MA 401 Fall, 2005 Final Exam - SOLUTIONS L. K. Norris

1. **(20 pts)** Prove the following part of the Sturm-Liouville theorem:

Let $\phi_m(x)$ and $\phi_n(x)$ be eigenfunctions corresponding to **<u>distinct</u>** *eigenvalues λ_m and λ_n, respectively, of the following singular Sturm-Liouville problem on [a,b]:*

$$\frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big)+\Big(q(x)+\lambda r(x)\Big)y=0$$

$$A_1y(a)+A_2y'(a)=0,\quad y(x)\ \textit{must be bounded at}\ \ x=b$$

where it is assumed that $(A_1)^2+(A_2)^2\neq 0$ and $p(b)=0$. Then:

$$\int_a^b r(x)\phi_m(x)\phi_n(x)dx=0$$

<u>SOLUTION:</u> ϕ_m and ϕ_n satisfy the following equations:

$$\frac{d}{dx}\Big(p(x)\frac{d\phi_m}{dx}\Big)+\Big(q(x)+\lambda_m r(x)\Big)\phi_m=0 \tag{1}$$

and

$$\frac{d}{dx}\Big(p(x)\frac{d\phi_n}{dx}\Big)+\Big(q(x)+\lambda_n r(x)\Big)\phi_n=0 \tag{2}$$

Multiply (1) by ϕ_n) and (2) by ϕ_m and then subtract the resulting equations. Note that the terms involving $q(x)$ will cancel, leaving

$$\frac{d}{dx}\Big(p(x)\frac{d\phi_m}{dx}\Big)\phi_n-\frac{d}{dx}\Big(p(x)\frac{d\phi_n}{dx}\Big)\phi_m+\Big((\lambda_m-\lambda_n)r(x)\Big)\phi_m\phi_n=0 \tag{3}$$

Next observe the following identity:

$$\frac{d}{dx}\Big(p(x)(\frac{d\phi_m}{dx}\phi_n-\frac{d\phi_n}{dx}\phi_m)\Big)=\frac{d}{dx}\Big(p(x)\frac{d\phi_m}{dx}\Big)\phi_n-\frac{d}{dx}\Big(p(x)\frac{d\phi_n}{dx}\Big)\phi_m$$

Thus the first two terms in (3) can be replaced by the total derivative on the left hand side of this identity. Carrying out the substitution we can rewrite (3) as

$$-\Big((\lambda_m-\lambda_n)r(x)\Big)\phi_m\phi_n=\frac{d}{dx}\Big(p(x)(\frac{d\phi_m}{dx}\phi_n-\frac{d\phi_n}{dx}\phi_m)\Big) \tag{4}$$

Integrating both sides from $x=a$ to $x=b$ we find

$$-(\lambda_m-\lambda_n)\int_a^b r(x)\phi_m\phi_n\ dx=\int_a^b\frac{d}{dx}\Big(p(x)(\frac{d\phi_m}{dx}\phi_n-\frac{d\phi_n}{dx}\phi_m)\Big)dx \tag{5}$$

Now consider only the right hand side of this equation. Applying the fundamental theorem of calculus we find

$$\begin{aligned}\int_a^b\frac{d}{dx}\Big(p(x)(\frac{d\phi_m}{dx}\phi_n-\frac{d\phi_n}{dx}\phi_m)\Big)dx&=\Big(p(x)(\frac{d\phi_m}{dx}\phi_n-\frac{d\phi_n}{dx}\phi_m)\Big)\Big|_a^b\\ &=\Big(p(b)(\frac{d\phi_m}{dx}(b)\phi_n(b)-\frac{d\phi_n}{dx}(b)\phi_m(b))\Big)-\Big(p(a)(\frac{d\phi_m}{dx}(a)\phi_n(a)-\frac{d\phi_n}{dx}(a)\phi_m(a))\Big)\\ &=-\Big(p(a)(\frac{d\phi_m}{dx}(a)\phi_n(a)-\frac{d\phi_n}{dx}(a)\phi_m(a))\Big)\quad\text{since } p(b)=0\end{aligned} \tag{6}$$

To show that the remaining terms also vanishes we recall that both ϕ_m and ϕ_n satisfy the boundary condition $A_1y(a)+A_2y'(a)=0$ where $(A_1)^2+(A_2)^2\neq 0$. Hence ϕ_m and ϕ_n satisfy

$$\begin{aligned}A_1\phi_m(a)+A_2\phi_m'(a)&=0\\ A_1\phi_n(a)+A_2\phi_n'(a)&=0\end{aligned}$$

Rewriting these two equations as a single matrix equation we find

$$\begin{pmatrix} \phi_m(a) & \phi'_m(a) \\ \phi_n(a) & \phi'_n(a) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since not both A_1 and A_2 can be zero we conclude that the 2 by 2 matrix in the above equation must have a vanishing determinant. This yields the equation

$$\phi_m(a)\phi'_n(a) - \phi_n(a)\phi'_m(a) = 0 \tag{7}$$

This equation shows that the right hand side of (6), and hence also the right hand side of (5), vanishes. Hence the left hand side of (5) must also be zero which leads to the equation

$$(\lambda_m - \lambda_n) \int_a^b r(x)\phi_m\phi_n \, dx = 0.$$

Since $\lambda_m \neq \lambda_n$ this yields the result

$$\int_a^b r(x)\phi_m\phi_n \, dx = 0.$$

as was to be shown.

2. **(10 pts)** Solve the following Sturm-Liouville problem:

Find numbers λ and non-trivial functions $y(x)$ that satisfy the ODE $9y'' - 2y' + \lambda y = 0$ and the boundary conditions $y(0) = 0 = y(1)$.

HINT: In problems this semester where we encountered SL problems with the ODE $y'' + \lambda y = 0$ we chose the ranges $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$. This was because the solutions of the characteristic equation for $y'' + \lambda y = 0$ are $r = \pm\sqrt{-\lambda}$, and hence the FORM of the general solution changes at $\lambda = 0$. In your problem when you apply the quadratic formula to solve the characteristic equation the radical will be more complicated than $\sqrt{-\lambda}$. You should use what is under the radical sign to decide the ranges of λ to investigate for the eigenvalues.

SOLUTION: The characteristic equation (assuming $y = e^{rx}$) for the given ODE is $9r^2 - 2r + \lambda = 0$. with solutions $r = \frac{1 \pm \sqrt{1-9\lambda}}{9}$. Hence we must study the three cases (I) $1 - 9\lambda = 0$, (II) $1 - 9\lambda = \omega^2 > 0$ and (III) $1 - 9\lambda = -\omega^2 < 0$,

Case I: $1 - 9\lambda = 0$: Then $r_1 = r_2 = 1/9$ and the general solution is $y = (A + Bx)e^{x/9}$. Then $y(0) = 0 \implies A = 0$ and the solution reduces to $y = Bxe^{x/9}$. The second boundary condition $y(1) = 0$ now yields $0 = Be^{1/9} \implies B = 0$. Hence $\lambda = 1/9$ (from $1 - 9\lambda = 0$) is NOT an eigenvalue.

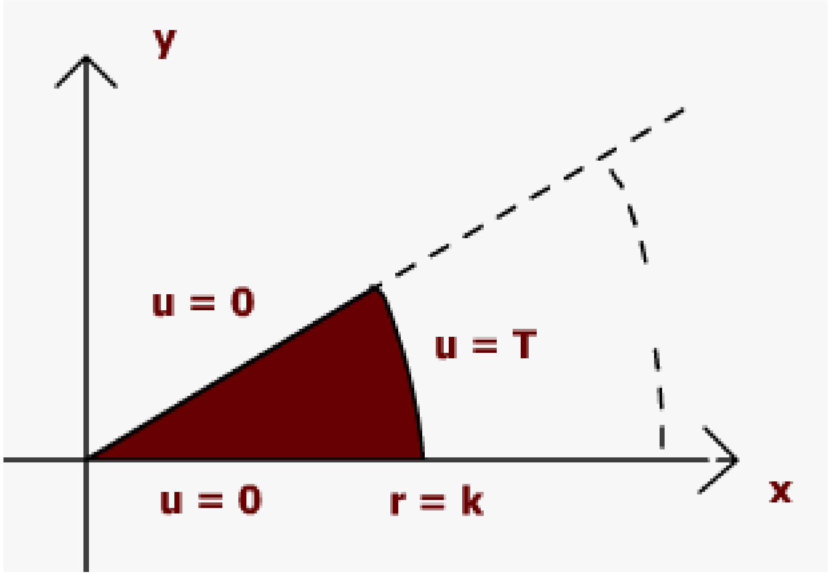
Case II: $1 - 9\lambda = \omega^2 > 0$, $\omega > 0$: In this case $r_1 = 1/9 + \omega/9$, $r_2 = 1/9 - \omega/9$ and the general solution is $y = Ae^{r_1 x} + Be^{r_2 x}$. Then $y(0) = 0 \implies B = -A$ and the solution reduces to $y = A\Big(e^{r_1 x} - e^{r_2 x}\Big)$. The second boundary condition $y(1) = 0$ now yields $0 = A\Big(e^{r_1} - e^{r_2}\Big) = Ae^{1/9}\Big(e^{\omega} - e^{-\omega}\Big) = 2Ae^{1/9}\Big(\sinh(\omega)\Big)$. Since $\omega > 0$ then $\sinh(\omega) \neq 0$ so $A = 0$. Hence there are no eigenvalues when $1 - 9\lambda = \omega^2 > 0$.

Case III: $1 - 9\lambda = -\omega^2 < 0$, $\omega > 0$: In this case $r = \frac{1 \pm i\omega}{9}$ and the general solution is $y = e^{x/9}\Big(A\sin(\omega x/9) + B\cos(\omega x/9)\Big)$. Then $y(0) = 0 \implies B = 0$ and the solution reduces to $y = Ae^{x/9}\sin(\omega x/9)$. The second boundary condition $y(1) = 0$ now yields $0 = Ae^{x/9}\sin(\omega/9) \implies \omega = 9n\pi$, with $n = 1, 2, 3, \ldots$. Hence when $1 - 9\lambda = -\omega^2 = -81n^2\pi^2 < 0 \implies \boxed{\lambda_n = \frac{1 + 9^2n^2\pi^2}{9}}$.

The corresponding eigenfunctions are $\boxed{y_n = e^{x/9}\sin(n\pi x) \quad n = 1, 2, \ldots}$.

3. **(30 pts)** Solve $\nabla^2 u(r,\theta) = 0$ to **find the steady-state temperature distribution** $u(r,\theta)$ in a wedge-shaped flat plate occupying the region $0 \le r \le k$, $0 \le \theta \le \alpha$ (polar coordinates) where $0 < \alpha < \pi/2$ and $k > 0$. The edges $\theta = 0$ and $\theta = \alpha$ are kept at zero temperature, and the arc $r = k$ $(0 < \theta < \alpha)$ is kept at constant temperature $T_0 \neq 0$. In this problem after you **find the formal solution** you are to set up and **evaluate** the appropriate integral(s) that occur(s) in the problem. NOTE: The Laplacian in cylindrical coordinates for a function $u(r,\theta)$ is

$$\nabla^2 u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$



SOLUTION: Here is the set up for the problem.

- **The PDE:** $\nabla^2 u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$
- **The Boundary Conditions:** $u(r,0) = 0$, $u(r,\alpha) = 0$ for $0 < r < k$.
  $u(0,\theta)$ exists and $u(k,\theta) = T_0 \neq 0$ for $0 < \theta < \alpha$.

Now separate variables using $u(r,\theta) = R(r)F(\theta)$, substitute into the equations and divide by $R(r)F(\theta)$ to obtain the following SL problem plus the equation $r^2R'' + rR' - \lambda R = 0$ for $R(r)$:

$$F'' + \lambda F = 0 \quad , \quad F(0) = 0 \quad \text{and} \quad F(\alpha) = 0$$

This is one of the standard SL problems we've solved this semester, with the results: The eigenvalues are $\boxed{\lambda_n = \frac{n^2\pi^2}{\alpha^2}}$ and the eigenfunctions are $\boxed{F_n(\theta) = \sin(\frac{n\pi\theta}{\alpha})}$ where $n = 1, 2, \ldots$. Using the new values of λ in the $R(r)$ equation we find

$$r^2R_n'' + rR_n' - \frac{n^2\pi^2}{\alpha^2}R_n = 0$$

We recognize this as an Euler equation, and the solution is $R_n(r) = a_n r^{\frac{n\pi}{\alpha}} + b_n r^{-\frac{n\pi}{\alpha}}$. For the solution to be bounded at $r = 0$ we take $b_n = 0$. Hence the formal solution to this problem is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\alpha}} \sin(\frac{n\pi\theta}{\alpha})$$

Applying the final boundary condition $u(k,\theta) = T_0 \neq 0$ we obtain

$$T_0 = \sum_{n=1}^{\infty} A_n k^{\frac{n\pi}{\alpha}} \sin(\frac{n\pi\theta}{\alpha})$$

We recognize this as a half-range Fourier sine series for the function T_0. Hence

$$A_n k^{\frac{n\pi}{\alpha}} = \frac{2}{\alpha}\int_0^\alpha T_0 \sin(\frac{n\pi\theta}{\alpha})d\theta \Longrightarrow A_n = \frac{2}{\alpha k^{\frac{n\pi}{\alpha}}}\int_0^\alpha T_0 \sin(\frac{n\pi\theta}{\alpha})d\theta$$

Since T_0 is a constant the integrals are easy to compute:

$$\boxed{A_n} = \frac{2T_0}{\alpha k^{\frac{n\pi}{\alpha}}}\int_0^\alpha \sin(\frac{n\pi\theta}{\alpha})d\theta = \frac{2T_0}{\alpha k^{\frac{n\pi}{\alpha}}}(-\frac{\alpha}{n\pi}\cos(\frac{n\pi\theta}{\alpha}))|_0^\alpha = \boxed{\frac{2T_0}{n\pi k^{\frac{n\pi}{\alpha}}}(1-(-1)^n)}$$

4. **(20 pts)** **<u>Solve the wave equation</u>** $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$ for sound waves $u(x,y,z,t)$ in a rectangular room occupying the region $0 \le x \le L_1$, $0 \le y \le L_2$ and $0 \le z \le L_3$ in the first octant. The boundary conditions are:

$$\begin{aligned} u(0,y,z,t) = 0 = u(L_1,y,z,t) &\quad \text{for} \quad 0 < y < L_2, \quad 0 < z < L_3, \ t > 0 \\ u(x,0,z,t) = 0 = u(x,L_2,z,t) &\quad \text{for} \quad 0 < x < L_1, \quad 0 < z < L_3, \ t > 0 \\ u(x,y,0,t) = 0 = u(x,y,L_3,t) &\quad \text{for} \quad 0 < x < L_1, \quad 0 < y < L_2, \ t > 0 \end{aligned}$$

and the initial conditions are

$$\begin{aligned} u(x,y,z,0) = f(x,y,z) &\quad \text{for} \quad 0 < x < L_1, \quad 0 < y < L_2, \quad 0 < z < L_3 \\ \frac{\partial u}{\partial t}(x,y,z,0) = 0 &\quad \text{for} \quad 0 < x < L_1, \quad 0 < y < L_2, \quad 0 < z < L_3 \end{aligned}$$

Fnd the **<u>formal solution</u>** and give **<u>appropriate formulas</u>** for the coefficients that occur in your solution.

[**HINT:** When you separate variables and translate the boundary conditions you will find THREE Sturm-Liouville problems to solve, involving THREE separation constants.]

<u>SOLUTION:</u> Separate variables with the assumption $u(x,y,z,t) = X(x)Y(y)Z(z)T(t)$. Substitution into the PDE yields

$$\frac{T''}{c^2T} = \frac{X''}{x} + \frac{Y''}{xy} + \frac{Z''}{z} \quad \Longrightarrow \frac{X''}{X} = \frac{T''}{c^2T} - \frac{Y''}{Y} - \frac{Z''}{Z}$$

Since the only x dependance is on the left hand side both sides must be equal to a constant, which we write as $-\lambda$, so

$$\boxed{X'' + \lambda X = 0 \ \ , X(0) = X(L_1) = 0} \text{ is the first SL problem, and } \frac{T''}{c^2T} - \frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

Next rewrite $\frac{T''}{c^2T} - \frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$ in the form

$$\frac{Y''}{Y} = \frac{T''}{c^2T} - \frac{Z''}{Z} + \lambda$$

Both sides must be a constant, which we write as $-\mu$, so

$$\boxed{Y'' + \mu Y = 0 \ \ , Y(0) = Y(L_2) = 0} \text{ is the second SL problem, and } \frac{T''}{c^2T} - \frac{Z''}{Z} + \lambda = -\mu$$

Next rewrite $\frac{T''}{c^2T} - \frac{Z''}{Z} + \lambda = -\mu$ in the form

$$\frac{Z''}{Z} = \frac{T''}{c^2T} + \lambda + \mu$$

Both sides must be a constant, which we write as $-\nu$, so

$$\boxed{Z'' + \nu Z = 0 \ \ , Z(0) = Z(L_3) = 0} \text{ is the third SL problem, and } \boxed{T'' + (\lambda + \mu + \nu)c^2 T = 0}.$$

Each of the three SL problems are standard ones we have solved this semester. The eigenvectors and eigenvalues for each are:

$$\begin{aligned}
\lambda_n &= \frac{n^2\pi^2}{(L_1)^2} \quad , \quad X_n = \sin(\frac{n\pi x}{L_1}) \quad , \ n = 1, 2, \ldots \\
\mu_m &= \frac{m^2\pi^2}{(L_2)^2} \quad , \quad Y_m = \sin(\frac{m\pi y}{L_2}) \quad , \ m = 1, 2, \ldots \\
\nu_k &= \frac{k^2\pi^2}{(L_3)^2} \quad , \quad Z_k = \sin(\frac{k\pi z}{L_3}) \quad , \ k = 1, 2, \ldots
\end{aligned}$$

To solve the remaining T equation let $\omega_{mnk}^2 = c^2\pi^2\left(\frac{n^2}{L_1^2} + \frac{m^2}{L_2^2} + \frac{k^2}{L_3^2}\right)$. The T equation is then

$$T''_{nmk} + \omega_{nmk}^2 T_{nmk} = 0 \quad \Longrightarrow T_{nmk} = A_{nmk}\sin(\omega_{nmk}t) + B_{nmk}\cos(\omega_{nmk}t)$$

The condition $\frac{\partial u}{\partial t}(x, y, z, 0) = 0$ which implies $T'(0) = 0$ which in turn implies $A_{nmk} = 0$, so the solution of the T equation reduces to $\boxed{T_{nmk} = B_{nmk}\cos(\omega_{nmk}t)}$.

The formal solution to the problem is thus

$$\boxed{u(x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B_{nmk}\sin(\tfrac{n\pi x}{L_1})\sin(\tfrac{m\pi y}{L_2})\sin(\tfrac{k\pi z}{L_3})\cos(\omega_{nmk}t)}$$

Finally we use the remaining initial condition $u(x, y, z, 0) = f(x, y, z)$ in the formal solution to find

$$\boxed{f(x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B_{nmk}\sin(\tfrac{n\pi x}{L_1})\sin(\tfrac{m\pi y}{L_2})\sin(\tfrac{k\pi z}{L_3})}$$

The formula for the coefficients is:

$$B_{nmk} = \frac{8}{L_1L_2L_3}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} f(x, y, z)\sin(\frac{n\pi x}{L_1})\sin(\frac{m\pi y}{L_2})\sin(\frac{k\pi z}{L_3})dx\ dy\ dz$$

5. **(20 pts)** Let k_1 be a real number such that $J_0(k_1) = 0$. Find the **formal solution** of the following boundary-value/initial-value problem with wave equation in polar coordinates:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 9\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \quad , \qquad 0 < r < 1 \ , \quad t > 0 \\
&\text{B.C. } \ u(0) \text{ is bounded} \ , \quad u(1, t) = 0 \ , \quad t > 0 \\
&\text{I.C. } \ u(r, 0) = 0.1J_0(k_1 r) \ , \quad \frac{\partial u}{\partial t}(r, 0) = 0 \quad 0 < r < 1
\end{aligned}$$

After finding the **formal solution** set up and **evaluate** the integrals for the coefficients that occur in the problem.

SOLUTION: Note first that the condition $J_0(k_1) = 0$, means that k_1 is one of the positive roots of $J_0(x)$. We will need this fact at the very end of the problem.

Separating variables with $u(r, t) = R(r)T(t)$ we obtain

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{1}{9}\frac{T''}{T} = -\lambda \quad \text{(The negative sign is inserted to get periodic T - solutions)}$$

Hence we have, after multiplying through by r^2, the SL Problem

$$r^2R'' + rR' + \lambda r^2 R = 0 \quad , \quad R(0) \ \text{ exists, and } \ R(1) = 0$$

and the T equation with one initial condition:

$$T'' + 9\lambda T = 0 \quad , \quad T'(0) = 0$$

To solve the R equation, replace λ with ω^2 to obtain

$$r^2R'' + rR' + \omega^2 r^2 R = 0 \quad , \quad R(0) \text{ exists, and } R(1) = 0$$

The solution of this equation, the parameterized Bessel equation of order zero, and the given boundary conditions were discussed in class. The solution is

$$R_j = J_0(\alpha_{0j} r) \quad , \quad j = 1, 2, 3, \ldots$$

where α_{0j} is the jth positive root of $J_0(x)$, and the eigenvalues are $\lambda_j = (\alpha_{0j})^2$.

The T-equation now becomes

$$T_j'' + (3\alpha_{0j})^2 T_j = 0 \quad , \quad T_j'(0) = 0$$

The general solution of this equation is $T_j = A_j \cos(3\alpha_{0j}t) + B_j \sin(3\alpha_{0j}t)$, and the initial condition $T'(0) = 0$ forces us to choose $B_j = 0$ for all $j = 1, 2, 3, \ldots$, so that $T_j = A_j \cos(3\alpha_{0j}t)$. Hence the formal solution to this problem is

$$\boxed{u(r,t) = \sum_{j=1}^{\infty} A_j \cos(3\alpha_{0j}t) J_0(\alpha_{0j} r)}$$

To find the coefficients A_j we apply the remaining initial condition, namely $u(r,0) = 0.1J_0(k_1 r)$, which implies $R(r) = 0.1J_0(k_1 r)$ for $t > 0$. Setting t=0 in the above solution and using the initial condition we arrive at the formula

$$0.1J_0(k_1 r) = \sum_{j=1}^{\infty} A_j J_0(\alpha_{0j} r)$$

This is a Bessel series expansion of the function given on the left-hand-side. Using the formulas we know for the coefficients we have

$$A_j = \frac{0.2}{1^2 (J_1(\alpha_{0j}))^2} \int_0^1 r\Big(J_0(k_1 r)\Big)\Big(J_0(\alpha_{0j})\Big)\, dr$$

As remarked at the beginning of the solution we know that k_1 is one of the roots of $J_0(x)$. This means that there is a positive interger, call it k, such that $\boxed{k_1 = \alpha_{0k}}$. We know that the eigenfunctions in this problem, namely $J_0(\alpha_{0j})$, are orthogonal with weight function r on the interval $[0, 1]$. Hence the above integral on the right hand side will be ZERO for all values of $j = 1, 2, ...\infty$ except for the one value when $j = k$. Hence, using the formula we know for the integral of the square of the eigenfunctions we obtain

$$A_j = \left\{ \begin{array}{cc} 0 & j \neq k \\ \frac{0.2}{1^2 (J_1(\alpha_{0k}))^2} \left(\frac{1}{2}(J_1(\alpha_{0k}))^2\right) = 0.1 & j = k \end{array} \right.$$

Hence only one term survives in the formal solution, which reduces to

$$\boxed{u(r,t) = 0.1J_0(k_1 r)\cos(3k_1 t)}$$

Summary of Solutions

1. **(20 pts)** See the solution given above.

2. **(10 pts)** The **eigenvalues** are $\lambda_n = \frac{1+9^2n^2\pi^2}{9}$ and the corresponding **eigenfunctions** are $y_n = a_n e^{x/9}\sin(n\pi x)$, $n \geq 1$.

3. **(30 pts)** Separate variables as $u(r,\theta) = R(r)F(\theta)$. Then the SL problem is $\boxed{F'' + \lambda F = 0 \quad , \quad F(0) = 0 \text{ and } F(\alpha) = 0}$.
   The eigenvalues are $\boxed{\lambda_n = \frac{n^2\pi^2}{\alpha^2}}$ and the eigenfunctions are $\boxed{F_n(\theta) = \sin(\frac{n\pi\theta}{\alpha})}$ where $n = 1, 2, \ldots$. The R equation is $r^2 R_n'' + rR_n' - \frac{n^2\pi^2}{\alpha^2}R_n = 0$ with general solution $R_n(r) = a_n r^{\frac{n\pi}{\alpha}} + b_n r^{-\frac{n\pi}{\alpha}}$. b_n must vanish to have the solution bounded at $r = 0$. The formal solution is $u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\alpha}}\sin(\frac{n\pi\theta}{\alpha})$. Applying the final boundary condition we get $\boxed{A_n = \frac{2T_0}{n\pi k^{\frac{n\pi}{\alpha}}}(1-(-1)^n)}$

4. **(20 pts)** Separate variables as $u(x,y,z,t) = X(x)Y(y)Z(z)T(t)$. One obtains three SL problems, namely

$$\boxed{X'' + \lambda X = 0 \quad , X(0) = X(L_1) = 0}$$

$$\boxed{Y'' + \mu Y = 0 \quad , Y(0) = Y(L_2) = 0}$$

$$\boxed{Z'' + \nu Z = 0 \quad , Z(0) = Z(L_3) = 0}$$

   and the T-equation $\boxed{T'' + (\lambda + \mu + \nu)c^2 T = 0}$. The formal solution and the formula for the coefficients are :

$$\boxed{u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B_{nmk}\sin(\tfrac{n\pi x}{L_1})\sin(\tfrac{m\pi y}{L_2})\sin(\tfrac{k\pi z}{L_3})\cos(\omega_{nmk}t)}$$

   The formula for the coefficients is:

$$B_{nmk} = \frac{8}{L_1L_2L_3}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} f(x,y,z)\sin(\frac{n\pi x}{L_1})\sin(\frac{m\pi y}{L_2})\sin(\frac{k\pi z}{L_3})dx\ dy\ dz$$

5. **(20 pts)** Separate variables as $u(r,r) = R(r)T(t)$. Then we obtain 1 SL problem,

$$r^2R'' + rR' + \lambda r^2 R = 0 \quad , \quad R(0) \text{ exists, and } R(1) = 0$$

   and the T equation with one initial condition:

$$T'' + 9\lambda T = 0 \quad , \quad T'(0) = 0$$

   The eigenfunctions of the SL problem are $J_0(\alpha_{0j})$ where α_{0j} is the jth positive root of $J_0(x)$, $j = 1, 2, \ldots$. Hence the formal solution to this problem is

$$\boxed{u(r,t) = \sum_{j=1}^{\infty} A_j\cos(3\alpha_{0j}t)J_0(\alpha_{0j}r)}$$

   The formula for the coefficients A_j is

$$A_j = \begin{cases} 0 & j \neq k \\ \frac{0.2}{1^2(J_1(\alpha_{0k}))^2}\left(\frac{1}{2}(J_1(\alpha_{0k}))^2\right) = 0.1 & j = k \end{cases}$$

   Hence only one term survives in the formal solution, which reduces to

$$\boxed{u(r,t) = 0.1J_0(k_1r)\cos(3k_1t)}$$