

**MATH 407, Sample Questions, November 2001**

- (1) Let  $H$  be a normal subgroup of order 4 in  $S_4$ .
- (i) What is the order of the factor group  $S_4/H$ ?
  - (ii) Prove that  $S_4/H$  is not cyclic.
  - (iii) Show that  $S_4/H$  is isomorphic to  $S_3$ .
- (2) Let  $G$  be a group of order 21, which has exactly one subgroup of order 3 and exactly one subgroup of order 7. Show that  $G$  is cyclic.
- (3)(a) What is the order of the group  $\mathbb{Z}_{12} \times \mathbb{Z}_9$ ?
- (b) What is the order of the factor group  $\mathbb{Z}_{12} \times \mathbb{Z}_9 / \langle (10, 6) \rangle$ ?
  - (c) Classify the abelian group  $\mathbb{Z}_{12} \times \mathbb{Z}_9 / \langle (10, 6) \rangle$  according to the fundamental theorem of finitely generated abelian groups, that is, find a familiar group  $G$  which is isomorphic to  $\mathbb{Z}_{12} \times \mathbb{Z}_9 / \langle (10, 6) \rangle$ .
- (4) Assume  $n$  is odd and  $\sigma \in S_n$  is a  $n$ -cycle. Prove that  $\sigma$  does not commute with any element of order 2. (Hint: show first that any element of order 2 is a product of disjoint transpositions).
- (5) Consider  $\mathcal{P}(X)$ , the power set of  $X$ . Define the operation  $+$  on  $\mathcal{P}(X)$  to be  $Y + Z = (Y - Z) \cup (Z - Y)$ , where  $Y$  and  $Z$  are subsets of  $X$ . Let  $\cap$  be the multiplication on  $\mathcal{P}(X)$ . Prove that  $\langle \mathcal{P}(X), +, \cap \rangle$  is a ring.
- (6) An element  $g$  of a group  $G$  is said to be *self-inverse* if  $g = g^{-1}$ . Show that every group of even order has at least one self-inverse element other than the identity.
- (7)(a) Find all zero divisors in  $\mathbb{Z}_5[i]$ .
- (b) Find all idempotent elements in  $\mathbb{Z}_5[i]$ .
- (8) Find all solutions to  $x^2 - 5x + 6 = 0$  in  $\mathbb{Z}_{12}$ .

*These sample questions are designed to familiarize you with various concepts which may be on the exam. Note that since this is essentially a homework set, the questions here may require more time and thought than would be expected on a timed inclass exam.*