Part II.

1. Symplectic Manifolds.

In Part I of these notes we have concentrated on the prototype of symplectic manifolds, namely the cotangent bundle of a manifold with its canonical symplectic structure. This fundamental arena needs to be generalized in order to discuss various applications in physics. In the simplest generalization one retains the cotangent bundle T^*M of a manifold M as the phase space, but generalizes the canonical symplectic structure. This generalization is a special case of a general symplectic manifold, which we now define.

DEFINITION #1: A **symplectic manifold** is a pair (N, ω) where N is an even-dimensional differentiable manifold and ω is a closed, non-degenerate 2-form on N. ω is a **symplectic structure** on N.

EXERCISE: Show that the condition that N be even dimensional cannot be dropped from the above definition.

Although we have up to now worked exclusively with the cotangent bundle T^*M of a manifold M with its canonical symplectic structure, it turns out that **locally** every symplectic manifold looks like a cotangent bundle. This follows from the following well-known theorem.

Theorem #1: (Darboux) Let (N, ω) be a 2n-dimensional symplectic manifold and let $n \in N$. Then there is a neighborhood U of n and a coordinate system (q^i, p_j) , $i, j = 1, 2, \ldots, n$, on U such that $\omega|_U = dp_i \wedge dq^i$.

Proof: See, for example, N. Woodhouse, *Geometric Quantization.*

The significance here is that many of the results we have worked out on T^*M carry over to a general symplectic manifold. In fact, any result on T^*M that does not depend explicitly on the bundle structure

$$T^*M \xrightarrow{\pi} M$$

will still hold on a general symplectic manifold.

As a first example of a symplectic manifold we will work out the details of the classical equations of motion for a particle in combined gravitational and electromagnetic fields in terms of the so-called **charged symplectic form** on the cotangent bundle T^*M over spacetime M.

DEFINITION #2: Let (M, g) be a 4-dimensional spacetime manifold with metric tensor g, and let F be a source-free Maxwell 2-form on M. Then

$$\omega_e = d\theta + \frac{e}{2}\pi^*(F) \quad , \ e \in \mathbb{R} \tag{1}$$

is a smplectic structure on $T^*M \xrightarrow{\pi} M$, the **charged symplectic 2 form**.

EXERCISE: Verify that ω_e is a symplectic structure on T^*M.

Recall that if A is any tensor field of type (r,s) on an n-dimensional manifold M then A defines a unique **function** $\hat{A} : LM \to T^r_s(\mathbb{R}^n)$ where LM denotes the bundle of linear frames over M. The value of the function $\hat{A}$ at a linear frame (p, e_i) , $p \in M$, is just the $T^r_s(\mathbb{R}^n)$ components of A with respect to the linear frame (e_i) at p. The function $\hat{A}$ has the **tensorial** transformation property with respect to change of frame (right translation) on the frame bundle.

In describing certain fundamental Hamiltonian systems on T^*M is it useful to note a similar correspondence between **symmetric contravariant tensor fields** on M and real-valued functions on T^*M.

Let A be a symmetric contravariant tensor field on a manifold M. Then associated with A is a unique function $\hat{A} : T^*M \to \mathbb{R}$ defined by

$$\hat{A}(x, \beta) := A(x)(\beta, \beta, \ldots, \beta) \quad , \ \forall (x, \beta) \in T^*M \ . \tag{2}$$

Note that since there is no structure group associated with the fiber bundle structure of T^*M , we cannot characterize $\hat{A}$ as having a "tensorial" transformation property on T^*M.

EXAMPLES:

a. If X is a vector field on M then the associated function $\hat{X}$ simply represents evaluation of covectors on the vector field X. The function $\hat{X}$ is sometimes denoted P_X and referred to as the **X-component of momentum**. The motivation for this terminology is contained in the following exercise.

EXERCISE: Let x^i be coordinates on $U \subset M$, and (q^i, p_j) be the corresponding natural canonical coordiantes on $\hat{U} \subset T^*M$. Show that the function on $\hat{U}$ defined by the basis vector fields $\frac{\partial}{\partial x^i}$ on U are the coordinate functions $p_i \;\; , \; i = 1, 2, \ldots, n$.

b. Let g denote a contravariant **metric tensor** field on M. Then the associated function $\hat{g} : T^*M \to \mathbb{R}$ is defined by

$$\hat{g}(x, \beta) = g(x)(\beta, \beta) \; . \tag{3}$$

Expressing the function $\hat{g}$ in a natural canonical coordinate system (q^i, p_j) we find

$$\hat{g} = g^{ij}(q^k) p_i p_j \quad , \quad g^{ij} = g(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) \; , \tag{4}$$

This function is proportional to the standard **free particle Hamiltonian** on T^*M. Observe that the functions $g^{ij}(q^k)$ on T^*M are the pull backs under the projection π of the functions

$$g^{ij}(x) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

on M.

LEMMA: Let (M, g, F) be a 4-dimensional spacetime with metric tensor field g and Maxwell 2-form F. The classical dynamics of a charged point particle of mass m and electric charge e in the spacetime is given by the Hamiltonian dynamics on T^*M of the Hamiltonian

$$H = \frac{1}{2m} \hat{g} \tag{5}$$

with respect to the charged symplectic 2-form ω_e.

Proof: We must show that the Hamilton equations of motion reduce to the Lorentz force law

$$m D_u(u) = e F \lrcorner \; u \tag{6}$$

where (a) D denotes the covariant derivative on M with respect to the Levi-Civita connection defined by g, and (b) u denotes the 4-velocity (unit tangent vector) of the particle.

We must first work out the general form of the Hamiltonian vector field X_H defined by the charged symplectic 2-form ω_e that follows from the defining equation

$$dH = -X_H \lrcorner \; \omega_e \; . \tag{7}$$

Writing

$$X_H = X^i \frac{\partial}{\partial q^i} + X_i \frac{\partial}{\partial p_i} \tag{8}$$

and expanding both sides of equation (7) we have

$$\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i = -(X_i - e F_{ij} X^j) dq^i + X^i dp_i \; . \tag{9}$$

Thus we find

$$X_H = (\frac{\partial H}{\partial p_i})\frac{\partial}{\partial q^i} - (\frac{\partial H}{\partial q^i} - eF_{ij}\frac{\partial H}{\partial p_j})\frac{\partial}{\partial p_i} \ . \tag{10}$$

The equations for the integral curves of X_H are:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = (1/m)g^{ij}p_j \tag{10-a}$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + eF_{ij}\frac{\partial H}{\partial p_j} = -(1/2m)g^{ab}_{,i}p_a p_b + eF_{ij}((1/m)g^{jk}p_k) \tag{10-b}$$

Solving (10-a) for p_j we find

$$p_j = mg_{jk}\dot{q}^k \tag{10-c}$$

Substituting this on both side of equation (10-b) we find

$$\frac{d}{dt}(mg_{ij}\dot{q}^j) = -(m/2)g^{ab}_{,i}g_{ak}g_{bl}\dot{q}^h\dot{q}^l + eF_{ij}\dot{q}^j \tag{10-d}$$

Using $\frac{d}{dt}g_{ij} = g_{ij,k}\dot{q}^k$ on the left hand side of this equation, and the eliminating the derivatives of g_{ij} on both sides using the definition of the Christoffel symbols Γ^i_{jk}, this equation can be put into the form

$$m(\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k) = eF^i{}_j\dot{q}^j \tag{10-e}$$

This is the relativistic form of the Lorentz force law.

2. Quantum Mechanics of a Particle

In this section we outline the basic assumptions and structure of the quantum mechanical theory of a particle, and in particular the assumptions underlying the **canonical quantization** approach to quantization.

The classical mechanics' picture of the dynamics of a particle is that of a trajectory in phase space. This trajectory is an integral curve of the Hamiltonian vector field X_H and it provides the unique coordinates and momenta $(q^i(t), p_j(t))$ of the particle for $t > 0$ given the initial values $(q^i(0), p_j(0))$. As we have seen in our discussions up to this point the underlying mathematical formalism for this classical deterministic picture is Hamiltonian dynamics on a symplectic manifold.

Heissenberg's uncertainty relation $\Delta q \Delta p \geq \hbar$ is one of the corner stones of the quantum mechanical picture of the dynamics of a particle that has evolved during the 20^{th} century. This principle asserts that if, for example, the position of a particle is known exactly so that $\Delta q = 0$, then the corresponding momentum coordinate in phase space is completely undetermined. The significant point here is that the uncertainty principle completely destroys the classical picture of a single trajectory in phase space, and implies that the mathematical formalism for describing states of a particle must be generalized. As is well-known the quantum theory replaces the "state = point in phase space" of classical physics with a "state = ray in a Hilbert space". More generally we have the following formal structure of the quantum mechanics of a particle.

Assumptions of the Quantum Mechanics of a Particle

A. To each physical system there corresponds a Hilbert space $\mathcal{H}$ with elements ψ and inner product $< \, , \, >$: $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $(\psi, \phi) \to < \psi, \phi >$. Furthermore,

A-1. Each **state** of the system is represented by a ray in the Hilbert space $\mathcal{H}$. Thus if c is any non-zero complex number then ψ and cψ represent the same state. It is often convenient to normalized state vectors to have length 1 so that $< \psi, \psi >= 1$.

A-2. The **dynamical observables** of the physical system correspond to Hermitian operators on the Hilbert space $\mathcal{H}$.

B. The result of a measurement of an observable A is one of the possible eigenvalues a of the corresponding operator. As a result of any measurement the physical system finds itself in the state represented by the corresponding eigenvector, which we denote by $|a> \in\mathcal{H}$. (Thus our notation is A$|a> =a|a>$. Moreover, we will write $< a|b >$ for the inner product $< |a>, |b>>$.)

C. If the system is in the state $|a>$ then the probability that a measurement of an observable B on the state $|a>$ yields the value b is

$$\mu(a,b) := |< a|b >|^2 \ . \tag{11}$$

Hence in every measurement the physical system makes a transition from an initial state $|a>$ to a final state $|b>$, and $\mu(a,b)$ is referred to as a **transition probability**.

D. If $\hat{A}$ and $\hat{B}$ are operators corresponding to classical dynamical observables A and B of the physical system, then $\hat{A}$ and $\hat{B}$ satisfy the **commutation relations**

$$[\hat{A},\hat{B}] = i\hbar\widehat{\{A,B\}}I \ , \tag{12}$$

where $\{A,B\}$ denotes the classical Poisson bracket of the corresponding dynamical observables and I denotes the identity operator.

REMARKS:

i. Assumption D above is at the heart of the **canonical quantization** scheme, and was first formulated by P.A.M. Dirac. One can ask if the prescription holds for all, or just some, dynamical observables. In Dirac's original formulation he assumed that the prescription holds for at least "the simpler" Poisson bracket relations. In fact the work of van Hove and Gröenewald show that in general the validity of the prescription is limited in scope. A general discussion of the results of van Hove and Gröenewald can be found in Abraham and Marsden, *Foundations of Mechanics* and in Guillemin and Sternber, *Symplectic Techniques in Physics.*

The basic mathematical idea underlying assumption D is that of a **representation** of the algebra of classical observables as linear Hermitian operators on a Hilbert space, and the geometric quantization theory attempts to make these ideas mathematically precise. Before proceeding we illustrate these ideas with the well-known example of quantization of the 1-dimensional harmonic oscillator.
The classical Hamiltonian of this physical system is

$$h = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \ , \tag{13}$$

where m is the mass of the particle and ω is the angular frequency of the oscillator. The classical Poisson brackets for the canonical coordinates (q,p) are

$$\begin{aligned} \{q,q\} &= 0 \ , \\ \{p,p\} &= 0 \ , \\ \{q,p\} &= 1 \ . \end{aligned} \tag{14}$$

Let us denote the operators corresponding to q and p by Q and P, respectively. Thus from (12) and (14) we have the **canonical commutation relations** (CCR)

$$\begin{aligned} [Q,Q] &= 0 \ , \\ [P,P] &= 0 \ , \\ [Q,P] &= i\hbar I \ . \end{aligned} \tag{15}$$

The fundamental dynamical observable in the classical theory is the total energy $E = h$, and in the quantum theory we obtain the corresponding operator H:

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2 \ . \tag{16}$$

The meaning of the operator Q^2 is of course

$$Q^2(\psi) = Q(Q(\psi)) \ , \tag{17}$$

with a similar interpretation for any product of linear operators.

We now assume there is a Hilbert space $\mathcal{H}$ on which the operators Q and P are Hermitian operators. The goal now is to construct a complete set of eigenvectors of the Hamiltonian operator H. In order to do this it is convenient to first introduce two additional operators a and $a^\dagger$, defined by

$$\begin{aligned} a &:= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}Q + \frac{i}{\sqrt{m\omega\hbar}}P\right) \ , \\ a^\dagger &:= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}Q - \frac{i}{\sqrt{m\omega\hbar}}P\right) \ . \end{aligned} \tag{18}$$

$a^\dagger$ is the Hermitian conjugate of a.

It now follows from (15)-(18) that

$$[a, a^\dagger] = I \ . \tag{19}$$

Using the above relations it is easy to show that

$$\begin{aligned} H &= \hbar\omega(a^\dagger a + \frac{1}{2}) \\ &= \hbar\omega(aa^\dagger - \frac{1}{2}) \\ &= \frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger) \ . \end{aligned} \tag{20}$$

We now assume that there is a Hilbert space $\mathcal{H}$, with inner product $< \ , \ >$, on which these operators act as linear operators. Note that the operators Q, P and H are Hermitian while a and $a^\dagger$ are not Hermitian. From the defining properties of the inner product and Hermitian conjugate we have, for any $\psi \in \mathcal{H}$,

$$< \psi, a^\dagger a\psi > = < a\psi, a\psi > \geq 0 \ . \tag{21}$$

Thus

$$< \psi, (\hbar\omega)a^\dagger a\psi > = \hbar\omega < a\psi, a\psi > \geq 0 \ . \tag{22}$$

We now suppose that $\psi_E \equiv |E>$ is an eigenvector of H with eigenvalue E so that

$$H\psi_E = E\psi_E \ . \tag{23}$$

Then using $\hbar\omega a^\dagger a = H - \frac{\hbar\omega}{2}I$ from (20) we have

$$\begin{aligned} < \psi_E, (\hbar\omega)a^\dagger a\psi_E > &= < \psi_E, (H - \frac{\hbar\omega}{2}I)\psi_E > \\ &= < \psi_E, H\psi_E > - \frac{\hbar\omega}{2} < \psi_E, \psi_E > \\ &= (E - \frac{\hbar\omega}{2}) < \psi_E, \psi_E > \ . \end{aligned} \tag{24}$$

We conclude from (23) and (24) that the energy eigenvalues are restricted by

$$E \geq \frac{\hbar\omega}{2} \ . \tag{25}$$

Now from (19) and (20) it follows that

$$\begin{aligned} [a, H] &= (\hbar\omega)a \ , \\ [a^\dagger, H] &= (-\hbar\omega)a^\dagger \ . \end{aligned} \tag{26}$$

Operating on both sides of equation (23) with the operator a we have

$$aH\psi_E = E(a\psi_E) \ . \tag{27}$$

Using the relation (26) to rewrite the left hand side we obtain

$$(Ha + (\hbar\omega)a)\psi_E = E(a\psi_E) \ .$$

Thus the Hilbert space vector $a\psi_E$ is also an eigenvector of the Hamiltonian operator H with eigenvalue $E - \hbar\omega$:

$$H(a\psi_E) = (E - \hbar\omega)a\psi_E \ . \tag{28}$$

A second application of the operator a to $a\psi_E$ leads to

$$H(a^2\psi_E) = (E - 2\hbar\omega)a^2\psi_E \ ,$$

and in general we have

$$H(a^n\psi_E) = (E - n\hbar\omega)a^n\psi_E \ . \tag{29}$$

Thus we may write

$$a^n\psi_E = \psi_{E-n\hbar\omega} \tag{30}$$

for these eigenvectors of H. But from the restriction (25) we must assume that there exists a lowest energy eigenfunction, say ψ_{E_0} that obeys the equation

$$a\psi_{E_0} = 0 \ . \tag{31}$$

Operating on the state $a\psi_{E_0}$ with $a^\dagger$ and using $a^\dagger a = \frac{1}{\hbar\omega}(H - \frac{\hbar\omega}{2}I)$ we obtain

$$\begin{aligned} 0 &= a^\dagger a\psi_{E_0} \\ &= \frac{1}{\hbar\omega}(H - \frac{\hbar\omega}{2}I)\psi_{E_0} \\ &= \frac{1}{\hbar\omega}(E_0 - \frac{\hbar\omega}{2})\psi_{E_0} \ . \end{aligned} \tag{32}$$

Hence the lowest eigenvalue E_0 is given by

$$E_0 = \frac{\hbar\omega}{2} \ , \tag{33}$$

and the energy spectrum of H is given by

$$E_n = (n + 1/2)\hbar\omega \qquad , \qquad n = 0, 1, 2, \ldots \ . \tag{34}$$