Covariant Field Theory

on

Frame Bundles of Fibered Manifolds†

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Abstract

We show that covariant field theory for sections of $\pi : E \to M$ lifts in a natural way to the bundle of vertically adapted linear frames $L_\pi E$. Our analysis is based on the fact that $L_\pi E$ is a principal fiber bundle over the bundle of 1-jets $J^1\pi$. On $L_\pi E$ the canonical soldering 1-forms play the role of the contact structure of $J^1\pi$. A lifted Lagrangian $\mathcal{L}:L_\pi E \to \mathbb{R}$ is used to construct modified soldering 1-forms, which we refer to as the Cartan-Hamilton-Poincaré 1-forms. These 1-forms on $L_\pi E$ pass to the quotient to define the standard Cartan-Hamilton-Poincaré $m$-form on $J^1\pi$. We derive generalized Hamilton-Jacobi and Hamilton equations on $L_\pi E$, and show that the Hamilton-Jacobi and canonical equations of Carathéodory-Rund and de Donder-Weyl are obtained as special cases. The manifold $L_\pi E$ emerges as a natural arena for a unified theory that contains, in addition to the sector for sections of $\pi$, dynamical sectors for a geometry for $M$ and a geometry for the fibers of $E$.

Keywords: symplectic geometry, $n$-symplectic geometry, multisymplectic geometry, frame bundle, Hamiltonian field theories, Poisson bracket, jet bundles, contact structure.

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1 Introduction

The Cartan-Hamilton-Poincaré (CHP) $m$-form is the central object in covariant Lagrangian field theory. The ingredients which go into the construction of this $m$-form are:

1. A Lagrangian $L : J^1\pi \to \mathbb{R}$, on the bundle of 1-jets of sections of $\pi : E \to M$, where $E$ is the configuration manifold of the theory,

2. A volume on the $m$-dimensional parameter space $M$,

3. The contact structure of $J^1\pi$.

It is the contact structure [17] in this mixture of ingredients that provides the geometrical foundation of the theory. In this paper we give a new geometrical formulation of the covariant field theory on $J^1\pi$ by lifting it to the bundle of vertically adapted linear frames $L_\pi E$ of $E$. We will show that the full depth of Lagrangian and Hamiltonian field theory on $J^1\pi$ has a useful geometrical representation on the bundle $L_\pi E$. In this representation the role of the contact structure of $J^1\pi$ is taken over by the canonical vector-valued soldering 1-form on $L_\pi E$. Introduction of a Lagrangian leads to the definition of a modified soldering form, and this vector-valued 1-form plays the role of the CHP-$m$-form. These structures pass to a certain quotient of $L_\pi E$ to give the standard structures on $J^1\pi$. The advantaged gained by this reformulation is that it allows us to utilize the natural geometry that is supported on $L_\pi E$, namely $n$-symplectic geometry, to further develop covariant field theory.

If $E$ is an arbitrary $n$-dimensional manifold, then the bundle of linear frames $\lambda : LE \to E$ supports a canonically defined $\mathbb{R}^n$-valued 1-form, the “soldering” 1-form. $n$-symplectic geometry on $LE$ is the generalized symplectic geometry that emerges upon taking the soldering 1-form $\theta$ as the generalized symplectic potential. This geometry, including the notions of $n$-symplectic observables, the corresponding generalized Hamiltonian vector-valued vector fields, and generalized Poisson and graded Poisson brackets, has been developed in a series of papers [5, 6, 12, 13, 14, 15]. A sketch of the basic structure of the theory can be found in section 2, but let us point out here that in [14] it was shown that the fundamentals of the canonical symplectic geometry on the cotangent bundle $T^*E$ can be constructed entirely in terms of the $n$-symplectic geometry on $LE$. 


When \( E \) has extra structure, in particular when \( \pi : E \to M \) is a fiber bundle as it is in Lagrangian field theory, then the \( n \)-symplectic geometry likewise inherits extra structure on \( LE \). In particular, the fiber structure of \( \pi : E \to M \) leads to a reduction of \( LE \) to the subbundle of vertically adapted linear frames \( L_\pi E \), with a corresponding reduction in the generality of \( n \)-symplectic observables. The structure group \( G_v \) of \( L_\pi E \) is the subgroup of \( GL(n) \) that is block lower triangular, corresponding to the convention that the last \( k = n - m \) vectors in each linear frame are required to be vertical. Following the model construction given in [14] Lawson showed [11, 6] that the multisymplectic geometry [7] on the affine cojet bundle \( J^1 \pi \) can also be derived directly from the \( n \)-symplectic geometry on \( L_\pi E \).

Turning our attention in this paper to the covariant field theory on \( J^1 \pi \), we will show that the geometrical foundations of the theory, namely the contact structure on \( J^1 \pi \), follows directly from the \( n \)-symplectic structure on \( L_\pi E \), while the CHP-form follows from a modified soldering form. The central idea on which the analysis is based is the following theorem.

**Theorem 1.1** Let \( \pi : E \to M \) be an \( m + k \) dimensional fiber bundle over the \( m \)-dimensional manifold \( M \). The vertically adapted frame bundle \( L_\pi E \) is a principal \( H = GL(m) \times GL(k) \) bundle over \( J^1 \pi \). In particular, \( J^1 \pi \cong L_\pi E/H \).

As a consequence of this theorem, which we prove in section 3, the canonical soldering forms on \( L_\pi E \) pass to the quotient to define the contact structure of \( J^1 \pi \) (see section 5).

Furthermore, this theorem leads to a decomposition of \( L_\pi E \). Once a Lagrangian is introduced, this decomposition will lead us to larger theory that is a type of Kaluza-Klein theory that includes a dynamical sector for a geometry of the parameter space ("spacetime") \( M \), a dynamical sector for a geometry of the fibers of \( E \), in addition to the original sector for the sections of \( E \). A simple picture of this development can be sketched out as follows.

Let \((x^i)\) be local coordinates on \( M \) and let \((y^A)\) be fiber coordinates on \( E \), so that \((z^\alpha) = (x^i, y^A)\) are adapted local coordinates on \( E \). With respect to such coordinates a general vertically adapted linear frame at a point in \( E \) will be of the form

\[
(e_i, e_A) = (v^j_i \frac{\partial}{\partial x^j} + v^B_i \frac{\partial}{\partial y^B}, v^A_B \frac{\partial}{\partial y^A})
\]

\(i, j = 1, \ldots, m, \ A, B = m + 1, \ldots, m + k\)

The first \( m \) vectors \((e_i)\) are non-vertical while the last \( k \) vectors \((e_A)\) are vertical with respect to \( \pi \). The matrices \((v^j_i)\) and \((v^B_A)\) are necessarily non-singular, while the matrix \((v^B_i) \in \mathbb{R}^{k \times m}\)
is arbitrary. Hence we may take the collection \((x^i, y^A, v^i_A, v^B_i, v^B_A)\) as local coordinates on \(L_\pi E\). We can represent an arbitrary adapted linear frame in terms of these local coordinates as the \((m + k) \times (m + k)\) matrix
\[
\begin{pmatrix}
v^j_i \\
v^B_i & v^B_A
\end{pmatrix}
\]
Using the notation \(\pi^i_j = (v^j_i)^{-1}\) and \(\pi^B_A = (v^B_A)^{-1}\), this matrix can be decomposed as follows:
\[
\begin{pmatrix}
v^j_i \\
v^B_i & v^B_A
\end{pmatrix} = \begin{pmatrix}
\delta^j_k \\
\pi^B_k v^B_a & \delta^B_C
\end{pmatrix} \begin{pmatrix}
v^k_i \\
0 & v^C_A
\end{pmatrix}
\]
(1.1)
The first factor is \(H\) invariant and defines a natural projection to \(J^1\pi\). We thus obtain the decomposition
\[
L_\pi E = J^1\pi \times_E (LVE \times_M LM)
\]
where \(LVE\) denotes the bundle of vertical frames of \(E\).

These results suggest that it may be useful to lift the covariant Lagrangian field theory on \(J^1\pi\) to \(L_\pi E\). In particular on \(L_\pi E\) we have available the \(n\)-symplectic geometry to use in studying the structure of field theories. We show in section 4 that for a lifted Lagrangian \(\mathcal{L} = \rho^\ast(L)\), the \(n\)-symplectic Hamiltonian vector fields defined by vertical vector fields on \(E\) may be thought of as variational vector fields. If \(X\) is such a vector field then \(X(\mathcal{L})\) gives the Euler-Lagrange operator to within a total divergence.

In section 6 we turn to the problem of constructing, on \(L_\pi E\), a lifted version of the CHP \(m\)-form. We show that in fact one can use a lifted Lagrangian to define an \(\mathbb{R}^n\)-valued CHP-form using the canonical \(\mathbb{R}^n\)-valued soldering form \(\theta\). The key to the construction is to use the fundamental vertical vector fields on \(L_\pi E\) together with the Lagrangian to give a global, invariant definition of the covariant momentum, which is essentially a frame bundle version of the Legendre transformation of classical theory. The result is that the \(\mathbb{R}^n\)-valued CHP-form is a modified, or non-canonical soldering form \(\theta_\mathcal{L}\). This new vector-valued CHP-form \(\theta_\mathcal{L}\) passes to the quotient to define the standard CHP-\(m\)-form on \(J^1\pi\).

As an application of the general formalism we derive in section 7 a generalized Hamilton-Jacobi differential equation and generalized Hamilton equations. Under appropriate assumptions these equations reproduce the Hamilton-Jacobi equations and Hamilton equations of the de Donder-Weyl [4, 16] and Carathéodory-Rund [2, 16] theories.
We recall that there is a certain degree of arbitrariness in Rund’s [16] canonical formalism for Carathéodory’s theory. We find that by identifying the canonical variables introduced here with the canonical variables in Rund’s formalism, the undetermined features of the Carathéodory-Rund theory can be given a natural interpretation on \(L_\pi E\), namely as the variables defining linear frames for \(M\). Looking again at the decomposition (1.1) we see now that the entries in the right-hand-factor represent a linear frame for \(M\) (the \((v^i_j)\) factor) together with a linear frame for the fibers of \(E\) (the \((v^B_A)\) factor). Thus by dropping the condition that the Lagrangian \(\mathcal{L} : L_\pi E \to \mathbb{R}\) be a lifted Lagrangian, one arrives at a theory where the solutions of the Euler-Lagrange field equations would determine not only a section of \(\pi\), but also a linear frame field for \(M\) together with a linear frame field for the fibers of \(E\). We present a model Lagrangian in section 9 that describes a Kaluza-Klein type theory, formulated in a natural way on \(L_\pi E\). Section 10 contains concluding remarks together with plans for applications and extensions of the results presented in this paper.

## 2 The Vertically Adapted Linear Frame Bundle \(L_\pi E\)

Let \(\pi : E \to M\) be a fiber bundle where \(M\) is \(m\)-dimensional and \(E\) is \(n = m + k\)-dimensional. Lower case latin indices are assumed to range over \(1 \ldots m\), upper case latin indices over \(m + 1 \ldots m + k\), and greek indices over \(1 \ldots m + k\). This convention will be used throughout the paper.

An adapted frame at \(e \in E\) is a frame where the last \(k\) basis vectors are vertical. Note that coordinate frames that come from adapted coordinates are adapted frames. The adapted frame bundle of \(\pi\), denoted \(L_\pi E\), consists of all adapted frames for \(E\).

\[
L_\pi E = \{ (e, \{e_i, e_A\}) : e \in E, \{e_i, e_A\} \text{ is a basis for } T_e E, \text{ and } d\pi(e_A) = 0 \}
\]

The canonical projection, \(\lambda : L_\pi E \to E\), is defined by \(\lambda(e, \{e_i, e_A\}) = e\).

\(L_\pi E\) is a reduced subbundle of \(LE\), the frame bundle of \(E\) (Lawson [11]). As such it is a principal fiber bundle over \(E\). Its structure group is \(G_v\), the nonsingular block lower triangular matrices.

\[
G_v = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in \text{GL}(m), B \in \text{GL}(k), C \in \mathbb{R}^{km} \right\}
\]
G acts on \( L_x E \) on the right by
\[
(e, \{e_i, e_A\}) \cdot \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \{(e, \{e_i A^i_j + e_A C^A_j, e_A B^A_B\})\}
\]

2.1 Coordinates

If \((x^i, y^A)\) are adapted coordinates on an open set \(U \subseteq E\), then one may induce several different coordinates on \(\lambda^{-1}(U)\). First consider the coframe or \(n\)-symplectic momentum coordinates \((x^i, y^A, \pi^i_j, \pi^A_j, \pi^A_B)\) on \(\lambda^{-1}(U)\) defined by
\[
x^i(e, \{e_i, e_A\}) = x^i(e) \quad \pi^i_j(e, \{e_i, e_A\}) = \epsilon^i_j(\frac{\partial}{\partial x^j}) \quad \pi^A_j(e, \{e_i, e_A\}) = \epsilon^A_j(\frac{\partial}{\partial y^j}) \\
y^A(e, \{e_i, e_A\}) = y^A(e) \quad \pi^A_j(e, \{e_i, e_A\}) = \epsilon^A_j(\frac{\partial}{\partial x^j})
\]

Here \((\epsilon^i, \epsilon^A)\) is the dual frame to \((e_i, e_A)\). We have as is customary retained the same symbols for the induced horizontal coordinates.

Secondly consider the frame or \(n\)-symplectic velocity coordinates \((x^i, y^A, v^i_j, v^A_j, v^A_B)\) on \(\lambda^{-1}(U)\) defined by
\[
x^i(e, \{e_i, e_A\}) = x^i(e) \quad v^i_j(e, \{e_i, e_A\}) = e_j(x^i) \quad v^A_j(e, \{e_i, e_A\}) = e_B(y^A) \\
y^A(e, \{e_i, e_A\}) = y^A(e) \quad v^A_j(e, \{e_i, e_A\}) = e_j(y^A)
\]

The \(v\) coordinates, viewed together as a block triangular matrix, form the inverse of the \(\pi\) coordinates above. The blocks have the following relations:
\[
v^i_j \pi^j_k = \delta^i_k \quad v^A_j \pi^j_k + v^A_B \pi^B_k = 0 \quad v^A_B \pi^B_C = \delta^A_C
\]

Lastly consider the following coordinates which are constructed from the previous two. Define \((x^i, y^A, u^i_j, u^A_j, u^A_B)\) on \(\lambda^{-1}(U)\) by
\[
x^i(e, \{e_i, e_A\}) = x^i(e) \quad u^i_j = \pi^i_j \quad u^A_j = v^A_i \pi^i_j = -v^A_B \pi^B_j \\
y^A(e, \{e_i, e_A\}) = y^A(e) \quad u^A_B = \pi^A_B
\]

It will turn out that the \(u^A_j\) coordinates are pull-ups of the standard jet coordinates on \(J^1 \pi\). As such, we will refer to these coordinates as Lagrangian coordinates.
Later in the paper we will need the following formulas for the fundamental vertical vector fields $E^*_\beta$ on $L_\pi E$ in Lagrangian coordinates.

\begin{align*}
E^*_j &= -u_k^i \frac{\partial}{\partial u^j_k} \\
E^*_B &= -u^A_C \frac{\partial}{\partial u^B_C} \\
E^*_A &= u^i_k v^B_A \frac{\partial}{\partial u^k_i} \tag{2.2}
\end{align*}

2.2 $n$-symplectic structure

$n$-symplectic geometry arises naturally on the frame bundle $LE$ of any $n$-dimensional manifold $E$. $LE$ supports a canonically defined $\mathbb{R}^n$-valued 1-form $\theta$, the soldering 1-form, and $n$-symplectic geometry is the geometry on $LE$ when one takes $d\theta$ as a vector-valued generalized symplectic form. We present here a sketch of the structure of the theory and refer the reader to the literature \cite{5, 6, 12, 13, 14, 15} for more details. See also the works of de León, Salgado et al. \cite{3} and Awane \cite{1}.

The intrinsic definition of the soldering 1-form $\theta$ parallels the definition of the canonical form on $T^*M$.

\begin{equation}
\theta_u(X) = e^\alpha (du \bar{\lambda}(X)) r_\alpha = \theta_u^\alpha(X) r_\alpha \tag{2.3}
\end{equation}

Here $u = (e, \{e_\alpha\}) \in LE$, $\bar{\lambda}: LE \to E$ is the canonical projection, and $\{r_\alpha\}$ is the standard basis for $\mathbb{R}^n$. In canonical coordinates,

\begin{equation}
\theta^\alpha = \pi^\alpha_\beta dx^\beta
\end{equation}

The above formula parallels the local coordinate formula $\vartheta = \rho_i dq^i$ for the canonical 1-form on $T^*M$.

Because the soldering 1-form $\theta$ is vector-valued, the natural structure equation for $n$-symplectic geometry takes the generalized form

\begin{equation}
d\hat{f}^{\alpha_1 \alpha_2 \cdots \alpha_p} - p! X_f^{\alpha_1 \alpha_2 \cdots \alpha_{p-1}} \bigwedge d\theta^{\alpha_p} \tag{2.4}
\end{equation}

Here $\hat{f} = (\hat{f}^{\alpha_1 \alpha_2 \cdots \alpha_p}) : LE \to \otimes^p \mathbb{R}^n$ is a vector-valued function on $LE$ and $X_f = (X_f^{\alpha_1 \alpha_2 \cdots \alpha_{p-1}})$ is the corresponding set of Hamiltonian vector fields. (Each superscript $\alpha_k$, $k = 1, 2, \ldots, p$, runs from 1 to $n$). Moreover, since the soldering form is equivariant under the free right action of the structure group $GL(n, \mathbb{R})$ on $LE$, the class of functions that can satisfy (2.4) is restricted. They divide naturally into vector-valued functions that map to either the symmetric
tensor spaces \((\otimes_s)^p \mathbb{R}^n\) or the anti-symmetric tensor spaces \((\otimes_a)^p \mathbb{R}^n\), where \(\otimes_s\) and \(\otimes_a\) denote the symmetric and anti-symmetric tensor products, respectively. There is a naturally defined Poisson bracket for both sets of observables, and the complete set of symmetric observables is a Poisson algebra with respect to the bracket, while the set of anti-symmetric observables is a graded Poisson algebra with respect to the bracket. These brackets, when restricted to the subsets of tensorial observables, are the frame bundle versions of the Schouten-Nijenhuis brackets [15]. On \(L_\pi E\) the allowable tensorial observables [11] correspond to contravariant tensor fields that are projectible to \(E\).

As a reduced subbundle of \(LE\), \(L_\pi E\) has the \(n\)-symplectic geometry obtained by restricting the soldering form. Since this soldering form is \(\mathbb{R}^{m+k}\)-valued, we will denote it \((\theta^i, \theta^A)\). Let \(u = (e, \{e_i, e_A\})\) be a point in \(L_\pi E\). If \(\lambda : L_\pi E \to E\) is the canonical projection and \(X \in T_uL_\pi E\), then \(\theta\) defined as in (2.3) above splits naturally into the two terms

\[
\theta_u(X) = \theta^i(X)r_i + \theta^A(X)r_A
\]

where \((e^i, e^A)\) is the dual frame and \((r_i, r_A)\) is the standard basis for \(\mathbb{R}^{m+k}\). In local momentum coordinates,

\[
\theta^i = \pi^i_j dx^j \quad \theta^A = \pi^A_j dx^j + \pi^A_B dy^B
\]

3 The Relationship between \(L_\pi E\) and \(J^1\pi\)

We will demonstrate three useful facts relating \(L_\pi E\) and \(J^1\pi\).

1. \(J^1\pi\) is an associated bundle to \(L_\pi E\) [11].

2. \(L_\pi E\) is a principal fiber bundle over \(J^1\pi\).

3. \(L_\pi E\) is a pull-back bundle over \(J^1\pi\) [11].

3.1 A special case

Consider the case where \(\pi\) is a trivial bundle. Let \(M = \mathbb{R}^m\) and \(E = \mathbb{R}^m \times F\) with \(F\) a \(k\)-dimensional manifold. Let \(\pi : \mathbb{R}^m \times F \to \mathbb{R}^m\) be the standard projection. It is known that
for this bundle each 1-jet corresponds to an \( m \)-tuple of tangent vectors to \( F \).

\[
J^1 \pi \cong \mathbb{R}^m \times (TF \oplus \cdots \oplus TF)
\]

It is clear that such a bundle is associated to \( L_\pi E \).

Let us examine \( L_\pi E \) in this case. We will make use of the other projection mapping \( \bar{\pi} : \mathbb{R}^m \times F \to F \). For each frame \( (u, \{e_i, e_A\}) \) in \( L_\pi E \), we decompose each vector into

\[
e_i = (v_i, w_i) \quad e_A = (v_A, w_A)
\]

where \( v_i = d_u \pi(e_i), \ w_i = d_u \bar{\pi}(e_i), \ v_A = d_u \pi(e_A), \) and \( w_A = d_u \bar{\pi}(e_A) \). Note that \( v_A = 0 \) by the definition of \( L_\pi E \), so we have

\[
e_i = (v_i, w_i) \quad e_A = (0, w_A)
\]

The \( k \) vectors \( \{w_A\} \) form a basis for \( T_{\bar{\pi}(u)} F \), and the \( m \) vectors \( \{v_i\} \) form a basis for \( T_{\pi(u)} \mathbb{R}^m \). The \( m \) vectors \( \{w_i\} \) are simply an \( m \)-tuple of vectors in \( T_{\bar{\pi}(u)} F \).

Decomposing all of \( L_\pi E \) in this way, we obtain

\[
L_\pi E \cong J^1 \pi \times_E (L \mathbb{R}^m \times LF)
\]

This is a bundle isomorphism over \( E = \mathbb{R}^m \times F \). From this decomposition, it is clear that \( L_\pi E \) is a pull-back bundle over \( J^1 \pi \). Furthermore, the fiber is the Lie group \( \text{GL}(m) \times \text{GL}(k) \).

### 3.2 The general case

Consider an arbitrary fiber bundle \( \pi : E \to M \). In this more general setting, a 1-jet is no longer simply an \( m \)-tuple of tangent vectors. There are three major ways of describing 1-jets, each with its own charm:

1. Equivalence classes of sections of \( \pi \).
2. Linear right-inverses to \( d_u \pi \).
3. Non-vertical \( m \)-dimensional subspaces of \( T_u E \).
One quick way to define the projection from $L_\pi E$ to $J^1\pi$ is to map each adapted frame to the span of its non-vertical elements.

$$(u, \{e_i, e_A\}) \mapsto (u, \text{span}\{e_i\})$$

However, we will benefit from starting with $J^1\pi$ as an associated bundle.

As stated earlier, the structure group of $L_\pi E$ is $G_v$, the nonsingular block lower triangular matrices. This group $G_v$ can be decomposed [11] into the product of two of its subgroups, $H$ and $J$, where

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \text{GL}(m), B \in \text{GL}(k) \right\}$$

and

$$J = \left\{ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} : C \in \mathbb{R}^{km} \right\}$$

Note that $J$ is Lie group isomorphic to the additive group $\mathbb{R}^{km}$.

We will show that $J^1\pi$ is a bundle associated to $L_\pi E$ with fiber $G_v/H$. Although $H$ is a closed Lie subgroup of $G_v$ it is not normal. As such $G_v/H$ does not have a natural group structure; it is manifold with a left $G_v$-action. For each coset $gH \in G_v/H$, we select the unique representative in $J$.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$$

By choosing these representatives, we identify $G_v/H$ with $J$ and hence $\mathbb{R}^{km}$. These identifications are diffeomorphisms.

Consider how the left $G_v$-action looks for our selected representatives.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C + B\xi & B \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ CA^{-1} + B\xi A^{-1} & I \end{pmatrix}$$

So the $G_v$-action appears affine when $G_v/H$ is identified with $\mathbb{R}^{km}$. Therefore it is prudent to use this identification to define an affine structure on $G_v/H$ modelled on $\mathbb{R}^{km}$. This $G_v$-invariant structure will pass to the fibers of the associated bundle, making it an affine bundle.

**Theorem 3.1**  
$L_\pi E \times_{G_v} (G_v/H) \cong J^1\pi$
Proof: The isomorphism maps each equivalence class \([e, \{e_i, e_A\}, \xi]\) to the linear map \(\phi : T_{\pi(e)}M \rightarrow T_eE\) defined by \(\phi(\hat{e}_i) = e_i + \xi^A e_A\), where we use the basis \(\hat{e}_i = d_{\pi}(e_i)\).

**Corollary 3.2** 
\(L_\pi E\) is a principal fiber bundle over \(J^1 \pi\) with fiber \(H\).

Proof: This fact follows directly from proposition 5.5 in reference [10].

We will denote the projection from \(L_\pi E\) to \(J^1 \pi\) by \(\rho\). It is given by
\[
\rho(e, \{e_i, e_A\}) = (e, \tau) \quad \text{where} \quad \tau(\hat{e}_i) = e_i
\]

We now show that the \(u_j^A\)-coordinates defined earlier are the pull-ups of the jet coordinates. If \((x^i, y^A)\) are adapted coordinates on an open set \(U \subseteq E\) and \(u = (e, \{e_i, e_A\}) \in \lambda^{-1}(U)\) then
\[
y_i^A \circ \rho(u) = y_i^A(e, \tau) = d_e y_i^A \circ \tau(\frac{\partial}{\partial x^i} \bigg|_{\pi(e)}) = d_e y_i^A(e_j \hat{e}_j \bigg|_{\pi(e)}) = d_e y_i^A(e_j \hat{e}_j \bigg|_{\pi(e)}) = \frac{\partial}{\partial x^i} y_i^A(u) = u_j^A(u)
\]

What remains to be shown is that \(L_\pi E\) is a pull-back bundle over \(J^1 \pi\). To see this, we will decompose each adapted frame in a manner similar to the trivial case covered earlier.

We can split each adapted frame \((u, e_i, e_A)\) into three pieces:

1. A point in \(LM\), \((\pi(u), \tilde{e}_i)\), where \(\tilde{e}_i = d_{\pi}(e_i)\)

2. A point in \(LVE\), \((u, e_A)\), where \(LVE\) is the bundle of vertical frames over \(E\)

3. A point in \(J^1 \pi\), \((u, \phi)\), where \(\phi : T_{\pi(u)}M \rightarrow T_uE\) is defined by \(\phi(\tilde{e}_i) = e_i\)

**Theorem 3.3** 
\(L_\pi E \cong J^1 \pi \times_E (LVE \times_M LM)\)

Proof: The isomorphism is given by \((u, e_i, e_A) \mapsto ((u, \phi), (u, e_A), (\pi(u), \tilde{e}_i))\). The inverse map is quite nice: \(((u, \phi), (u, f_A), (p, f_i)) \mapsto (u, \phi(f_i), f_A)\)
4 Prolongations of Vector Fields to $L_\pi E$

Definition 4.1 A Lagrangian on $L_\pi E$ is a function $\mathcal{L} : L_\pi E \rightarrow \mathbb{R}$. A Lagrangian on $L_\pi E$ is lifted if it satisfies the auxiliary conditions

$$E^*_j(L) = 0 \quad E^*_B(L) = 0$$

Remark Using (2.2) one can show that these conditions imply that $\mathcal{L}$ is constant on the fibers of $\rho : L_\pi E \rightarrow J^1\pi$, and thus is the pull up of a function on $J^1\pi$. We will assume that our Lagrangians are lifted until section 9 where we will drop this assumption in order to study the extra $\text{GL}(m) \times \text{GL}(k)$ degrees of freedom in this bundle structure.

In order to see the role played by the canonical $n$-symplectic structure on $L_\pi E$ in Lagrangian field theory, we consider a variation of a local section $\phi : M \rightarrow E$. The variation of $\phi$ can be defined by a vector field $f$ on $E$ that projects to the zero vector field on $M$, so that in adapted local coordinates $f$ has the form $f = f^A \partial_A$. The associated tensorial function $\hat{f} : L_\pi E \rightarrow \mathbb{R}^{m+k}$ is given in local coordinates on $L_\pi E$ by $\hat{f} = \hat{f}^\alpha \hat{\gamma}_\alpha$, where

$$(\hat{f}^\alpha) = (\hat{f}^i, \hat{f}^A) = (0, f^B \pi_B^A)$$

The $n$-symplectic Hamiltonian vector field $X_{\hat{f}}$ determined by $\hat{f}$ is the unique solution of equation (2.4) with $p = 1$. Thus $X_{\hat{f}}$ is defined by

$$d\hat{f}^\alpha = -X_{\hat{f}} \downarrow d\theta^\alpha$$

and in local coordinates it has the form [12]

$$X_j = f^A \partial_A - \frac{\partial f^A}{\partial x^j} \pi_A^B \frac{\partial}{\partial \pi_j^B} - \frac{\partial f^A}{\partial y^C} \pi_A^B \frac{\partial}{\partial \pi_C^B}$$

Transforming to Lagrangian coordinates we find

$$X_j = f^A \partial_A + \left( \frac{\partial f^A}{\partial x^j} + u^B_j \frac{\partial f^A}{\partial y^B} \frac{\partial}{\partial u_j^A} \right) - \left( \frac{\partial f^A}{\partial y^C} u^B_A \frac{\partial}{\partial u_C} \right)$$

$$= \left( f^A \partial_A + \frac{df^A}{dx^j} \frac{\partial}{\partial u_j^A} \right) - \left( \frac{\partial f^A}{\partial y^C} u^B_A \frac{\partial}{\partial u_C} \right) \quad (4.5)$$
Lemma 4.2 Let $f$ be a vertical vector field on $E$. The projection of the associated Hamiltonian vector field $X_f$ on $L_\pi E$ to $J^1\pi$ is the prolongation $j(f)$ of $f$ to $J^1\pi$.

Proof The vector fields $\frac{\partial}{\partial u^i}$ are vertical with respect to $\rho$, and $\rho_*(\frac{\partial}{\partial u^i}) = \frac{\partial}{\partial y^i}$. ■

This lemma shows that the Hamiltonian vector field $X_f$ on $L_\pi E$ is a lift of the prolongation of $f$ to $J^1\pi$. That $X_f$ actually has the properties of the prolongation of $f$ with respect to Lagrangians follows from the following lemma. We let

$$E_A(\cdot) = \frac{\partial(\cdot)}{\partial y^A} - \frac{d}{dx^i} \left( \frac{\partial(\cdot)}{\partial u^A_i} \right)$$

denote the Euler-Lagrange operator in local coordinates on $L_\pi E$.

Lemma 4.3 If $X_f$ is the $n$-symplectic Hamiltonian vector field on $L_\pi E$ of a vertical vector field $f$ on $E$, and if $\mathcal{L}$ is a lifted Lagrangian on $L_\pi E$, then

$$X_f(\mathcal{L}) = f^A E_A(\mathcal{L}) + \frac{d}{dx^i} (f^A p_A^i)$$

(4.6)

Proof The proof is a straightforward calculation using (4.5). ■

After introducing the CHP 1-forms in the section 6 we will use (4.6) to lift the variational principle to $L_\pi E$.

Remark As mentioned in section 2.2 there are other observables in n-symplectic geometry on $L_\pi E$ beyond those corresponding to vertical vector fields on $E$. In particular there is the Poisson algebra of all vertical symmetric contravariant tensor fields on $E$, as well as the graded Poisson algebra of all vertical antisymmetric contravariant tensor fields on $E$. The associated (equivalence classes of) vector-valued Hamiltonian vector fields on $L_\pi E$ also project to tensor fields on $J^1\pi$. Since these vector-valued Hamiltonian vector fields generalize the natural lift of a vector field from $E$ to $L_\pi E$, their projections to $J^1\pi$ can be taken as the prolongation of the tensor fields on $E$ to $J^1\pi$. These ideas will be elaborated in more detail elsewhere.
5 The Contact Structure

The contact structure on $J^1\pi$ amounts to a natural splitting of the tangent and cotangent spaces to $E$. For every $(e, \tau) \in J^1\pi$ there is a natural splitting of $T_e E$ and $T^*_e E$ into horizontal and vertical subspaces. This is usually encoded via the linear projections onto the vertical and horizontal. Saunders [17] envisions the contact structure as linear endomorphisms of the pullback vector bundles $J^1\pi \times_E (TE)$ and $J^1\pi \times_E (T^*E)$. These maps can be defined invariantly as follows. For $(e, \tau) \in J^1\pi$, $X \in T_e E$, and $\omega \in T^*_e E$.

\begin{align*}
h(X) &= \tau \circ d_e \pi(X) \\
v(X) &= X - h(X) \\
h^t(\omega) &= \omega \circ \tau \circ d_e \pi \\
v^t(\omega) &= \omega - h^t(\omega)
\end{align*}

Guillemin and Sternberg [8] prefer to think of the contact structure as $TE$-valued 1-forms on $J^1\pi$. To achieve this, they compose the $h$ and $v$ above with $d_{(e,\tau)} \pi_{1,0}$, where $\pi_{1,0} : J^1\pi \to E$.

In local coordinates, the contact structure looks like a pair of (1,1) tensor fields on $E$, except that they depend on jet coordinates.

\begin{align*}
h &= dx^k \otimes \left( \frac{\partial}{\partial x^k} + y^A \frac{\partial}{\partial y^A} \right) \\
v &= (dy^B - y^j dx^j) \otimes \frac{\partial}{\partial y^B}
\end{align*}

Depending on interpretation, the expressions above can be the horizontal and vertical projections for either $J^1\pi \times_E (TE)$ or $J^1\pi \times_E (T^*E)$. They can also be interpreted as $TE$-valued 1-forms on $J^1\pi$.

5.1 The Contact Structure viewed from $L_{\pi}E$

The contact structure arises on $J^1\pi$ because each 1-jet $(e, \tau)$ allows us to decompose $T_e E$ into a direct sum. Similarly, the soldering form arises on $L_{\pi}E$ because each adapted frame $u = (e, e_i, e_A)$ allows us to represent $T_e E$ as $\mathbb{R}^{m+k}$. So the contact structure is analogous to the soldering form. Recall that

\begin{align*}
\theta_u(X) &= e^i(d_u \lambda(X))r_i + e^A(d_u \lambda(X))r_A = \theta^i_u(X)r_i + \theta^A_u(X)r_A
\end{align*}

and that in local coordinates,

\begin{align*}
\theta^i &= \pi^i_j dx^j \\
\theta^A &= \pi^A_j dx^j + \pi^A_B dy^B
\end{align*}
Consider the following $TE$-valued one-forms on $L_\pi E$

$$\theta_h(u) = \theta^i(u) \otimes e_i, \quad \theta_v(u) = \theta^A(u) \otimes e_A$$

In local coordinates,

$$\theta_h = \pi^i_k dx^k \otimes v^l_i (\frac{\partial}{\partial x^l} + u^A_i \frac{\partial}{\partial y^A}) = dx^k \otimes \left( \frac{\partial}{\partial x^k} + u^A_k \frac{\partial}{\partial y^A} \right)$$

$$\theta_v = \pi^A_B (dy^B - u^B_j dx^j) \otimes v^C_A \frac{\partial}{\partial y^C} = (dy^B - u^B_j dx^j) \otimes \frac{\partial}{\partial y^B}$$

These objects are strikingly similar to the contact structure of $J^1\pi$. In fact, they pass to the quotient to give the contact structure on $J^1\pi$. The contact structure is known to appear in “various guises” [17]; the soldering form on $L_\pi E$ is another, perhaps more potent, version.

We also remark that the contact structure falls trivially from the following theorem

**Theorem 5.1** Let $\lambda : P \to E$ be a principal fiber bundle with structure group $G$, let $H \subseteq G$ be a closed lie subgroup, and let $F$ be a manifold with a left $G$-action. Then

$$P \times_H F \cong (P/H) \times_E (P \times_G F)$$

**Proof:** First note that by Proposition 5.5 in reference [10], $P/H \cong P \times_G (G/H)$ and $\rho : P \to P/H$ is a principal bundle. So $P \times_H F$ is a bundle associated to $\rho$. This makes sense–if $F$ has a left $G$-action then it has a left $H$-action. The isomorphism map is

$$(p, f)H \mapsto (pH, (p, f)G)$$

It is well-defined and a smooth diffeomorphism. ■

**Corollary 5.2**

$$L_\pi E \times_H \mathbb{R}^{m+k} \cong J^1\pi \times_E TE$$

$$L_\pi E \times_H (\mathbb{R}^{m+k})^* \cong J^1\pi \times_E T^*E$$

The natural splitting of the fibers $\mathbb{R}^{m+k}$ and $(\mathbb{R}^{m+k})^*$ is $H$-invariant and passes to the quotient to form the contact structure.
6 The Cartan-Hamilton-Poincaré Forms

One associates [8, 7] with a given Lagrangian $L$ on $J^1\pi$ the Cartan-Hamilton-Poincaré (CHP)-$m$-form $\theta_L$, which one may use to reexpress the action integral of the Lagrangian. This form can be defined directly [8] on $J^1\pi$, or it can be defined [7] on $J^1\pi$ as the pull back of the canonical multisymplectic form on $J^1\ast\pi$, the affine dual of $J^1\pi$. Although the CHP-form on $L\pi E$ can be defined in terms of the $n$-tangent structure on $L\pi E$, we will define this form directly in terms of invariant quantities on $L\pi E$. We will first define CHP-1-forms, from which the CHP-$m$-form will be constructed.

The fundamental vertical vector fields $E_A^i$ are given in Lagrangian coordinates in (2.2). If $\mathcal{L}: L\pi E \to \mathbb{R}$ is a lifted Lagrangian on $L\pi E$, then it is the pull-up under $\rho$ of a Lagrangian $L$ on $J^1\pi$. Hence, since $\rho_*(\frac{\partial}{\partial u^i}) = \frac{\partial}{\partial y^i}$, we have

$$E_A^i(\mathcal{L}) = u^i_k v^B_A \frac{\partial \mathcal{L}}{\partial u^B_k} = u^i_k v^B_A \frac{\partial L}{\partial y^B_k}$$

This leads us to the following definition:

**Definition 6.1** Let $\mathcal{L}: L\pi E \to \mathbb{R}$ be a Lagrangian on $L\pi E$. The covariant momenta of $\mathcal{L}$ are

$$\mathcal{P}_A^i = E_A^i(\mathcal{L}) = (u^i_k v^B_A) p^k_B$$

where $p^k_B = \frac{\partial \mathcal{L}}{\partial u^k}$ denotes the canonical momenta of the Lagrangian.

**Remark** Notice that the covariant momenta ($\mathcal{P}_A^i$) are globally defined tensorial objects on $L\pi E$, while the canonical momenta ($p^k_B$) are only defined locally.

We are now in a position to give a global definition of the CHP-form on $L\pi E$. We first define the related 1-forms.

**Definition 6.2** Let $\mathcal{L}: L\pi E \to \mathbb{R}$ be a lifted Lagrangian on $L\pi E$, and $\tau(m)$ a positive function of $m$, the dimension of $M$. The CHP-1-forms $\theta^i_L$ on $L\pi E$ are

$$\theta^i_L = \tau(m) \mathcal{L} \theta^i + E_A^i(\mathcal{L}) \theta^A$$

$$\theta^A_L = \theta^A$$

(6.7) 

(6.8)
Remark The positive function $\tau(m)$ in this definition is included to allow for various theories to occur as special cases. We will see that the choice $\tau(m) = 1$ yields the canonical theory of Carathéodory-Rund, and $\tau(m) = \frac{1}{m}$ yields the canonical theory of de Donder-Weyl.

Remark The collection of forms $(\theta^a_L) = (\theta^i_L, \theta^A_L)$, where $\theta^A_L = \theta^A$, is a modified, or non-canonical soldering form if $L > 0$. This follows from the easily verifiable properties $X \cdot \theta^a_L = 0$ for all $\alpha = 1, 2, \ldots, n$ if and only if $X$ is vertical with respect to $\lambda: L \pi E \to E$ and $R^*_g \theta_L = g^{-1} \cdot \theta_L$.

Working out the local coordinate form of the CHP-1-forms in Lagrangian coordinates we find

\[
\theta^i_L = -H^i_j dx^j + P^A_B dy^B \quad (6.9)
\]
\[
\theta^A_L = P^A_j dx^j + P^A_B dy^B \quad (6.10)
\]

where

\[
H^i_j = u^i_k (p^k_B u^B_j - \tau(m)L \delta^k_j) \quad (6.11)
\]
\[
P^i_B = u^i_k p^k_B \quad (6.12)
\]
\[
P^A_j = -u^A_B u^B_j \quad (6.13)
\]
\[
P^A_B = u^A_B \quad (6.14)
\]

We will refer to the $H^i_j$ as the components of the covariant Hamiltonian, and to the $P^i_B$ as the components of the covariant canonical momentum. If we define symbols $h^k_j$ by the formula

\[
h^k_j = p^k_B u^B_j - \tau(m)L \delta^k_j \quad (6.15)
\]

then the covariant Hamiltonian (6.11) can be expressed as $H^i_j = u^i_k h^k_j$. Setting $\tau(m) = 1$ we find that $h^i_j$ has the form of Carathéodory’s Hamiltonian [2, 16] tensor. Similarly, setting $\tau = \frac{1}{m}$ we find that $h = h^i_i$ yields the Hamiltonian in the de Donder-Weyl theory [4, 16].

Finally we show how the CHP-1-forms can be used to construct the CHP-m-form on $J^1 \pi$. 

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Proposition 6.3 Let \((B_i, B_A)\) denote the standard horizontal vector fields of any torsion-free linear connection on \(\lambda : L_\pi E \to E\), and let \(\text{vol}\) denote the pull up to \(L_\pi E\) of a fixed volume \(m\)-form on \(M\). Set \(\text{vol}_i = B_i \triangledown \text{vol}\). Then when \(\tau(m) = \frac{1}{m}\) the \(m\)-form
\[
\theta_L := \theta^i \wedge \text{vol}_i
\]
passes to the quotient to define the CHP-\(m\)-form \(\Theta_L\) on \(J^1\pi\).

Proof The vector fields \(B_i\) have the local coordinate form \(B_i = v^a_i \frac{\partial}{\partial z^a} + V\) where \(V\) is vertical with respect to \(\lambda : L_\pi E \to E\). Using this and formulas (6.7) through (6.12) one can show that
\[
\theta_L = -(p^j B_i u^B_j - \mathcal{L}) \text{vol} + p^j dy^A \wedge \left( \frac{\partial}{\partial x^j} \triangledown \text{vol} \right)
\]
The right-hand-side is constant on the fibers of \(\rho : L_\pi E \to J^1\pi\) and is in fact the pull-up \(\rho^*(\Theta_L)\) of the CHP-\(m\)-form \(\Theta_L\) on \(J^1\pi\).

Remark The above geometrical construction of the CHP-\(m\)-form is analogous to the geometrical construction given by Guillemin and Sternberg [8].

6.1 The Variational Principle on \(L_\pi E\)

We now lift the variational principle from \(J^1\pi\) to \(L_\pi E\). This is a simple procedure since we are using a lifted Lagrangian and only varying a section of \(\pi\). Let \(\phi : M \to E\) be a section of \(\pi\) and \(j\phi\) its 1-jet prolongation to \(J^1\pi\). For any section \(\xi : J^1\pi \to L_\pi E\) we have that
\(u = \xi \circ j\phi : M \to L_\pi E\) is a section of \(\pi \circ \lambda : L_\pi E \to M\).

The action integral on \(J^1\pi\) lifts nicely since \(\mathcal{L} = L \circ \rho\).
\[
\int_M j\phi^*(L)\text{vol} = \int_M u^*(\mathcal{L})\text{vol}
\]
We recall [7] that the action integral is extremized by \(\phi\) iff \(j\phi^*(W \triangledown d\Theta_L) = 0\) for all vector fields \(W\) on \(J^1\pi\). However, this condition can be weakened to \(j\phi^*(j(f) \triangledown d\Theta_L) = 0\) for all vertical vector fields \(f\) on \(E\).

Now for any such vertical vector field \(f\), consider its \(J^1\pi\) prolongation \(j(f)\) and its \(n\)-symplectic Hamiltonian vector field \(X_f\) on \(L_\pi E\) (also a kind of prolongation). From (4.2)
we know $\rho_*(X_j) = j(f)$. From proposition (6.3) we now have $X_j \perp d\theta_\xi = X_j \perp \rho^*(d\Theta_L)$. It follows that

$$u^*(X_j \perp d\theta_\xi) = j\phi^*(j(f) \perp d\Theta_L)$$

We conclude that the action integral is extremized by $\phi$ iff $u^*(X_j \perp d\theta_\xi) = 0$ for all vertical vector fields $f$ on $E$.

7 The Generalized Hamilton-Jacobi Equation

As an application of our general formalism we derive the Carathéodory-Rund and de Donder-Weyl Hamilton-Jacobi equations. By analogy with the time independent Hamilton-Jacobi theory (see, for example, reference [18]) we seek Lagrangian submanifolds of $L_\pi E$. However, since the dimension of $L_\pi E$ is in general not twice the dimension of $E$, a new definition is needed. For our purposes here we will consider $n = m + k$ dimensional submanifolds of $L_\pi E$ that arise as sections of $\lambda$. In particular we consider sections $\sigma : E \to L_\pi E$ that satisfy

$$\sigma^*(d\theta^\alpha_L) = 0 \quad (7.16)$$

We will refer to this equation as the generalized Hamilton-Jacobi equation.

Since $\sigma^*(d\theta^\alpha_L) = d(\sigma^*(\theta^\alpha_L))$ the condition (7.16) asserts that the 1-forms $\sigma^*(\theta^\alpha_L)$ are locally exact, and we express this as

$$\sigma^*(\theta^\alpha_L) = dS^\alpha \quad (7.17)$$

in terms of $m + k$ new functions $S^\alpha$ defined on open subsets of $E$. For convenience we will denote objects on $L_\pi E$ pulled back to $E$ using $\sigma$ with an over-tilde. Thus, for example, $\tilde{H}^i_j = H^i_j \circ \sigma$ and $\tilde{P}^A_i = P^A_i \circ \sigma$. Then we get from (6.11)–(6.14) and (7.17)

(a) $\tilde{H}^i_j = -\frac{\partial S^i}{\partial x^j}$,  \quad (b) $\tilde{P}^i_A = \frac{\partial S^i}{\partial y^A}$ \quad (7.18)

(a) $\tilde{u}^A_B \tilde{u}^B_j = -\frac{\partial S^A}{\partial x^j}$,  \quad (b) $\tilde{u}^A_B = \frac{\partial S^A}{\partial y^B}$ \quad (7.19)

Recalling that $H^i_j = P^i_B u^A_j - \tau(m)\mathcal{L}u^i_j$ and $P^i_A$ are functions of the coordinates $x^i, y^A, u^i_j$ and $u^A_i$, equations (7.18) can be combined into the single equation

$$H^i_j(x^a, y^B, u^a_i, u^B_i, \frac{\partial S^i}{\partial y^B}) \circ \sigma = -\frac{\partial S^i}{\partial x^j} \quad (7.20)$$
Similarly combining equations (7.19) we obtain
\[ \frac{dS^A}{dx^j} = 0 \]

We next consider special cases of these \textbf{generalized Hamilton-Jacobi equations.}

\section{7.1 The Theory of Carathéodory and Rund}

We note from (6.11), (6.12), and (6.15) that \( H^i_j = u^i_k h^k_j \) and \( P^i_A = u^i_k p^k_A \), where the matrix of functions \( (u^i_j) \) is GL(m)-valued. Using the notation \( P^i_j = -H^i_j \) and \( \breve{u}^i_j = u^i_j \circ \sigma \) we may rewrite (6.11) and (6.12) in the form
\[ \breve{P}^i_j = -\breve{u}^i_k h^k_j , \quad \breve{P}^i_A = \breve{u}^i_k p^k_A \] (7.21)

If we take \( t(m) = 1 \) then these equations are the equations defining the \textit{canonical momenta} in Rund’s canonical formalism for Carathéodory’s geodesic field theory (see equations (1.22), page 389 in [16], with the obvious change in notation). In this situation equation (7.20) can be identified with the Rund’s Hamilton-Jacobi equation for Carathéodory’s theory (see equation (3.29) on page 240 in [16]). We recall [16] that one can derive the Euler-Lagrange field equations from this Hamilton-Jacobi equation.

In (7.21) we have the result that the arbitrary non-singular matrix-valued functions \( (\breve{u}^i_j) \) that occur in Rund’s canonical formalism for Carathéodory’s theory have a geometrical interpretation in the present setting. Specifically they correspond to the coordinates for linear frames for \( M \). These defining relations are derived from Rund’s \textit{transversality condition}, and this condition has the elegant reformulation here as the kernel of \( (\theta_L^i) \).

We will say that a vector \( X \) at \( e \in E \) is transverse to a solution surface through \( e \) that is defined by a given Lagrangian \( \mathcal{L} \), if \( X = d\lambda(\hat{X}) \), where \( \hat{X} \in T_u(L \pi E) \) satisfies \( \hat{X} \perp \theta_L^i = 0 \), for some \( u \in \lambda^{-1}(e) \). \( \hat{X} \) thus satisfies the equations
\[ 0 = -H^i_j X^j + P^i_A X^A = u^i_k \left( -h^k_j X^j + p^k_A X^A \right) \]
\[ X^j = \hat{X}(x^i) , \quad X^A = \hat{X}(y^A) \]
from which we infer
\[ 0 = -h^k_j X^j + p^k_A X^A \] (7.22)
This is Rund’s transversality condition for the theory of Carathéodory when we take \( t(m) = 1 \) (see equation (1.10), page 388 in [16]). The canonical momenta \( P_j^i \) and \( P_A^i \) are defined by Rund to be solutions of

\[
0 = P_j^i X^j + P_A^i X^A
\]  

(7.23)

when \((X^j, X^A)\) satisfy (7.22). Rund’s solutions of these equations are given in (7.21). Looking at (7.21), (7.22) and (7.23) we see that the introduction of the \( u_j^i \) in (7.21) amounts to the introduction of the \( \text{GL}(m) \) freedom for linear frames for \( M \).

### 7.2 de Donder-Weyl Theory

Returning to (7.20) let us reduce this equation by making several assumptions. We suppose that \( \mathcal{L} \) is regular (in the usual sense on \( J^1 \pi \)), that the section \( \sigma \) is such that \( \tilde{u}_j^i = \delta_j^i \), and we make the choice \( t(m) = \frac{1}{m} \). Now summing \( i = j \) in (7.20) we obtain

\[
\tilde{h}(x^i, y^B, \frac{\partial S^i}{\partial y^B}) = -\frac{\partial S^i}{\partial x^i}
\]

where \( \tilde{h} = \tilde{p}_A^i \tilde{u}_i^A - \tilde{\mathcal{L}} \). This equation is the Hamilton-Jacobi equation of the de Donder-Weyl theory, as presented by Rund (see equation (2.31) on page 224 in [16]). We recall [16] that one can derive in this case also the Euler-Lagrange field equations from the de Donder-Weyl Hamilton-Jacobi equation.

### 8 Hamilton’s Equations

The structure of equations (6.9) - (6.12) suggests that one should be able to derive generalized Hamilton equations if the canonical momenta \( p_A^i = \frac{\partial \mathcal{L}}{\partial u_i^A} \) can be introduced as part of a local coordinate system on \( L_x E \). Part of the original philosophy used in developing \( n \)-symplectic geometry in reference [12] was to switch from scalar equations to tensor equations, motivated by the fact that the soldering 1-form is vector-valued. In particular, the basic structure equation (2.4) in \( n \)-symplectic geometry is tensor-valued. We show next that

\[
\nu^*(\eta \perp d\theta^i_C) = 0
\]

(8.24)
where $u : M \to L_\pi E$ is a section of $\pi \circ \lambda$, and $\eta$ is any vector field on $L_\pi E$, yields generalized canonical equations that contain known canonical equations as special cases. We consider here only $d\theta^L$ since by Proposition (6.3) it alone is needed to construct the CHP-$m$-form on $J^{1,\pi}$.

We need the following definition in order to introduce the canonical momenta as part of a coordinate system on $L_\pi E$.

**Definition 8.1** A Lagrangian $\mathcal{L}$ on $L_\pi E$ is **regular** if the $(m+k) \times (m+k)$ matrix

$$
(E^i_A \circ E^j_B (\mathcal{L}))
$$

is non-singular.

Working out the terms of this matrix in Lagrangian coordinates using (2.2) we obtain

$$
E^i_A \circ E^j_B (\mathcal{L}) = u^i_j u^k_i u^l_j (\frac{\partial^2 \mathcal{L}}{u^l_k u^m_l})
$$

It is clear that this definition is equivalent to the standard definition of regularity on $J^{1,\pi}$.

We now consider the transformation of coordinates from the set $(x^i, y^A, u^i_j, u^A_k, u^A_B)$ to the new set $(\bar{x}^i, \bar{y}^A, \bar{u}^i_j, p^j_i, \bar{u}^A_B)$ where

$$
\bar{x}^i = x^i, \quad \bar{y}^A = y^A, \quad \bar{u}^i_j = u^i_j, \quad \bar{u}^A_B = u^A_B, \quad p^j_i = \frac{\partial \mathcal{L}}{\partial u^A_i}
$$

Computing the Jacobian one finds that the new barred functions will be a proper coordinate system whenever the Lagrangian is regular. For the remainder of this section we shall assume that $\mathcal{L}$ has this property, despite the fact that many important examples (see [7]) have non-regular Lagrangians. Moreover, for simplicity we will drop the bars on the new coordinates.

In the generalized canonical equation (8.24) we now take $\eta = \frac{\partial}{\partial p^j_A}$. We find the result

$$
0 = \left( \frac{\partial H^j_k}{\partial p^j_A} \circ u \right) + (u^j_i \circ u) \left( \frac{\partial (y^A \circ u)}{\partial x^k} \right)
$$

Using $H^j_k = u^j_i h^i_k$ and the fact that $(u^j_i)$ is a non-singular matrix valued function, this last equation reduces to

$$
\frac{\partial h^j_k}{\partial p^j_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^k} \delta^j_i
$$
This is our first set of **generalized Hamilton equations**. Notice that by summing \( j = k \) in this equation we obtain

\[
\frac{\partial h}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^i}
\]

(8.25)

Upon setting \( t(m) = \frac{1}{m} \) we obtain half of the de Donder-Weyl canonical equations. Under suitable but complicated conditions these equations, with \( t(m) = 1 \), will also reproduce part of Rund’s canonical equations for the theory of Carathéodory.

In the generalized canonical equation (8.24) we now take \( \eta = \frac{\partial}{\partial y^i} \). We find

\[
0 = u^* \left( d(u^i_k p^k_A) + u^i_k \frac{\partial h^k_j}{\partial y^A} dx^j \right)
\]

Using an “over bar” notation to denote objects pulled back to \( M \) by \( u \) we may write this as

\[
\frac{\partial}{\partial x^j} \left( \bar{u}^i_k p^k_A \right) = -\bar{u}^i_k \left( \frac{\partial h^k_j}{\partial y^A} \right) \circ u
\]

(8.26)

This is our second set of **generalized Hamilton’s equations**.

Notice that what is non-standard in (8.26) is the appearance of the derivatives of the functions \( \bar{u}^i_j = u^i_j \circ u \). If, however, the section \( u : M \to L_\pi E \) is such that the \( \bar{u}^i_j \) are constants, then these equations reduce to

\[
\frac{\partial (\bar{p}^i_A)}{\partial x^j} = -\frac{\partial h^k_j}{\partial y^i} \circ u
\]

Setting \( \tau(m) = \frac{1}{m} \) and summing \( k = j \) in this equation we obtain

\[
\frac{\partial (\bar{p}^i_A)}{\partial x^i} = -\frac{\partial h}{\partial y^i} \circ u
\]

These equations, together with equations (8.25) when \( \tau(m) = \frac{1}{m} \), are the complete canonical equations in the de Donder-Weyl theory.

### 9 A Model Theory on \( L_\pi E \)

Equation (8.26) clearly suggests that what is needed in this new formalism on \( L_\pi E \) are dynamical equations for the coordinates \( u^i_j \) of frames for the parameter space \( M \). We note that our fundamental canonical variables \( P^i_j = -H^i_j \) and \( P^i_A \) on \( L_\pi E \) are functions of the variables \((x^i, y^A, u^i_j, u^A_i)\), but they do not depend on the other vertical coordinates \( u^A_B \). On
the other hand the complete set of momentum variables \((P_i^j, P_A^i, P_A^A, P_B^A)\) defined in (6.11)--(6.14) depend on all the coordinates on \(L\pi E\). It is evident that a complete Lagrangian field theory on \(L\pi E\) would thus need to supply field equations for the coordinates \(u^i_j\) of frames for \(M\) as well as equations for the coordinates \(u^A_B\) of vertical frames for \(E\), in addition to the standard field equations for sections of \(\pi\). To do this in as simple a way as possible we introduce the following total Lagrangian:

\[
\mathcal{L}_{\text{TOTAL}} = \mathcal{L}_E + \mathcal{L}_{M,V}
\]

Here \(\mathcal{L}_{M,V}\) is a Lagrangian for the \(\rho\)-vertical variables \(u^i_j\) and \(u^A_B\), and \(\mathcal{L}_E\) is to be a generalization of \(\rho^*(\mathcal{L})\) that now includes a coupling to the vertical coordinates.

For an example suppose that \(\pi : E \to M\) is a vector bundle over spacetime \(M\). We can take \(\mathcal{L}_{M,V}\) to be the Lagrangian for a higher-dimensional Kaluza-Klein metric tensor on \(E\), but written in \(n\)-tuple form using \(u^\alpha_\beta\). The term \(\mathcal{L}_E\) can then be taken to be the usual Lagrangian for a section of \(\pi\), but with the usual “fixed” metric tensors now replaced by the dynamical metric written in terms of \(n\)-tuples.

As a second example suppose that \(\pi : E \to M\) is a principal bundle over spacetime \(M\). Then one can take \(\mathcal{L}_{\text{TOTAL}}\) to be a Lagrangian for the Yang-Mills generalization of the Kaluza-Klein theory as formulated, for example, by Hermann [9].

We note that sections of \(LM\) may be considered as 1-jets of non-singular maps from \(\mathbb{R}^m \to M\) since \(LM \subset J^1(\mathbb{R}^m, M)\). Similarly since \(LV \subset J^1(\mathbb{R}^k, E)\), sections of \(LV\) may be considered as 1-jets of maps from \(\mathbb{R}^k \to E\). From a jet bundle point of view we may therefore consider the total Lagrangian \(\mathcal{L}_{\text{TOTAL}}\) as defined on a subset of the space

\[
J^1\pi \times_E (J^1(\mathbb{R}^k, E) \times_M J^1(\mathbb{R}^m, M))
\]

It seems clear then that \(L\pi E\) is a natural arena for unified field theories in which the determination of a geometry for \(M\), and a geometry for the fibers of \(E\), are both part of the dynamical problem. Field equations would produce, for our examples given above, a metric of the Kaluza-Klein type for tangents to \(E\) together with a field (section of \(\pi\)) defined on \(M\). If one fixes the \(M\) and \(V\) gauges by reducing \(H\) to \(\{I\}\) and thereby reducing \(\mathcal{L}_{\text{TOTAL}}\) to \(\rho^*(\mathcal{L})\), then one arrives at the generalized Lagrangian field theory on \(J^1\pi\) lifted to \(L\pi E\) that was discussed earlier.
10 Conclusions

In this paper we have reformulated covariant field theory for sections of $\pi: E \to M$ on the bundle of vertically adapted linear frames $L\pi E$. The advantage gained by this reformulation is that it allows us to utilize the natural geometry that is supported on $L\pi E$, namely $n$-symplectic geometry, to further develop covariant field theory. We have concentrated on demonstrating that $L\pi E$ with its canonical $n$-symplectic structure provides an appropriate arena for formulating covariant field theory, leaving aside the development of the modified $n$-symplectic geometry defined by the Cartan-Hamilton-Poincaré (CHP) 1-forms to future papers.

To this end we showed that covariant field theory on $J^1\pi$ lifts in a natural way to $L\pi E$. The analysis was based on the theorem, presented in section 3, that $J^1\pi$ is a principal fiber bundle $\rho: L\pi E \to J^1\pi$ over the bundle of 1-jets of sections of $\pi$. This theorem was used to show that the soldering 1-forms $\theta^\alpha$ on $L\pi E$ play the role of the contact structure on $J^1\pi$. The soldering 1-forms and a lifted Lagrangian $\mathcal{L} = \rho^*(L)$ were then used to constructed modified soldering 1-forms $\theta^\alpha_{\mathcal{L}}$ on $L\pi E$, the CHP 1-forms. These CHP 1-forms were shown to pass to the quotient to define the standard CHP m-form on $J^1\pi$. Further we used the CHP 1-forms to derive generalized Hamilton-Jacobi and generalized canonical equations on $L\pi E$, and then showed that the Hamilton-Jacobi and canonical equations in the theories of Carathéodory-Rund and de Donder-Weyl are contained as special cases. What we did not do was develop the explicit structure of the modified $n$-symplectic geometry, including allowable observables, Hamiltonian vector fields, and Poisson and graded Poisson brackets, that one should be able to define using the CHP 1-forms. Nor did we develop the variational principle on $L\pi E$ for a Lagrangian that is not lifted from $J^1\pi$. These problems we leave to future publications.

As we studied the structure of $L\pi E$ it became apparent that the extra degrees of freedom in $L\pi E$ that are not in $J^1\pi$ need not be ghost degrees of freedom, but may have direct physical significance. Part of the extra degrees of freedom, namely coordinates for the frames of $M$, were identified with the undetermined elements in the Carathéodory-Rund canonical formalism. The manifold $L\pi E$ emerged as a natural arena for a unified theory that contains
in addition to the sector for sections of $\pi$, dynamical sectors for a geometry for $M$ and a geometry for the fibers of $E$. In fact it was shown that the manifold $L_\pi E$ is a subset of the jet space $J^1 \pi \times_E (J^1(\mathbb{R}^k, E) \times_M J^1(\mathbb{R}^m, M))$. We argued in section 9 that one may easily write down on $L_\pi E$ a model Lagrangian, in $n$-tuple form, for a type of higher dimensional Kaluza-Klein theory. We intend to pursue these ideas in future work.
References


