MA 401.001 Fall 2005 Test #3 LK Norris

1. (**10 points**) Solve Poisson's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ on the rectangle $0 < x < a$ and $0 < y < b$ with zero boundary data, i.e.

$$u(0, y) = 0 = u(a, y) \ , \ 0 < y < b \quad \text{and} \quad u(x, 0) = 0 = u(x, b) \ , \quad 0 < x < a$$

<u>**Solution:**</u> As discussed in class and in the textbook we assume a solution of the form $\phi(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{nm} \sin(\frac{n\pi x}{z}) \sin(\frac{m\pi y}{b})$ so that the zero boundary conditions are satisfied automatically. To find the coefficients E_{nm} we insert this into the ODE and find

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-\frac{n^2}{a^2} - \frac{m^2}{b^2})\pi^2 E_{nm} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{b})$$

We recognize this as a double Fourier series for the function $f(x, y)$, so the coefficients are determined by

$$E_{nm} = \frac{4}{((-\frac{n^2}{a^2} - \frac{m^2}{b^2})ab\pi^2)} \int_0^a \int_0^b f(x, y) \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{b}) dy\ dx$$

2. (**25 points**) The general solution of the ODE $y'' + \omega^2 y = 0$ is

$$y = A \sin(\omega x) + B \cos(\omega x)$$

<u>**Use the power series method**</u> to derive this general solution. [HINT: The Taylor series for the basic solutions are $\sin(\omega x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n+1}}{(2n+1)!}$ and $\cos(\omega x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n}}{(2n)!}$

<u>**Solution:**</u> We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ so that $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Substituting into the ODE yields

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \omega^2 a_n x^n = 0$$

Shifting indices in the first summation (m=n-2 $\Longrightarrow$ n=m+2) we can rewrite this as

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} \omega^2 a_n x^n = 0$$

Combining the two summations (no terms need to be extracted) and setting the coefficients of the various powers of x equal to zero we find

$$(n+2)(n+1) a_{n+2} x^n + \omega^2 a_n = 0 \Longrightarrow a_{n+2} = -\frac{\omega^2 a_n}{(n+2)(n+1)} \ , \ n >= 0.$$

We compute a few terms to see the pattern:

$$a_2 = -\frac{\omega^2}{2} a_0 \ , \quad a_4 = -\frac{\omega^2}{4 \cdot 3} a_2 = (-1)^2 \frac{\omega^4}{4!} a_0 \ , \quad a_6 = -\frac{\omega^2}{6 \cdot 5} a_4 = (-1)^3 \frac{\omega^6}{6!} a_0$$

Hence the even indexed terms have the general form

$$\boxed{a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0}$$

Similarly for the odd indexed terms we find:

$$a_3 = -\frac{\omega^2}{3\cdot 2}a_1\ , \quad a_5 = -\frac{\omega^2}{5\cdot 4}a_3 = (-1)^2\frac{\omega^4}{5!}a_1\ , \quad a_7 = -\frac{\omega^2}{7\cdot 6}a_5 = (-1)^3\frac{\omega^6}{7!}a_1$$

Since a_1 is arbitrary I will replace it with $\omega\bar{a}_1$ so that the powers of ω and the factorial in the denominator are the same. Thus for the odd indexed coefficients we find the general term Hence the even indexed terms have the general form

$$\boxed{a_{2n+1} = (-1)^n \frac{\omega^{2n+1}}{(2n+1)!}\bar{a}_1}$$

Hence for the solution we find

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0\ even}^{\infty} a_n x^n + \sum_{n=0\ n\ odd}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\
&= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n}}{(2n)!} x^{2n} + \bar{a}_1 \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n+1}}{(2n+1)!} x^{2n+1} \\
&= a_0 \cos(\omega x) + \bar{a}_1 \sin(\omega x)
\end{aligned}$$

3. (**25 points**) The Legendre polynomials $P_n(x)$ are the bounded solutions on the interval $[-1, 1]$ of the equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. The first few are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ and $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

   (a) The Legendre polynomials are known to be orthogonal with weight function 1 on the interval $[-1, 1]$. **Show by direct integration** that $P_2(x)$ and $P_4(x)$ are orthogonal with weight function 1 on the interval $[-1, 1]$.

   Solution:

$$\begin{aligned}
\int_{-1}^{1} P_2(x)P_4(x)dx &= \int_{-1}^{1} \frac{1}{2}(3x^2-1)18(35x^4 - 30x^2 + 3)dx \\
&= 19\int_{-1}^{1} (105x^6 - 90x^4 + 9x^2 - 35x^4 + 30x^2 - 3)dx \\
&= 38\int_{0}^{1} (105x^6 - 90x^4 - 35x^4 + 39x^2 - 3)dx \\
&= 38\left(\frac{105}{7} - \frac{90}{5} - \frac{35}{5} + \frac{39}{3} - 3\right) \\
&= 38\,(15 - 18 - 7 + 13 - 3) \\
&= 0
\end{aligned}$$

   (b) Use the formula $\int_{-1}^{1}(P_n(x))^2 dx = \frac{2}{2n+1}$ to derive the formula for the coefficients c_n in the Legendre expansion

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

for a piecewise smooth function $f(x)$.

Solution: Fix an integer $k \geq 0$ and multiply both sides of the above equation by $P_k(x)$ and integrate the result to obtain

$$\int_{-1}^{1} f(x)P_k(x)dx = \sum_{n=0}^{\infty} \int_{-1}^{1} c_n P_n(x)P_k(x)dx$$

The integrals on the right hand side, as n varies from $n = 0$ to $n = \infty$, are all zero except for the one when $n = k$. Using the given result we find

$$\int_{-1}^{1} f(x)P_k(x)dx = c_k \frac{2}{2k+1} \Longrightarrow \boxed{c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x)P_k(x)dx}$$

(c) Find the Legendre series expansion of the function $f(x) = \frac{3}{2}x - \pi$.

Solution: Since $P_0(x) = 1$ and $P_1(x) = x$, the Legendre series expansion of the given function is

$$f(x) = \boxed{-\pi P_0(x) + \frac{3}{2}P_1(x)}$$

4. (**25 points**) A solid sphere of radius a is axially symmetric so that the steady-state temperature distribution $u(r, \theta)$ in spherical coordinates is governed by the PDE

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot(\theta)\frac{\partial u}{\partial \theta}\right) = 0$$

subject to the boundary conditions $u(a, \theta) = f(\theta)$ and $\lim_{r\to 0^+} |u(r, \theta)| < \infty$. Derive the formal solution to this problem. Be sure to give the formula(s) for the coefficients that may arise in the solution.

Solution: Separate variables with $u(r, \theta) = R(r)F(\theta)$. Substituting into the ODE give

$$\frac{r^2 R'' + 2rR'}{R} + \frac{F'' + \cot(\theta)F'}{F} = 0$$

Both terms must be a constant, so set the first term equal to λ and the second term equal to $-\lambda$. This yields the two equations

$$r^2 R'' + 2rR' - \lambda R = 0 \qquad F'' + \cot(\theta)F' + \lambda F = 0$$

The second equation can be transformed into Legendre's differential equation, and we know that it will have bounded solutions if and only if $\lambda = n(n + 1)$ for $n \geq 0$. Thus the bounded solutions of the second equation for $F(\theta)$ are the Legendre polynomials $F_n(\theta) = P_n(\cos(\theta))$.

Next, using $\lambda = n(n + 1)$ in the first equation for $R(r)$ we obtain the ODE $r^2R'' + 2rR' - n(n+1)R = 0$. This is an Euler differential equation, and as shown in class (you can work it out if you don't remember) is $R_n(r) = a_n r^n + b_n \frac{1}{r^{2n+1}}$. Since the solutions must be bounded at $r = 0$ we must choose $b_n = 0$ for all n. Hence, putting the results back together we find

$$u_n(r, \theta) = A_n r^n P_n(\cos(\theta)) \quad , \quad n \geq 0$$

By multiplying the right-hand-side by $\frac{a^n}{a^n}$ and absorbing the factor of a^n in the numerator into the coefficient A_n we can also write this solution as

$$u_n(r, \theta) = A_n \left(\frac{r}{a}\right)^n P_n(\cos(\theta)) \quad , \quad n \geq 0$$

However, either form of the solution is acceptable. The formal solution to the problem is thus

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos(\theta))$$

Finally applying the boundary condition $u(a,\theta) = f(\theta)$ we find

$$f(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos(\theta))$$

This is a Legendre series expansion for the function $f(\theta)$, and the coefficients are given by

$$\boxed{A_n = \frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos(\theta)) \sin(\theta) d\theta}$$

5. (**15 points**) Consider Bessel's differential equation of order n: $x^2y'' + xy' + (x^2 - n^2)y = 0$ with $x > 0$.

   (a) Show that the point $x_0 = 0$ is a regular singular point for the ODE.
   <u>**Solution:**</u> A point x_0 is a regular singular point of the ODE $y'' + p(x)y' + q(x)y = 0$ if it is a singular point of the ODE but $xp(x)$ and $x^2q(x)$ are analytic at x_0. In this case $x_0 = 0$ and $p(x) = \frac{1}{x}$ and $q(x) = \frac{x^2-n^2}{x^2}$, so:

   $$xp(x) = 1 \quad \text{and} \quad x^2q(x) = x^2 - n^2$$

   Both of these functions are analytic at $x_0 = 0$ and so $x_0 = 0$ is a regular singular point of the ODE.

   (b) Find and solve the "indicial equation" for a Frobenius solution of the given ODE.
   <u>**Solution:**</u> Assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ so that $y' = \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$. Substituting into the ODE gives

   $$0 = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} -n^2 a_n x^{n+r}$$

   The indicial equation is the coefficient of the lowest power of x set equal to zero, with $a_n \neq 0$. The lowest power of x occurs with the $n = 0$ terms in the first, second and fourth sums. So we extract the n=0 terms from those sums to obtain

   $$\left(r(r-1) + r - n^2\right) a_0 = 0 \quad , \quad a_0 \neq 0$$

   or

   $$\boxed{\left(r^2 - n^2\right) = 0}$$

   This is the indicial equation and the solutions are $\boxed{r = \pm n}$.