MA 242.012 Fall 2004 Test #2 LK Norris

SOLUTIONS

1. (5 pts) By using paths \( y = mx \) into the origin, or any other method, show that the following limit does NOT exist:

\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}
\]

**SOLUTION:**

\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{x^2}{x^2 (1 + m^2)} = \frac{1}{1 + m^2}
\]

Since the limit depends on the value \( m \) of the slope, the limit does not exist.

2. (5 pts) Determine the largest set on which the following functions is continuous:

\[
f(x, y, z) = \frac{xyz}{x^2 + y^2 - z}
\]

**SOLUTION:** This function is a rational function and rational functions are continuous on their domains. The domain of this rational function is the set of all points \((x,y,z)\) where \(z \neq x^2 + y^2\).

3. (10 pts.) Explain, quoting appropriate facts and theorems, why the function

\[
f(x, y) = \frac{xy}{x^2 - y}
\]

is differentiable at the point \((2, 1, 1)\).

**SOLUTION:** Computing \(x\) and \(y\) partial derivatives we find \(f_x = -\frac{y(x^2 + y)}{(x^2 - y)^2}\) and \(f_y = \frac{x^3}{(x^2 - y)^2}\). Both of these functions are rational functions and hence are continuous on their respective domains. Notice that both have the same denominator, namely \((x^2 - y)^2\) which is not zero at \((2, 1)\), so \((2, 1)\) is in the domain of both \(f_x\) and \(f_y\) and hence both are continuous at \((2, 1)\). Moreover \((x^2 - y)^2\) is not zero inside a
circle of radius 0.1 centered at (2, 1). Hence both partial derivatives exist near (2, 1) and are continuous at (2, 1), so the given function is differentiable at (2, 1).

4. (10 pts.) Let \( f(r, \theta) = e^r \cos(\theta) \) and let \( r = st \) and \( \theta = \sqrt{s^2 + t^2} \). Use the chain rule to compute \( \frac{\partial f}{\partial s} \) and \( \frac{\partial f}{\partial t} \).

**SOLUTION:**

\[
\frac{\partial f}{\partial s} = f_r r_s + f_\theta \theta_s = (e^r \cos(\theta))(t) + (-e^r \sin(\theta))(\frac{s}{\sqrt{s^2 + t^2}})
\]

\[
\frac{\partial f}{\partial t} = f_r r_t + f_\theta \theta_t = (e^r \cos(\theta))(s) + (-e^r \sin(\theta))(\frac{t}{\sqrt{s^2 + t^2}})
\]

5. (40 pts.) Consider the function \( f(x, y, z) = x^2 y^3 z^4 \).

(a) Find the directional derivative of \( f \) at the point \( P_0 = (1, 1, 1) \) in the direction from \( P_0 \) to the point \( P_1 = (-2, 3, 0) \).

**SOLUTION:** The unit vector from \( P_0 \) to \( P_1 \) is \( \vec{u} = \frac{1}{ \sqrt{14} } [-3, 2, -1] \).

The gradient of the function is \( \nabla f = [2xy^3z^4, 3x^2y^2z^3, 4x^2y^3z^2] \), and evaluating at the point (1,1,1) we get \( \nabla f(P_0) = [2, 3, 4] \). Hence the directional derivative is

\[
D_{\vec{u}} f(P_0) = [2, 3, 4] \cdot \frac{1}{ \sqrt{14} } [-3, 2, -1] = -\frac{4}{\sqrt{14}}
\]

(b) What is the maximum rate of change of \( f \) at the point \( P_0 \)?

**SOLUTION:** The magnitude of the gradient of the function at the point: \( ||\nabla f(P_0)|| = \sqrt{29} \)

(c) Give the unit vector that points the direction of the maximum rate of change of \( f \) at \( P_0 \).

**SOLUTION:** The direction of the gradient, namely:

\[
\frac{\nabla f(P_0)}{||\nabla f(P_0)||} = \frac{1}{\sqrt{29}}[2, 3, 4]
\]

(d) Find an equation for the tangent plane to level surface of \( f \) that passes through the point \( P_0 \).

**SOLUTION:** The gradient vector at \( P_0 \) is the normal vector for the plane. Hence:

\[
2(x - 1) + 3(y - 1) + 4(z - 1) = 0
\]
6. \(20\) pts. Find all critical points of the function

\[ f(x, y) = 2x^3 + xy^2 + 2x^2 + 2y^2 \]

and then use the second derivative test to classify each critical point as corresponding to either a local maximum, local minimum or saddle point.

**SOLUTION:** There are two critical points at \((0, 0)\) and \((-\frac{2}{3}, 0)\). Computing the discriminant \(D = f_{xx}f_{yy} - (f_{xy})^2\) and \(f_{xx}\) at these two points we find \((0, 0)\) corresponds to a local minimum (\(D(0, 0) > 0\) and \(f_{xx}(0, 0) > 0\)), and \((-\frac{2}{3}, 0)\) corresponds to a saddle point (\(D(0, 0) < 0\)) of the graph of \(f\).

7. \(10\) pts. Consider the function of two variables

\[ f(x, y) = x^2 + xy + y^2 - y + 1 \]

Find the absolute maximum and absolute minimum values of this function along the line segment \(y = x\) for \(-1 \leq x \leq 1\).

**SOLUTION:** Evaluating \(f\) along the line segment \(y = x\) we obtain a function of \(x\) alone, namely \(g(x) = f(x, x) = 3x^2 - x + 1\) defined on the closed interval \([-1, 1]\). Since \(g\) is a polynomial it is continuous everywhere and hence \(g\) attains an absolute max and absolute min on the closed interval. To find critical points solve \(g'(x) = 6x - 1 = 0\) so \(x = \frac{1}{6}\) is the only critical point.

\[ g(-1) = 5 \, , \, g(1) = 3 \, , \, g(1/6) = \frac{11}{12} \]

Hence the **absolute maximum value** of the function on the closed interval is \(\frac{11}{12}\) and the **absolute minimum value** of the function is \(\frac{11}{12}\).