 = s_\beta \tilde{w} w_1 \theta \tilde{w}^{-1} = s_\beta \tilde{w} w_1 \tilde{w}^{-1} \theta$ ,  $r_2 = s_\beta w_2 \theta$ ,  $w_0 = w \tilde{w}^{-1}$  and  $\Phi_\beta = \Phi_0(s_\beta) = \{\gamma \in \Phi \mid (\beta, \gamma) = 0\}$ . Now  $r_1(\beta) = r_2(\beta) = w_0(\beta) = \beta$ , so by [Car72, 2.5.5]  $w_0 \in W(\Phi_\beta) = W_0(s_\beta)$ . Since  $\theta(\beta) = -\beta$  both  $r_1\theta = s_\beta \tilde{w} w_1 \tilde{w}^{-1}$  and  $r_2\theta = s_\beta w_2$  are contained in  $W(\Phi_\beta)$ . Since  $w_0 r_1 w_0^{-1} = r_2$  it follows by induction that there exists  $\tilde{r} \in W_1(\theta) \cap W_0(s_\beta)$  such that  $\tilde{r} r_1 \tilde{r}^{-1} = r_2$ . Now  $r = \tilde{r} \tilde{w} \in W_1(\theta)$  and  $r w_1 \theta r^{-1} = w_2 \theta$ , which proves the result.  $\square$

We can now show the following:

**Proposition 8.12.** *Let  $T$  be a  $\theta$ -stable maximal torus  $T$  with  $T^-$  a maximal  $\theta$ -split torus of  $G$ , let  $\Delta$  be a  $\theta^-$ -basis of  $\Phi = \Phi(T)$ ,  $\Delta_0 = \Delta_0(-\theta)$  and let  $\theta', \varphi'$  and  $V'$  be as in 4.9. Then we have the following.*

- (i) *If  $w_1, w_2 \in \varphi(V)$  (or  $\varphi'(V')$ ) are involutions, then  $w_1$  and  $w_2$  are in the same twisted  $W$ -orbit if and only if  $w_1$  and  $w_2$  are  $W$ -conjugate.*
- (ii) *Every  $W$ -orbit in  $\varphi(V)$  contains an involution  $ww_{\Delta_0}^0$  with  $w$  a  $\Delta_0$ -standard involution.*
- (iii) *All  $\Delta_0$ -standard involutions are contained in both  $\varphi(V)$  and  $\varphi'(V')$ .*

*Proof.* (i). We may assume that  $w_1$  and  $w_2$  are  $\Delta_0$ -standard. Assume first that  $w_1$  and  $w_2$  are  $W$ -conjugate. Since  $T^-$  is a maximal  $\theta$ -split torus, we may assume that  $T_{w_i}^- \subset T^-$  ( $i = 1, 2$ ) and hence  $w_1, w_2 \in W(T) \cap W(T^-)$ . By Proposition 8.11  $w_1$  and  $w_2$  are conjugate under  $W_1(\theta)$ . Let  $w \in W_1(\theta)$  such that  $ww_1w^{-1} = w_2$ . Since  $W_1(\theta) = W(H, T)$  it follows that  $\theta(w) = w$  and hence  $ww_1\theta(w^{-1}) = ww_1w^{-1} = w_2$ .

For the converse statement assume  $w \in W$  such that  $ww_1\theta(w^{-1}) = w_2$ . In this case  $w((T^-)_{w_1}^+) = (T^-)_{w_2}^+$  and  $w((T^-)_{w_1}^- T^+) = (T^-)_{w_2}^- T^+$ . So  $ww_1\theta w^{-1} = w_2\theta$ . Then by Proposition 8.11  $w_1$  and  $w_2$  are  $W$ -conjugate, which proves the result for  $\varphi(V)$ . The result for  $\varphi'(V')$  follows with a similar argument.

(ii) is immediate from Corollary 8.9.

(iii). Let  $w \in W$  be a  $\Delta_0$ -standard involution. By [Hel91, Corollary 4.10]  $w$  corresponds to a  $H$ -conjugacy class of  $\theta$ -stable maximal tori of  $G$ . In particular there

exists  $x \in G$  such that  $x^{-1}Tx$  is  $\theta$ -stable and  $n = x\theta(x)^{-1} \in N_G(T)$  is a representative of  $w$ . So it follows that  $w \in \varphi(V)$ . To show that  $w \in \varphi'(V') = \varphi(V)w_{\Delta_0}^0$  it suffices to show that the involution  $w w_{\Delta_0}^0 \in W(-\theta)$  is also contained in  $\varphi(V)$ . By [Hel91, Theorem 4.6] there exists  $w_2 \in W$  such that  $w_1 = w_2 w w_{\Delta_0}^0 w_2^{-1}$  is a  $\Delta_0$ -standard involution. So by the above result we have  $w_1 \in \varphi(V)$ . Since  $w_1$  and  $w w_{\Delta_0}^0$  are  $W$ -conjugate, it follows from Proposition 8.11 that there exists  $w_3 \in W$  such that

$$w_1 \theta = w_3 w w_{\Delta_0}^0 \theta w_3^{-1}.$$

But then  $w_1 = w_3 w w_{\Delta_0}^0 \theta w_3^{-1} \theta = w_3 w w_{\Delta_0}^0 \theta (w_3^{-1})$  and hence  $w w_{\Delta_0}^0 = w_3^{-1} w_1 \theta (w_3) \in \varphi(V)$ . This proves the result.  $\square$

*Remark 8.13.* It follows from this result that in the case that  $T^-$  is a maximal  $\theta$ -split torus of  $G$ , we can use representatives from the  $W$ -conjugacy classes of involutions in the Weyl group to represent the twisted  $W$ -orbits in  $\varphi(V)$  and  $\varphi'(V')$ . A classification of  $\mathcal{L}_\Delta(V)$  and the corresponding  $\theta$ -stable maximal tori in  $\mathcal{T}_\Delta$  can be derived from [Hel91, sect. 7].

With only very few exceptions, the isomorphism class of an involution  $w_\Pi^0 \in \mathcal{L}_\Delta(V)$  is determined by the type of the root system  $\Phi_\Pi$  spanned by  $\Pi$ . Only in the case that  $(G, \theta)$  is  $\theta$ -split can it happen that two involutions  $w_{\Pi_1}^0$  and  $w_{\Pi_2}^0$  are not  $W$ -conjugate, while the root systems  $\Phi_{\Pi_1}$  and  $\Phi_{\Pi_2}$  ( $\Pi_1, \Pi_2 \subset \Delta$ ) are of the same type. In these cases it is easy to find two subsets  $\Pi_1, \Pi_2 \subset \Delta$ , such that the root systems  $\Phi_{\Pi_1}$  and  $\Phi_{\Pi_2}$  are of the same type and  $w_{\Pi_1}^0$  and  $w_{\Pi_2}^0$  not  $W$ -conjugate. For more details, see [Hel91, sect. 7].

**8.14. Image of  $\varphi$ .** We can now show that the question of computing the image of  $\varphi : V \rightarrow \mathcal{L}$  comes down to the involutions  $w_\Pi^0$  representing the orbits in  $\varphi(V)/W$ . First some more notation. Let  $\Delta$  be a basis of  $\Phi = \Phi(T)$  and let  $w_0, \theta = \theta w_0$ ,  $\varphi'$  and  $V'$  be as in 4.9. If  $s = \varphi(v)$ ,  $v' = \delta(v)$  and  $s' = \varphi'(v') = s w_0$  then we write  $W_s = \{w \in W \mid w * s = s\}$ ,  $W_{s'} = \{w \in W \mid w *' s' = s'\}$ ,  $W_v = \{w \in W \mid w \cdot v = v\}$  and  $W_{v'} = \{w \in W \mid w \cdot' v' = v'\}$ . These sets are of importance in the characterization of the image and fibers of  $\varphi$  and  $\varphi'$ . We have the following relation between these subsets of  $W$ .

**Lemma 8.15.** *Let  $\theta, \theta', v, s = \varphi(v)$ ,  $v' = \delta(v)$  and  $s' = \varphi'(v')$  be as above. Then we have the following.*

- (i)  $W_s = W_{s'}$
- (ii)  $W_v = W_{v'}$

*Proof.* Let  $w \in W$ . From (7) it follows that

$$w * s = s \iff w *' s' = (w * s) w_0 = s w_0 = s',$$

which proves (i).

(ii) follows with a similar argument.  $\square$

If  $w_\Pi^0 \in \mathcal{L}_\Delta$ , then we also write  $W(\Pi)$  for  $W_{w_\Pi^0}$ . We have now the following characterization of the orbits  $W *' w_\Pi^0 \subset \mathcal{L}_{\theta'}$  and  $W * w_\Pi^0 w_0^{-1} \subset \mathcal{L}$ .

**Lemma 8.16.** *Let  $\theta, \theta', \mathfrak{I}$  and  $\mathfrak{I}_{\theta'}$  be as above. If  $w_{\Pi}^0 \in \mathfrak{I}_{\Delta}$ ,  $W(\Pi)$  as above and  $w_1, \dots, w_n$  minimal coset representatives of  $W/W(\Pi)$ , then we have the following.*

- (i)  $W *' w_{\Pi}^0 = \{w_1 *' w_{\Pi}^0, \dots, w_n *' w_{\Pi}^0\}$ .
- (ii)  $W * w_{\Pi}^0 w_0^{-1} = \{w_1 * w_{\Pi}^0 w_0^{-1}, \dots, w_n * w_{\Pi}^0 w_0^{-1}\}$ .

**8.17.** In order to calculate  $\varphi'(V')$  (or  $\varphi(V)$ ) we need to determine first the subsets  $\Pi \in \Lambda_{\Delta}^0$  and compute the corresponding subgroups  $W(\Pi)$ . In 8.13 we showed how an easy classification of  $\varphi'(V')$  can be obtained by starting with a  $\theta$ -stable maximal torus  $T$  with  $T_{\theta}^-$  a maximal  $\theta$ -split torus of  $G$  and a  $\theta^-$ -basis  $\Delta$  of  $\Phi(T)$ .

Before we discuss how to compute the corresponding subgroups  $W(\Pi)$  of  $W$ , we first recall the following. A pair  $(G, \theta)$  is called  *$\theta$ -split* if there exists a  $\theta$ -split maximal torus of  $G$ . The pair  $(G, \theta)$  is called *quasi  $\theta$ -split* if there exists a  $\theta$ -split Borel subgroup of  $G$ . This basically means that the Satake diagram of  $(G, \theta)$  consists of only white dots, with possibly a diagram automorphism. Note that if  $(G, \theta)$  is  $\theta$ -split and  $T$  is a  $\theta$ -stable maximal torus with  $T_{\theta}^-$  a maximal  $\theta$ -split torus of  $G$ , then  $\theta|_{\Phi(T)} = -\text{id}$ . This means that if  $s \in \mathfrak{I}_{\theta}$  and  $w \in W$ , then  $w * s = ws\theta(w^{-1}) = wsw^{-1}$ . So  $\mathfrak{I}_{\theta}$  consists of the set of involutions in  $W(T)$  and the  $W(T)$ -orbits in  $\mathfrak{I}_{\theta}$  are precisely the  $W(T)$ -conjugacy classes of involutions in  $W(T)$ . We summarize this in the following result.

**Lemma 8.18.** *Let  $(G, \theta)$  be  $\theta$ -split,  $B$  a  $\theta$ -split Borel subgroup of  $G$ ,  $T \subset B$  a  $\theta$ -split maximal torus of  $G$  and  $\mathfrak{I}$  the set of twisted involutions in  $W(T)$ . Then  $\mathfrak{I}$  consists of the set of involutions in  $W(T)$ .*

We can now show that to compute the subgroups  $W(\Pi)$  of  $W$  as above, it suffices to consider  $\mathfrak{I}_{\text{id}}$  or  $\mathfrak{I}_{-\text{id}}$ . The  $W$ -orbits in all other cases can be identified with orbits for these two cases. This can be seen as follows.

**8.19.** Let  $T$  be a maximal torus of  $G$ ,  $\Phi = \Phi(T)$ ,  $W = W(T)$ ,  $\theta \in \text{Aut}(G, T)$  an involution with  $T_{\theta}^-$  a maximal  $\theta$ -split torus and  $\Delta$  a  $\theta^-$ -basis of  $\Phi$ . Write  $\theta = \theta^* w_{\Delta(-\theta)}^0$  as in 6.3 and let  $\mathfrak{I}^0 \subset W$  denote the set of involutions in  $W$ . Assume  $\Phi$  is irreducible. Then we have three cases.

- (1)  **$-\text{id} \in \mathbf{W}$ .** Then  $\Phi$  is of type  $B_n, C_n, D_{2n}, E_7, E_8, F_4$  or  $G_2$ . In this case  $\theta' = \theta^* = \text{id}$ ,  $\mathfrak{I}_{\theta'} = \mathfrak{I}^0 = \mathfrak{I}_{-\text{id}}$  and  $\mathfrak{I}_{\theta} = \mathfrak{I}^0 w_{\Delta(-\theta)}^0$ . So we can compute either  $\varphi(V)$  or  $\varphi'(V')$ . If  $w_{\Pi}^0 \in \mathfrak{I}_{\Delta}(V')$ , then  $W(\Pi)$  is the commutator subgroup for the involution  $w_{\Pi}^0$  which can easily be computed using LiE or other symbolic manipulation programs (see also 8.35).
- (2)  **$-\text{id} \notin \mathbf{W}$  and  $\theta' = \text{id}$ .** In this case  $\Phi$  is of type  $A_n, D_{2n+1}$  or  $E_6$  and  $\theta$  is an inner automorphism of  $G$ . Since  $\theta' = \theta^* = \theta w_{\Delta(-\theta)}^0 = \text{id}$  it follows that  $\mathfrak{I}_{\theta'} = \mathfrak{I}^0 = \mathfrak{I}_{\theta} w_{\Delta(-\theta)}^0$ . In this case it is easier to compute  $\mathfrak{I}_{\theta'}$  and  $\varphi'(V')$  instead of  $\varphi(V)$ . The computation of the groups  $w_{\Pi}^0$  is similar as in (1).
- (3)  **$-\text{id} \notin \mathbf{W}$  and  $\theta' \neq \text{id}$ .** In this case  $\Phi$  is again of type  $A_n, D_{2n+1}$  or  $E_6$ , but  $\theta$  is now an outer automorphism of  $G$ . In this case it is easier to switch to  $-\text{id}$  instead of  $\theta'$ . Since  $\theta = \theta' w_{\Delta(-\theta)}^0$  we get  $\theta w_{\Delta(-\theta)}^0 w_{\Delta}^0 = \theta' w_{\Delta}^0 = -\text{id}$ . So

$\mathfrak{I}_\theta = \mathfrak{I}_{-\text{id}} w_{\Delta(-\theta)}^0 w_\Delta^0 = \mathfrak{I}^0 w_{\Delta(-\theta)}^0 w_\Delta^0$  and  $\mathfrak{I}^0$  follows as in (2). In this case it is easier to compute  $\mathfrak{I}_\theta$  and  $\varphi(V)$  instead of  $\mathfrak{I}_{\theta'}$  and  $\varphi'(V')$ .

It follows from the above observations that by switching from the standard pair to another pair  $(B, T)$  one can reduce the characterization of  $\varphi(V)$  from twisted involutions to involutions in  $W$ . A similar result follows from Lemma 6.4 and Proposition 6.6 by considering minimal and maximal elements in a  $W$ -orbit in  $\mathfrak{I}$ . We summarize this in the following result.

**Lemma 8.20.** *Let  $T_\Pi \in \mathcal{T}_\Delta$ ,  $W = W(T_\Pi)$ ,  $\Phi = \Phi(T_\Pi)$  and assume  $\Phi$  is irreducible. Let  $\Delta_1$  be a  $\theta$ -basis of  $\Phi$ ,  $\Delta_2$  a  $\theta^-$ -basis of  $\Phi$ ,  $w_0(\theta)$  the longest element of  $W_0(\theta)$  with respect to  $\Delta_0(\theta) \subset \Delta_1$  and  $w_0(-\theta)$  the longest element of  $W_0(-\theta)$  with respect to  $\Delta_0(-\theta) \subset \Delta_2$ . If  $\mathfrak{I}_1 = \mathfrak{I}_\theta \cdot w_0(\theta) \subset W$  and  $\mathfrak{I}_2 = \mathfrak{I}_\theta \cdot w_0(-\theta) \subset W$ , then  $\mathfrak{I}_1$  or  $\mathfrak{I}_2$  consists of the set of involutions in  $W$ .*

*Proof.* As in 6.3 let  $\theta_1^* \in \text{Aut}(X, \Phi, \Delta_1)$  such that

$$\theta = -\text{id} \theta_1^* w_0(\theta).$$

Similarly let  $\theta_2^* \in \text{Aut}(X, \Phi, \Delta_2)$  such that

$$\theta = \theta_2^* w_0(-\theta).$$

From Lemma 6.4 it follows that  $\theta_1^* = \text{id}$  or  $\theta_2^* = \text{id}$ . If  $\theta_1^* = \text{id}$ , then  $\theta w_0(\theta) = -\text{id} \theta_1^* = -\text{id}$ . But then  $\mathfrak{I}_1 = \mathfrak{I}_\theta \cdot w_0(\theta) = \mathfrak{I}_{\theta w_0(\theta)} = \mathfrak{I}_{-\text{id}}$  and  $\mathfrak{I}_{-\text{id}}$  consists of the set of involutions in  $W$ .

Similarly if  $\theta_2^* = \text{id}$ , then  $\theta w_0(-\theta) = \theta_2^* = \text{id}$ . So  $\mathfrak{I}_2 = \mathfrak{I}_\theta \cdot w_0(-\theta) = \mathfrak{I}_{\theta w_0(-\theta)} = \mathfrak{I}_{\text{id}}$ , which is the set of involutions in  $W$ . This proves the result.  $\square$

**8.21. Fibers of  $\varphi$ .** As for the image, the classification of the fibers of  $\varphi(V)$  or  $\varphi'(V')$  also reduces to a problem related to the involutions  $w_\Pi^0 \in \mathfrak{I}_\Delta(V')$ . This can be seen as follows. Let  $f : \mathfrak{I}_{\theta'} \rightarrow \mathbb{N} \cup 0$  be given by

$$f(w) = \begin{cases} |\varphi'^{-1}(w)| & \text{if } w \in \varphi'(V'); \\ 0 & \text{if } w \in \mathfrak{I}_{\theta'} - \varphi'(V'). \end{cases}$$

For the fibers of  $\varphi$  we can define a similar function. Let  $w \in \mathfrak{I}_{\theta'}$  and write

$$w = s_1 \dots s_h w_\Pi^0 \theta'(s_h) \dots \theta'(s_1)$$

as in Proposition 3.12. Then we have the following.

**Lemma 8.22.** *Let  $f : \mathfrak{I}_{\theta'} \rightarrow \mathbb{N} \cup 0$  be as above and  $w = s_1 \dots s_h w_\Pi^0 \theta(s_h) \dots \theta(s_1) \in \mathfrak{I}_{\theta'}$ . Then  $f(w) = f(w_\Pi^0)$ .*

*Proof.* Let  $\varphi'^{-1}(w_\Pi^0) = \{v_1, \dots, v_n\}$  and  $s = s_1 \dots s_h$ . If  $w \in W$  and  $v' \in V'$ , then by Lemma 2.17 we have  $\varphi'(w \cdot v') = w *' \varphi'(v')$ . It follows that  $\varphi'^{-1}(s w_\Pi^0 \theta'(s^{-1})) = \varphi'^{-1}(s *' w_\Pi^0) = \{s \cdot v_1, \dots, s \cdot v_n\}$ , which proves the result.  $\square$

Before we give another characterization of fibers of  $\varphi$  and  $\varphi'$  we need first some more notation. Let  $v \in V$ ,  $x = x(v)$  and  $T_1 = x^{-1}Tx$ . Then  $T_1$  is  $\theta$ -stable. As in 3.6 let  $\omega_v : W(T) \rightarrow W(T_1)$  be the automorphism determined by  $\text{Int}(x^{-1})$ . Denote the image of  $N_H(T_1) = N(T_1) \cap H$  in  $W(T_1)$  by  $W_H(T_1)$ .

*Remark 8.23.* From Lemma 8.22 and the above characterizations of  $\mathfrak{L}_{\theta'}/W$  it follows that it suffices to determine the fibers of the elements  $w_{\Pi}^0 \in \mathfrak{L}_{\Delta}(V')$ . For  $w_{\Pi}^0 \in \mathfrak{L}_{\Delta}(V')$  let  $v_{\Pi} \in \varphi'^{-1}(w_{\Pi}^0)$  and  $T_{\Pi} = x(v_{\Pi})^{-1}Tx(v_{\Pi})$  as in 8.5. To classify the fibers of the elements  $w_{\Pi}^0$  one needs to determine for each  $T_{\Pi} \in \mathcal{T}_{\Delta}$  the group  $N_H(T_{\Pi})$ . Then the fiber follows from  $W_{w_{\Pi}^0}/W_{v_i} = W(\Pi)/W_{v_{\Pi}}$ . Note that by Proposition 6.6 the size of this set equals the number of  $\theta^-$ -orders on  $\Phi(T_{\Pi})$ , which are conjugate under  $W(T_{\Pi})$ , but not under  $W(H, T_{\Pi})$ .

This observation leads to the following characterization of the fibers of  $\varphi$  and  $\varphi'$ .

**Proposition 8.24.** *Let  $T$  be a  $\theta$ -stable maximal torus with  $T^-$  a maximal  $\theta$ -split torus of  $G$ , let  $v \in V$ ,  $x = x(v)$ ,  $T_1 = x^{-1}Tx$ ,  $s = \varphi(v)$ ,  $W_s = \{w \in W \mid w * s = s\}$  and  $W_v = \{w \in W \mid w \cdot v = v\}$ . Then we have the following.*

- (i)  $\varphi(w \cdot v) = \varphi(v)$  if and only if  $w \in W_s$ .
- (ii)  $W_v = \omega_v^{-1}(W_H(T_1))$ .
- (iii)  $\varphi^{-1}(s) = \{w \cdot v \mid w \in W_s\}$  and  $|\varphi^{-1}(s)| = |W_s/W_v|$ .
- (iv) The map  $\varphi : V \rightarrow \mathfrak{L}$  is injective if and only if there is a unique closed orbit.
- (v) The map  $\varphi : V \rightarrow \mathfrak{L}$  is surjective if and only if  $(G, \theta)$  is quasi  $\theta$ -split.

*Proof.* (i), (ii), (iii) and (iv) are immediate from the previous results.

(v). Let  $\Delta$  be a  $\theta^-$ -basis of  $\Phi(T)$  and let  $\Delta_0 = \Delta_0(-\theta)$ . By Corollary 8.9 every  $W$ -orbit in  $\varphi'(V')$  contains a  $\Delta_0$ -standard involution. On the other hand every  $W$ -orbit in  $\mathfrak{L}_{\theta'}$  has a representative in  $\mathfrak{L}_{\Delta}^0$ . So  $\varphi' : V' \rightarrow \mathfrak{L}_{\theta'}$  is surjective if and only if every  $W$ -orbit in  $\mathfrak{L}_{\theta'}$  contains a  $\Delta_0$ -standard involution. The latter is the case if and only if  $(G, \theta)$  is quasi  $\theta$ -split. This proves the result.  $\square$

*Remarks 8.25.* (1). The above results also holds for an arbitrary  $\theta$ -stable maximal torus  $T$ , but the proof is somewhat more complicated. We will only use the above result in the case that  $T^-$  is a maximal  $\theta$ -split torus of  $G$ .

(2). That the map  $\varphi : V \rightarrow \mathfrak{L}$  is not necessarily injective can be seen from example 6.8, where there are three orbits; two closed orbits and the open orbit.

**8.26. Reduction to simply connected groups.** One important aspect which has not been discussed so far is the dependence of the orbit decomposition on the choice of the open subgroup  $H$  of  $G_{\theta}$ . In the following we will discuss the relation between the orbits for  $H$  and  $H^0$  and show that for  $H^0$  we can reduce to the case that  $G$  is semisimple and simply connected. First some notation. Let  $T$  be a  $\theta$ -stable maximal torus and  $\mathcal{V} = \tau^{-1}(N_G(T))$  as in 2.4. For an open subgroup  $H$  of  $G_{\theta}$  we write  $V(G, H)$  for the set of  $T \times H$ -orbits on  $\mathcal{V}$ . For  $H^0 = G_{\theta}^0$  we will also write  $V^0(G)$  for  $V(G, H^0)$ . It follows from Proposition 2.5 that  $V(G, H)$  parameterizes the double cosets  $BxH$ . The finite group  $H/H^0$  is an elementary abelian 2-group which acts on  $V^0(G)$  in the obvious way. The relation between the orbits for  $H$  and  $H^0$  is stated in the following result.

**Lemma 8.27.** *Let  $H$  be an open subgroup of  $G_{\theta}$  and let  $V(G, H)$  and  $V^0(G)$  be as above. Then  $V(G, H)$  is the set of  $H/H^0$  orbits on  $V^0(G)$ .*

To compute  $V^0(G)$  we can restrict to the case that  $G$  is semisimple and simply connected. That it suffices to look at semisimple groups follows from the following result.

**Lemma 8.28.** *The inclusion  $[G, G] \rightarrow G$  induces a bijection  $V^0([G, G]) \rightarrow V^0(G)$ .*

To show that we can restrict to the case of simply connected groups we first note the following. If  $\lambda : \tilde{G} \rightarrow [G, G]$  is a simply connected covering of  $[G, G]$ , then by [Ste68, 9.16] the involution  $\theta|_{[G, G]}$  can be lifted to an involution  $\tilde{\theta}$  of  $\tilde{G}$ . Since  $\tilde{G}$  is semisimple and simply connected it follows from [Ste68, 8.2] that the fixed point group  $\tilde{G}_{\tilde{\theta}}$  of  $\tilde{\theta}$  is connected. So  $V^0(\tilde{G}) = V(\tilde{G}, \tilde{G}_{\tilde{\theta}})$ . The above observations lead to the following result.

**Proposition 8.29.** *Let  $\lambda : \tilde{G} \rightarrow [G, G]$  be a simply connected covering of  $[G, G]$  and let  $\tilde{\theta}$  be as above. Then we have the following.*

- (i)  $\lambda$  induces a bijection  $V^0(\tilde{G}) \rightarrow V^0([G, G])$ .
- (ii)  $V(\tilde{G}, \tilde{G}_{\tilde{\theta}}) = V^0(\tilde{G}) \simeq V^0(G)$ .
- (iii)  $V(G, H)$  can be canonically identified with the set of orbits of  $H/H^0$  acting on  $V^0(\tilde{G})$ .

In a sense this result reduced the computation of the orbits to the case that  $G$  is semisimple and simply connected.

**8.30. Open orbit versus closed orbits.** To compute the diagram of all the orbits in  $V$  it makes sense to start at either the top or the bottom of the diagram. The top represents the unique open orbit, while the bottom consists of the closed orbits. The first is related to a  $\theta$ -stable maximal torus containing a maximal  $\theta$ -split torus, while the second is related to a  $\theta$ -stable maximal torus containing a maximal torus of  $G_{\theta}$ . Most of the combinatorial data in [Spr84] and [RS90] is formulated with respect to a fixed closed orbit. In the previous sections we generalized these results to the setting of a maximal torus containing a maximal  $\theta$ -split torus and we also proved a number of additional results. In the following we list a few other advantages of working in the setting of a  $\theta$ -stable maximal torus  $T$  containing a maximal  $\theta$ -split torus  $A$  rather than working with a closed orbit.

- (a) The unique open orbit is related to a  $\theta^+$ -order on  $\Phi(T)$ .
- (b) Most of the combinatorial data of the symmetric variety is related to a maximal torus containing a maximal  $\theta$ -split torus. This includes a classification of these symmetric varieties with their restricted root systems, see [Hel88]. Also there exist nice descriptions of  $G$ ,  $H$ ,  $A$  (the maximal  $\theta$ -split torus) and the maximal torus  $T \supset A$ .
- (c) It is easier to find the set of representatives  $\mathfrak{L}_{\Delta}(V')$  for  $\varphi'(V')/W$ . This is mainly due to the fact that all real roots are  $\theta$ -singular and therefore the involutions  $w_{\Pi}^0$  in the characterization of the twisted involutions in Proposition 3.12 are easily determined. If one starts from a closed orbit it is still possible

to find a set of representatives  $w_{\Pi}^0$  for the classes in  $\varphi'(V')/W$  from the results in [Hel91], but it requires a lot of additional work, because one needs to distinguish between compact imaginary roots and  $\theta$ -singular imaginary roots.

- (d) For both the  $\theta^-$ -order and the  $\theta^+$ -order on  $\Phi(T)$ , the involutions  $\theta'$  and  $w_0$  follow from the description given in Corollary 7.2, the classification of involutions of  $G$  as in [Hel88] and the classification of involutions in  $W(T)$  as in [Hel91].

Note that to compute the orbit closures in the above setting it is often easier to use the opposite Bruhat order as in Corollary 5.26.

*Remark 8.31.* The description of the image and fibers of  $\varphi$  as in this section leads to an algorithm to classify the orbits in  $\varphi(V)$  and  $V$ . In principle one could also classify  $\varphi(V)$  and  $V$  by using the Bruhat descendants as in 5.28, although this would be computationally very costly. The best option would be to combine both methods. In the following we will describe an algorithm, which combines both these methods. Naturally the detailed description of the root systems as in [Bou68] is essential in all this.

**8.32. An algorithm to classify  $V$ .** Combining the results in this paper we get now an algorithm to classify the orbits in  $V^0(G)$  together with the orbit closures. Each step of this algorithm can be implemented in LiE or a number of other programs. For a further discussion of this, see 8.35. First we describe the algorithm in the following. Assume that  $G$  is semisimple and simply connected and assume that the pair  $(G, \theta)$  is irreducible. In the following let  $T$  be a  $\theta$ -stable maximal torus such that  $T^-$  is a maximal  $\theta$ -split torus of  $G$  and fix a  $\theta^-$ -basis  $\Delta$  of  $\Phi = \Phi(T)$ . Let  $V = V^0(G)$ . First we determine  $\mathfrak{I}_{\text{id}} = \mathfrak{I}_{-\text{id}}$ . For this we do the following.

- (1) Get a list of the involutions  $w_{\Pi}^0$  in  $\mathfrak{I}_{\Delta}^0$ , which are determined by the type of  $\Phi_{\Pi}$  in almost all cases. This follows from [Hel91, Table II].
- (2) For each  $w_{\Pi}^0 \in \mathfrak{I}_{\Delta}^0$  compute the group  $W(\Pi)$ , which is in these cases precisely the commutator subgroup of  $w_{\Pi}^0$ . For a discussion on this, see also 8.19.
- (3) Find minimal coset representatives of  $W/W(\Pi)$ . This classifies the  $W$ -orbit of  $w_{\Pi}^0$  in  $\mathfrak{I}_{\theta'}$ , using Lemma 8.16.

These three steps classify  $\mathfrak{I}_{\text{id}} = \mathfrak{I}_{-\text{id}}$ . The next step is to determine  $\theta'$  and  $w_0$  and to classify  $\varphi(V)$  or  $\varphi'(V')$  by identifying the corresponding  $W$ -orbits in  $\mathfrak{I}_{\text{id}}$  or  $\mathfrak{I}_{-\text{id}}$ .

- (4) Determine the involutions  $\theta'$  and  $w_0$ . This follows from Corollary 7.2, the classification of involutions of  $G$  in [Hel88, sect. 4] and the classification of involutions of  $W(T)$  in [Hel91, sect. 7]. If  $\theta' = \text{id}$ , then compute  $\varphi'(V')$  and if  $\theta' \neq \text{id}$ , then compute  $\varphi(V)$  by identifying  $\varphi(V)$  as a subset of  $\mathfrak{I}_{-\text{id}} w_{\Delta_0(-\theta)}^0 w_{\Delta}^0$ .
- (5) Find  $\mathfrak{I}_{\Delta}(V') \subset \mathfrak{I}_{\Delta}^0$  or  $\mathfrak{I}_{\Delta}(V) \subset \mathfrak{I}_{\Delta}^0 w_{\Delta_0(-\theta)}^0 w_{\Delta}^0$ . For  $\mathfrak{I}_{\Delta}(V')$  this easily follows from Proposition 8.8 and the classification in [Hel91, Table IV]. For  $\mathfrak{I}_{\Delta}(V)$  note that we can choose  $\Delta_0(-\theta)$ -standard involutions as representatives for the  $W$ -orbits in  $\varphi(V)$ . Since these involutions commute with  $w_{\Delta_0(-\theta)}^0 w_{\Delta}^0$  a list of these also follows from Proposition 8.8 and the classification of involutions of  $W(T)$  in [Hel91, Table IV].

- (6) Determine  $\varphi'(V') \subset \mathcal{I}_{\theta'}$  or  $\varphi(V) \subset \mathcal{I}_{-\text{id}w_{\Delta_0(-\theta)}^0 w_{\Delta}^0}$ , by finding the  $W$ -orbits of the involutions in  $\mathcal{I}_{\Delta}(V')$  or  $\mathcal{I}_{\Delta}(V)$ . For this we use (1) and (2).

After this one can finally find  $V$  as follows:

- (7) Determine  $|\varphi'^{-1}(w_{\Pi}^0)|$  for each involution  $w_{\Pi}^0$  in  $\mathcal{I}_{\Delta}(V')$  or  $\mathcal{I}_{\Delta}(V)$ . Then, using Lemma 8.22 we find the  $W$ -orbits in  $V$  and  $V'$ .

The above steps give a complete list of all the elements of  $V$ . This last point of the algorithm will need some additional fine-tuning for each absolutely irreducible pair  $(G, \theta)$ .

To find the diagrams  $(\mathcal{I}_{\theta'}, \leq_1)$  and  $(V, \leq)$  as in 5.22 and 5.4 we use Bruhat descendants as in 5.28. First we consider the diagram for  $(\mathcal{I}_{\theta'}, \leq_1)$ .

- (8) Find admissible sequences for the involutions  $w_{\Pi}^0 \in \mathcal{I}_{\Delta}^0$  and identify them as admissible subsequences of a fixed admissible sequence  $\mathbf{s}$  for  $w_{\Delta}^0$ .
- (9) Use the Bruhat descendants as in 5.28 and the minimal coset representatives as in (4) to identify the elements of a  $W$ -orbit with admissible subsequences of  $\mathbf{s}$ .

Finally we can now find the diagram of  $(V, \leq)$  as follows.

- (10) Identify  $\varphi'(V')$  in the diagram of  $(\mathcal{I}_{\theta'}, \leq_1)$  using (6). Then the diagram of  $(V, \leq)$  can be obtained by combining each of the nodes in  $\varphi'(V')$  with the fibers of  $\varphi'$  as in (7), using Proposition 5.31.

From the dimension formulas and the above algorithm, it is easy to get a list of all the orbits of a given dimension.

We will illustrate this algorithm with an example.

*Example 8.33.* Let  $G = SL_n(k)$  and  $\theta(g) = {}^t g^{-1}$ . Then  $H = SO_n(k)$ . Let  $B$  be the Borel subgroup of upper triangular matrices,  $A = T$  the group of diagonal matrices,  $\Phi = \Phi(T)$  the root system of  $T$  with respect to  $G$ ,  $\Phi^+$  the set of positive roots of  $\Phi$  related to  $B$ ,  $\Delta$  the corresponding basis of  $\Phi$ ,  $W = W(T)$  the Weyl group of  $T$  and  $X = X^*(T)$  the group of characters of  $T$ . The torus  $A$  is both a maximal  $\theta$ -split torus of  $G$  and a maximal torus of  $G$ . Moreover the orbit  $BH$  is open in  $G$ . Let  $w_0 \in W$  such that  $\theta(\Phi^+) = w_0(\Phi^+)$ ,  $\theta' = \theta w_0$ ,  $v_0 \in V$  and  $n_0 = x(v_0)\theta(x(v_0))^{-1} \in N_G(T)$  such that  $n_0$  induces  $w_0$  in  $W$  and let  $\zeta$  be the involution of  $G$  given by  $\zeta(x) = n_0^{-1}\theta(x)n_0$ ,  $x \in G$ . By Corollary 7.2  $w_0 = w_{\Pi}^0 w_{\Delta}^0$ , where  $\Pi = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\}$ . Since  $G$  is  $\theta$ -split, we have  $\Pi = \emptyset$  and  $w_0 = w_{\Delta}^0$ . The action of  $\theta'$  on  $(X, \Phi)$  is given by  $\theta'(\chi) = \theta w_0(\chi) = -w_0(\chi)$  ( $\chi \in X$ ). So  $\theta'$  acts on  $W$  by  $\theta'(w) = w_0 w w_0^{-1}$ . The set  $\mathcal{I}$  consists of the set of involutions in  $W$ . Let  $\mathcal{I}_{\theta'} = \mathcal{I} \cdot w_0$  as in Lemma 4.10.

From [Hel91, Table II] it follows that the involutions  $w_{\Pi}^0$  are all of type  $A_1 + \dots + A_1$ , ( $r$  times) and there is only one conjugacy class for each type. The Weyl group  $W$  is isomorphic to the symmetric group  $S_n$ . We identify  $S_n$  with the group of permutation matrices, which is contained in  $GL_n(k)$ , but not in  $SL_n(k)$ .

Since  $G$  is  $\theta$ -split it follows from Proposition 8.24 that the map  $\varphi : V \rightarrow \mathcal{I}$  is surjective. The fibers of  $\varphi$  can be described as follows.

**Lemma 8.34.** *Identify  $W$  with  $S_n$  and let  $s \in \mathcal{I}$ . Let  $\mathcal{I}^f \subset \mathcal{I}$  be the set of involutions in  $W$  which have a fixed point on  $\{1, \dots, n\}$ , and  $\mathcal{I}^{nf} \subset \mathcal{I}$  be the set of involutions in  $W$  which have no fixed point on  $\{1, \dots, n\}$ . Then we have the following.*

- (i) *If  $s \in \mathcal{I}^{nf}$ , then  $|\varphi^{-1}(s)| = 2$ .*
- (ii) *If  $s \in \mathcal{I}^f$ , then  $|\varphi^{-1}(s)| = 1$ .*
- (iii) *If  $n$  is odd, then  $\mathcal{I}^{nf} = \emptyset$ , so  $\varphi$  is an isomorphism.*

This result also follows from a straightforward, but lengthy, matrix calculation.

An implementation of the above algorithm in LiE combined with this result is illustrated in Table 1. We use there the following notation. Let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  and write  $s_i = s_{\alpha_i}$  for the reflection defined by  $\alpha_i \in \Delta$ . In the table we list the minimal coset representatives of  $W/W(\Pi)$  for each of the involutions  $w_\Pi^0 \in \mathcal{I}_\Delta \simeq \mathcal{I}_\Delta(V)$ . We also list the admissible sequences for  $w_0$ . If  $(s_{i_1}, \dots, s_{i_k})$  is such an admissible sequence for  $w_0$  and  $\mathbf{a} = (a_0, a_1, \dots, a_k)$  the corresponding sequence in  $\mathcal{I}_{\theta'}$ , then we will write  $\bar{s}_i$  if  $s_{i_r} * a_{r-1} \neq a_{r-1}$  and  $s_i$  otherwise.

Table 1. Calculation of  $\mathcal{I}$  and  $V$

	$w_\Pi^0$	$W/W(\Pi)$	$ \varphi^{-1}(w) $	Adm. sequences $w_0$
$n = 2$	id	id	2	$(s_1)$
	$w_0 = s_1$	id	1	$(s_1)$
$n = 3$	id	id	1	$(\bar{s}_1, s_2)$
		$s_1$	1	$(\bar{s}_2, s_1)$
		$s_2$	1	
	$w_0 = s_2s_1s_2$	id	1	
$n = 4$	id	id	2	$(\bar{s}_1, \bar{s}_2, s_3, s_1)$
		$s_1$	2	$(\bar{s}_1, \bar{s}_2, s_1, s_3)$
		$s_2s_1$	2	$(s_2, \bar{s}_1, \bar{s}_3, s_2)$
	$s_2$	id	1	$(s_2, \bar{s}_1, \bar{s}_2, s_3)$
		$s_1$	1	$(s_2, \bar{s}_3, \bar{s}_1, s_2)$
		$s_3$	1	$(s_2, \bar{s}_3, \bar{s}_2, s_1)$
		$s_1s_3$	1	
		$s_2s_1$	1	
		$s_2s_3$	1	
$w_0 = s_1s_2s_1s_3s_2s_1$	id	1		

**8.35. Computational considerations.** The above algorithm can be implemented on a computer for each symmetric variety  $G/H$ , as in [Hel88]. Since the structures of the various symmetric varieties  $G/H$  differ, one will have to use some specific properties of each of the symmetric varieties  $G/H$  to optimize the algorithm in each case. There are 33 types of symmetric varieties of which 16 are infinite families. The eventual goal is to be able to compute the orbits of all finite cases and the infinite families up to a certain dimension. We should note that some of the finite cases are extremely large and it will be hard to compute these. The highest dimension of the infinite cases that can be handled will depend on several factors. First of all it

depends of course on the processor used, but more importantly, it depends on the efficiency of both the algorithm and its implementation.

Another aspect we should note is that any implementation of the above algorithm will require a considerable amount of work. This is mainly due to the fact that there are 33 types of symmetric varieties, which have to be dealt with independently. The amount of work needed to implement the algorithm can be reduced considerably by using one of the available symbolic manipulation programs for which a lot of Weyl group algorithms already have been implemented. For example one could use the excellent package of Stembridge (see [Ste92]), who has implemented several Weyl group algorithms in Maple. Even more suited is the package LiE (see [MLL92]). Using this package the calculation of  $W(\Pi)$  as in step (2) and the calculation of the minimal coset representatives in  $W/W(\Pi)$  as in (3) are easily implemented. Another reason that LiE is a good choice is that the source code (written in C) is available, so one can also optimize its algorithms to suit the above calculation of  $\mathcal{I}_{\theta'}$  and  $V$ . An implementation in LiE should be able to handle all the finite dimensional cases. If a more efficient implementation of the algorithm is needed, then one must write an independent program.

*Remark 8.36.* In the case that  $G$  is not simply connected one also has the added complication of the fact that  $H$  does not need to be connected. Although this has no influence on the calculation of the subset  $\varphi(V)$  in the diagram of  $(\mathcal{I}_{\theta'}, \leq_1)$  as in 5.22, the actual diagrams for  $(V, \leq)$  as in 5.4 can be different. An example of this is the case of  $G = PGL_n(k)$  and with  $\theta$  the involution given by  $\theta(g) = {}^t g^{-1}$ . In this case  $G_{\theta}/G_{\theta}^0$  is a group of order 2. The group  $SL_n(k)$  as in the above example is a simply connected covering group of  $G$ . By Proposition 8.29 the orbits in  $V^0(G)$  correspond to those in  $V^0(SL_n(k))$ . The group  $G_{\theta}/G_{\theta}^0$  acts on  $V^0(G)$  by permuting the fibers of the twisted involutions  $s \in \mathcal{I}^{nf}$  and by acting trivially on  $\varphi^{-1}(\mathcal{I}^f)$ .

*Remark 8.37.* Since  $\mathcal{I}_{\theta'}$  basically consists of conjugacy classes of involutions in  $W$ , one might wonder if there are easier methods to describe all the involutions in the Weyl group. This is in fact true. The conjugacy classes of involutions in the Weyl group are described in [Car72] and his description is easily implemented on a computer. For  $\Phi$  of type  $A_n$  one can also use another approach and use the Robinson-Schensted correspondence to describe all the involutions by tableaux. For  $\Phi$  of type  $B_n$  something similar is possible, but for the other types nothing is known along these lines. However the main disadvantage of both these methods is that a priori there is no obvious way to describe the orbit closures, while this follows from 5.28 using an inductive procedure.

**8.38. Open problems.** There are several open mathematical questions whose solution would make the algorithm much more efficient. Also there are some related open problems that deserve further attention. In the following we discuss some of these problems.

- (1) In general the algorithm in this paper does a good job in computing the orbits, however in its present form the algorithm is not yet very efficient in computing the orbit closures. The present algorithm uses an inductive procedure to

compute a reduced expression for a twisted involution. This procedure is similar to the one typically used for computing reduced expressions for elements of the Weyl group (see [Sno90] and [Hum90]). To compute this reduced expression for a twisted involution  $w$  involves an amount of work, which is approximately proportional to the length of  $w$  times the rank of the root system. The big disadvantage of this method is that it carries out the same or similar calculations several times in a row. For the Weyl group Casselman (see [Cas94]) described recently a much more efficient method to compute these reduced expressions (and the underlying Bruhat order) by using automata. He expands on the ideas laid out in [BH93]. It is likely that the ideas in [Cas94] can be generalized to the setting of twisted involutions, but for this further research will be needed. This has the potential to drastically reduce the amount of computational work involved. We hope to address this problem in a forthcoming paper.

- (2) The classification of  $B$  orbits on  $G/H$ , as described above, provides the first step towards a solution of the similar problem over the real numbers: namely the classification of orbits of minimal parabolic  $\mathbb{R}$ -subgroups acting on semisimple symmetric spaces. To a large extent one can follow the same approach as in the case of algebraically closed fields, but there are some additional problems with the fibers of  $\varphi$ . One of these problems is that there is no longer a unique open orbit. In this case the conjugacy classes of the involutions  $w_{\Pi}^0$  in the image of  $\varphi$  correspond to the  $H_k$ -conjugacy classes of  $\theta$ -stable maximal  $k$ -split tori. A classification of these can be found in [Hel97, Hel98]. A similar description of the combinatorial data follows from [HW93]. Once the case of algebraically closed fields is implemented on a computer, one can start to think about implementing the much more difficult problem of orbits of minimal parabolic  $\mathbb{R}$ -subgroups on real symmetric  $k$ -varieties.
- (3) In the study of infinite-dimensional representations of semisimple complex Lie algebras, the Kazhdan-Lusztig polynomials describe the combinatorial relations between the characters of the highest weight modules (see [KL79]). For the characters of the irreducible representations of a connected semisimple Lie group there is a similar set of polynomials that describe the combinatorial formalism behind a DGM extension (see [Vog83]). Once the algorithm to compute the orbits and their closures has been implemented, one of the natural next steps is to try and compute these polynomials.
- (4) A computer algebra package to compute the structure and geometry of the orbit decomposition as described in this paper could be extended to a more general computer algebra package for computations related to symmetric varieties. Since most of the fine structure of these symmetric varieties is related to the structure and geometry of the orbit decomposition as described in this paper, the results in this paper will provide a good foundation for the development of such a computer algebra package. Other aspects of symmetric varieties that could be included in such a package are for example the restricted root systems with Weyl groups and multiplicities, but also the polynomials

as in (3). Hopefully the results in this paper will eventually lead to such a computer algebra package.

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