

**DEF:** A function  $f(x)$  is **analytic** at  $x_0$  if it has a power series expansion at  $x_0$  that converges to  $f(x)$  on an open interval containing  $x_0$ .

**DEF:** A point  $x_0$  is an **ordinary point** of the ODE  $y'' + p(x)y' + q(x)y = 0$  if the functions  $p(x)$  and  $q(x)$  are analytic at  $x_0$ . Otherwise  $x_0$  is a **singular point** of the ODE.

**DEF:** A singular point  $x_0$  of the ODE  $y'' + p(x)y' + q(x)y = 0$  is a **regular singular point** of the ODE if the functions  $xp(x)$  and  $x^2q(x)$  are analytic at  $x_0$ . Otherwise  $x_0$  is an **irregular singular point** of the ODE.

**Theorem:** Let  $x_0 = 0$  be a regular singular point of the ODE  $y'' + p(x)y' + q(x)y = 0$ . Then one solution of the ODE is given by a Frobenius series  $y(x) = x^p \sum_{n=0}^{\infty} a_n x^n$  with  $a_0 \neq 0$ . Here  $p$  is a solution of the indicial equation which is the coefficient of the lowest power of  $x$  set equal to zero when the above form for the solution is substituted into the ODE.

**Theorem:** Let  $x_0 = 0$  be a regular singular point of the ODE  $y'' + p(x)y' + q(x)y = 0$ , and let  $r_1$  and  $r_2$  be the two roots of the indicial equation, with  $r_1 \geq r_2$  when both are real. Then the ODE has two linearly independent solutions  $y_1$  and  $y_2$ , where:

**Case I:** If  $r_1 - r_2 \neq \text{an integer}$  then with  $x > 0$

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, a_0 \neq 0 \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} b_n x^n, b_0 \neq 0$$

**Case II:** If  $r = r_1 = r_2$  then with  $x > 0$

$$y_1 = x^r \sum_{n=0}^{\infty} a_n x^n, a_0 \neq 0 \quad \text{and} \quad y_2 = y_1(x) \ln(x) + x^r \sum_{n=1}^{\infty} b_n x^n$$

**Case III:** If  $r_1 - r_2 = \text{a positive integer}$  then with  $x > 0$

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, a_0 \neq 0 \quad \text{and} \quad y_2 = k y_1(x) \ln(x) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n, b_0 \neq 0$$

Bessel's equation of order  $p \geq 0$  is  $x^2 y'' + x y' + (x^2 - p^2) y = 0$ , and  $x_0 = 0$  is a regular singular point of Bessel's equations. When  $p = n$  is an integer, then two linearly independent solutions of the equation are, for each  $n$ , given by  $J_n(x)$  and  $Y_n(x)$ , where

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

Here are some important properties of the Bessel functions of the first kind (the  $J_n$ ) and the second kind (the  $Y_n$ ).

1. For each  $n \geq 0$ ,  $J_n(x)$  is bounded at zero while  $Y_n(x)$  is **unbounded** at  $x = 0$ . In particular  $J_0(0) = 1$  while  $J_n(0) = 0$  for  $n \geq 1$ .
2. For each  $n \geq 0$ ,  $J_n(x)$  crosses the x-axis infinitely many times. The zeros of each Bessel function are listed in tables and are also given in Maple. Thus we assume that we can always find as many solutions of  $J_n(x) = 0$  as we wish.