

Properties of Twisted Involutions in Signed Permutation Notation

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Abstract

In algebraic contexts Weyl group elements are usually represented in terms of generators and relations, where representation is not unique. For computational purposes, a more combinatorial representation for elements of classical Weyl groups as signed permutation vectors was introduced in [5]. This paper characterizes some special classes of Weyl group elements using this notation. These classes are especially useful for the study of symmetric spaces and their representations.

1 Introduction

Weyl groups are defined as reflection groups of root systems. That is, given a root system Φ in a Euclidian vector space $V = \mathbb{R}^n$, for each vector $\alpha \in \Phi$, let s_α be the reflection through α . The Weyl group is the group generated by these reflections. In algebraic contexts, Weyl group elements are usually

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given in terms of generators and relations. While this is useful for many theoretical purposes, it is not efficient for computational problems since representation of an element as a product of generators is not unique. For the classical Weyl groups the standard basis vectors e_i in \mathbb{R}^n will be mapped to $\pm e_j$ (where i and j may be equal). We use this to represent the Weyl group elements in a more combinatorial manner, as *signed permutations*.

In this paper we give several results characterizing special types of Weyl group elements. An involution $\hat{\theta}$ of the root system induces an involution θ on its Weyl group by $\theta(w) := \hat{\theta}w\hat{\theta}^{-1}$. If θ is an involution of W then the set $\mathcal{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ is called the set of θ -twisted involutions in W . This set is important in the study of orbits of minimal parabolic k -subgroups acting on symmetric k -varieties, see [7, 8]. The geometry of these orbits and their closures induce a poset structure on the set \mathcal{I}_θ . Understanding this poset structure is key to understanding the structure of the orbits. In order to describe the orbits we need to have a characterization of the twisted involutions. In this paper we give an explicit characterization of these elements, when written in signed permutation notation. For $\theta = \text{id}$, the identity map we get the set $\mathcal{I}_{\text{id}} = \{w \in W \mid w = w^{-1}\}$.

Many of these results are relatively easy to state and easy to prove. This is in contrast to similar results for Weyl groups using generators and relations. Thus these results provide additional evidence of the usefulness of this more combinatorial approach. This work will be used in a computational package for Weyl groups being developed by the first two authors. The signed permutation notation for the classical Weyl groups and the results in this paper can also be used to simplify the algorithms for symmetric spaces such as those found in [1, 2] and [3]. Those papers depend heavily on the presentation of the Weyl group in terms of generators and relations, but can be reframed.

2 Preliminaries and Notation

2.1 Classical Weyl Groups and Signed Permutations

In the Classical Weyl groups (those with root systems corresponding to Dynkin Diagrams of type A, B, C, and D,) each root is the sum or difference of at most two standard basis vectors e_i . Weyl groups of root systems of type B and C are the same, hence from now on we only discuss type B . An element $w \in W$ can be completely described by its action on the e_i . In particular, $w(e_i) = \pm e_j$ for all i, j not necessarily different than i . For a nonzero real number a , define $\text{sgn}(a) = +$ if $a > 0$ and $\text{sgn}(a) = -$ if $a < 0$. We represent $w \in W$ by the vector (a_1, a_2, \dots, a_n) , where $w(e_i) = \text{sgn}(a_i)e_{|a_i|}$. That is, the vector corresponds to a signed

permutation matrix where the i th column has its ± 1 in the a_i th row, and the sign of a_i equals the sign of that unique nonzero entry. Equivalently, starting with the permutation of the standard basis vectors in standard notation

$$\begin{pmatrix} 1 & 2 & \dots & n & -1 & -2 & \dots & -n \\ a_1 & a_2 & \dots & a_n & -a_1 & -a_2 & \dots & -a_n \end{pmatrix},$$

the signed permutation is just the first n places in the bottom row. Hence the Weyl groups can be seen as subgroups of S_{2n} .

The standard generators for Weyl groups correspond to reflections. The Weyl group of type A_{n-1} is generated by the transpositions $s_1 = (1, 2), s_2 = (2, 3), s_3 = (3, 4), \dots, s_{n-1} = (n-1, n)$. Note this is S_n , the usual group of permutations on n elements. The Weyl group of type B_n is generated by the transpositions $s_1 = (1, 2), s_2 = (2, 3), s_3 = (3, 4), \dots, s_{n-1} = (n-1, n), s_n = (n, -n)$. Note that if $w \in W$ of type B_n then $w(-i) = -w(i)$ for any $i = 1, \dots, n$, justifying the use of only the first n positions in the signed permutation notation. The Weyl group of type D_n is generated by the transpositions $s_1 = (1, 2), s_2 = (2, 3), s_3 = (3, 4), \dots, s_{n-1} = (n-1, n)$, and the element $s_n = (n, -(n-1))(n-1, -n)$ which is a product of transpositions. In this case as well, $w(-i) = -w(i)$ for any $i = 1, \dots, n$. To differentiate between the elements s_n for B_n and D_n we will denote the signed permutation $(n, -(n-1))(n-1, -n)$ in D_n by \hat{s}_n .

The following proposition from [6] describes multiplication by basis elements in signed permutation notation.

Proposition 2.2. *For $1 \leq i < n$,*

- (i) $(a_1, a_2, \dots, a_n)s_i = (a_1, a_2, \dots, a_{i+1}, a_i, \dots, a_n)$.
- (ii) *If $a_k = i$, and $a_l = i + 1$ then $s_i(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_{k-1}, \text{sgn}(a_k)|a_l|, a_{k+1}, \dots, a_{l-1}, \text{sgn}(a_l)|a_k|, a_{l+1}, \dots, a_n)$,*
- (iii) $(a_1, a_2, \dots, a_n)s_n = (a_1, a_2, \dots, a_{(n-1)}, -a_n)$.
- (iv) $(a_1, a_2, \dots, a_n)s_{\hat{n}} = (a_1, a_2, \dots, -a_n, -a_{(n-1)})$.
- (v) *If $a_l = n$ then $s_n(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, -a_l, \dots, a_n)$.*
- (vi) *If $a_k = n - 1$, and $a_l = n$ then $s_{\hat{n}}(a_1, a_2, \dots, a_k, \dots, a_l, \dots, a_n) = (a_1, a_2, \dots, -\text{sgn}(a_k)|a_l|, \dots, -\text{sgn}(a_l)|a_k|, \dots, a_n)$.*

2.3 Involutions of Weyl groups

As shown in [4] it is sufficient to consider only those involutions $\hat{\theta}$ that fix the basis of the root system, and hence involutions θ that fixes the set of generators of W . For type A_{n-1} the only non-trivial involution that fixes

the basis of root system is defined by $\hat{\theta}(i) = (n + 1 - i)$, for $1 \leq i \leq n$. For type B_n there are no non-trivial involutions that fixes the basis of root system. For type D_n the only non-trivial involution that fixes the basis of root system is defined by $\hat{\theta}(i) = i$, for $1 \leq i \leq n - 1$, and $\hat{\theta}(n) = -n$.

3 Results

We characterize several special types of elements in the signed permutation notation. We begin by characterizing inverses.

Proposition 3.1. $(a_1, a_2, \dots, a_n)^{-1} = (b_1, b_2, \dots, b_n)$, if $|b_{|a_i|}| = i$, and the $\text{sgn}(b_i) = \text{sgn}(a_{b_i})$.

For example, $(4, -1, 2, -3)^{-1} = (-2, 3, -4, 1)$. Consider (a_1, a_2, \dots, a_n) as denoting the position and sign of the unique ± 1 in each column of the matrix A , then the inverse element is easily read off as the position and sign of the unique ± 1 in each row of A .

Theorem 3.2. (i) For $w \in W$ a Weyl group of type A_{n-1} , $\theta(w) = w$, if and only if $w = (a_1, a_2, \dots, a_n)$ satisfies $\hat{\theta}(a_i) = a_{n-i+1}$ for all i .

(ii) For $w \in W$ a Weyl group of type D_n , $\theta(w) = w$, if and only if $w = (a_1, a_2, \dots, a_n)$ satisfies $a_n = \pm n$.

Proof. Recall that $\theta(w) = \hat{\theta}w\hat{\theta}$. For type A_{n-1} , $\theta((a_1, a_2, \dots, a_n)) = (\hat{\theta}(a_n), \hat{\theta}(a_{n-1}), \dots, \hat{\theta}(a_1))$ and result (i) follows. For type D_n , if $a_k = \pm n$ then $\theta((a_1, a_2, \dots, a_n)) = (a_1, a_2, \dots, -a_k, \dots, -a_n)$ and result (ii) follows. \square

It is difficult to tell whether an element given by generators is in \mathcal{I}_{Id} or \mathcal{I}_θ . On the other hand, we can quickly see if an element given as a signed permutation vector is in \mathcal{I}_{Id} .

Lemma 3.3. An element $w = (a_1, a_2, \dots, a_n)$ is in \mathcal{I}_{Id} if and only if $|a_{|a_i|}| = i$ for all i , and $\text{sgn}(a_i) = \text{sgn}(a_{|a_i|})$.

Proof. This follows immediately from the characterization of inverses in Proposition 3.1 above. \square

Next we give a characterization of the signed permutation vectors that are contained in \mathcal{I}_θ .

Theorem 3.4. Let W be Weyl group of type A_n , and θ the nontrivial diagram involution. An element $w = (a_1, a_2, \dots, a_n) \in W$ is in \mathcal{I}_θ if and only if $a_{(n+1)-a_i} = (n+1) - i$ for all i .

Proof. First, if $w^{-1} = (b_1, \dots, b_n)$ then $b_{a_i} = i$ for all i . Additionally, if $\theta(w) = (c_1, \dots, c_n)$ then $c_i = (n+1) - a_{n+1-i}$, or equivalently, $c_{n+1-i} = n+1 - a_i$, for all i . Now, $a_{n+1-a_i} = n+1 - i$ if and only if $b_{n+1-i} = c_{n+1-i} = n+1 - a_i$. \square

Theorem 3.5. *Let W be Weyl group of type D_n , and θ the nontrivial diagram involution. An element $w = (a_1, a_2, \dots, a_n)$ is in \mathcal{I}_θ if and only if each of the following conditions holds:*

- (i) For $i \neq n$, if $a_i = n$ then $a_n = -i$.
- (ii) For $i \neq n$, if $a_i = -n$ then $a_n = i$.
- (iii) For $i \neq n$, and $j \neq n$, if $a_i = j$ then $a_j = i$.
- (iv) For $i \neq n$ and $j \neq n$, if $a_i = -j$ then $a_j = -i$.
- (v) No restrictions if $a_i = \pm i$ (even if $i = n$)

Proof. First, note that if $w^{-1} = (b_1, \dots, b_n)$ then $b_{|a_i|} = i \operatorname{sgn} a_i$ for all i . Additionally, $\theta(w) = (c_1, \dots, c_n)$ is completely determined by

$$c_i = a_i \begin{cases} \operatorname{sgn}(a_i) & \text{if } i \neq n \text{ and } |a_i| \neq n \\ -\operatorname{sgn}(a_i) & \text{if one of } i = n \text{ or } |a_i| = n \\ \operatorname{sgn}(a_i) & \text{if } i = n \text{ and } |a_i| = n \end{cases}$$

Suppose $c_i = b_i$. This implies $|a_{|a_i|}| = i$ for all i , that is $|a_i| = j$ if and only if $|a_j| = i$. As well,

$$\operatorname{sgn}(a_i) = \begin{cases} \operatorname{sgn}(a_{a_i}) & \text{if } i \neq n \text{ and } |a_i| \neq n \\ -\operatorname{sgn}(a_i) & \text{if one of } i = n \text{ or } |a_i| = n \\ \operatorname{sgn}(a_i) & \text{if } i = n \text{ and } |a_i| = n \end{cases}$$

\square

In [5] we developed an algorithm for computing the twisted (and non-twisted) involutions. The algorithm generates the \mathcal{I}_θ and \mathcal{I}_{id} posets discussed in the introduction. Figure 1 gives the poset of the elements of \mathcal{I}_θ for the Weyl group of type A_4 . From each element there is an edge for each group generator. The edges will correspond to one of two possible operations. The following lemmas give the two possibilities. In Figure 1, the operation in Lemma 3.6 is designated by a solid line while the operation in Lemma 3.7 is designated by a dashed line.

Lemma 3.6. *Given a Weyl group W , $w \in W$, an involution θ and a basis element s_i .*

- (i) *If $w \in \mathcal{I}_{id}$ then $s_i w s_i \in \mathcal{I}_{id}$.*

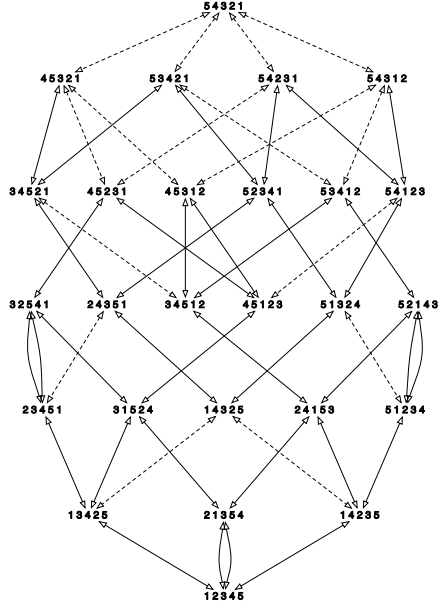


Figure 1: Poset of \mathcal{I}_θ for type A_4

(ii) If $w \in \mathcal{I}_\theta$ then $s_i w \theta(s_i) \in \mathcal{I}_\theta$.

Lemma 3.7. Given a Weyl group W , $w \in W$, an involution θ and a basis element s_i .

(i) If $w \in \mathcal{I}_{id}$ and $s_i w s_i = w$, then $s_i w \in \mathcal{I}_{id}$

(ii) If $w \in \mathcal{I}_\theta$ and $s_i w \theta(s_i) = w$, then $s_i w \in \mathcal{I}_\theta$

The next theorems characterizes when each of the above operations will be used. These lead to improved algorithms for computing the involution posets.

Theorem 3.8. Let W be a Weyl group of type A_{n-1}, B_n or D_n . For $i < n$, if $w \in W$ then $s_i w s_i = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{|a_i|, |a_{i+1}|\} = \{i, i+1\}$ and $\text{sgn}(a_i) = \text{sgn}(a_{i+1})$.

For Weyl group of type B_n , then $s_n w s_n = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $a_n = \pm n$. For Weyl group of type D_n , then $s_{\hat{n}} w s_{\hat{n}} = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, and $\text{sgn}(a_{n-1}) = \text{sgn}(a_n)$.

Proof. From Proposition 2.2 if $i < n$ then $s_i w$ acts on w by switching the elements which are equal to $\pm i$ and $\pm(i+1)$ but leaving their signs in the old

positions, while ws_i switches the elements in the i th and $(i+1)$ st positions and their signs as well. Hence $s_iws_i = w$ if and only if $\{a_i, a_{i+1}\} = \{i, i+1\}$ and $\text{sgn}(a_i) = \text{sgn}(a_{i+1})$. Similarly, multiplying on the right and left by s_n changes the sign of the element in the n th position, and the sign of the element that is $\pm n$. Finally, multiplying on the left and by $s_{\hat{n}}$ switches the elements in the $(n-1)$ th and n th positions and those values are $n-1$ and n so $s_{\hat{n}}ws_{\hat{n}} = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, with perhaps some further condition on the signs. To see that signs must be the same, compute according to the rules of proposition 2.2.

$$\begin{aligned} s_{\hat{n}}(a_1, a_2, \dots, a_n)s_{\hat{n}} &= s_{\hat{n}}(a_1, a_2, \dots, -a_n, -a_{(n-1)}) \\ &= (a_1, a_2, \dots, \text{sgn}(a_n)|a_{n-1}|, \text{sgn}(a_{n-1})|a_n|) \end{aligned}$$

□

The next theorem gives the twisted case.

Theorem 3.9. *Let W be a Weyl group of type A_{n-1} . For $i < n$, if $w \in W$ then $s_iw\theta(s_i) = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{a_{(n-i)}, a_{(n-i+1)}\} = \{i, i+1\}$.*

For W of type D_n , $i < n-1$, $s_iw\theta(s_i) = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{|a_i|, |a_{i+1}|\} = \{i, i+1\}$ and $\text{sgn}(a_i) = \text{sgn}(a_{i+1})$. For W of type D_n , and $i = n-1$ or $i = n$, then $s_iw\theta(s_i) = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, and $\text{sgn}(a_{n-1}) = -\text{sgn}(a_n)$.

Proof. For the A_{n-1} case, observe $\theta(s_i) = s_{n-i}$. Proceed similarly to the proof of theorem 3.8. From Proposition 2.2 if $i < n$ then s_iw acts on w by switching the elements which are equal to i and $(i+1)$, while $w\theta(s_i) = ws_{n-i}$ switches the elements in the $(n-i)$ th and $(n-i+1)$ st positions. Hence $s_iw\theta(s_i) = w$ if and only if $\{a_{(n-i)}, a_{(n-i+1)}\} = \{i, i+1\}$.

For the D_n case, note first that $\theta(s_i) = s_i$ if $i < n-1$ so in these cases $s_iw\theta(s_i) = s_iws_i$ and we get the same result as in theorem 3.8. Now $\theta(s_{n-1}) = s_{\hat{n}}$. Both s_{n-1} and $s_{\hat{n}}$, operate on the elements $(n-1)$ and n and the $(n-1)$ st and n th positions. So that $s_{\hat{n}}ws_{n-1} = w$ and $s_{n-1}ws_{\hat{n}} = w$ if and only if $w = (a_1, \dots, a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, with perhaps some further condition on the signs. To see that signs must be opposite, compute according to the rules of proposition 2.2.

$$\begin{aligned} s_{n-1}(a_1, a_2, \dots, a_n)s_{\hat{n}} &= s_{n-1}(a_1, a_2, \dots, -a_n, -a_{(n-1)}) \\ &= (a_1, a_2, \dots, -\text{sgn}(a_n)|a_{n-1}|, -\text{sgn}(a_{n-1})|a_n|). \end{aligned}$$

A similar calculation shows the $s_{\hat{n}}ws_{n-1} = w$ also requires that $\text{sgn}(a_{n-1}) = -\text{sgn}(a_n)$. □

While both of the above theorems hold for all $w \in W$, we are in fact most interested in the case when $w \in \mathcal{I}_{\text{id}}$, or $w \in \mathcal{I}_\theta$. It is interesting to note that one of the conditions for w in Theorem 3.9 is that $\text{sgn}(a_{n-1}) = -\text{sgn}(a_n)$ which matches the criterion in Theorem 3.5, namely, if $w \in \mathcal{I}_\theta$ then $\text{sgn}(a_{n-1}) = -\text{sgn}(a_n)$.

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