MA 242

test #2 solutions spring, 2000  l. k. norris

1. (30 pts) let \( f(x, y, z) = x^2 - 3xy + xyz \) and \( P_0 = (3, 2, 1) \).

(a) Find the directional derivative of \( f(x, y, z) \) at the point \( P_0 \) in the direction of the vector \( \vec{v} = \mathbf{i} + \mathbf{j} - \mathbf{k} \).

SOLUTION: The gradient of \( f \) is \( \nabla f = \langle 2x - 3y + yz, -3x + xz, xy \rangle \), and at \( P_0 \) this gives the vector \( \nabla f(P_0) = \langle 2, -6, 6 \rangle \). The unit in the direction of the vector \( \vec{v} = \mathbf{i} + \mathbf{j} - \mathbf{k} \) is \( \vec{u} = \frac{1}{\sqrt{3}} < 1, 1, -1 > \). Hence the directional derivative at \( P_0 \) in the direction of the vector \( \vec{v} \) is
\[
D_{\vec{u}} f(P_0) = \vec{u} \cdot \nabla f(P_0) = \frac{1}{\sqrt{3}} < 1, 1, -1 > \cdot < 2, -6, 6 > = -10/\sqrt{3}
\]

(b) What is the maximum rate of change of \( f(x, y, z) \) at \( P_0 \)?

SOLUTION: The greatest rate of increase of \( f \) at \( P_0 \) equals the magnitude of the gradient vector at \( P_0 \). Hence
\[
\text{greatest rate of increase of } f \text{ at } P_0 = |\nabla f(P_0)| = \sqrt{4 + 36 + 36} = \sqrt{76}
\]

(c) In what direction (unit vector!) does \( f \) change most rapidly at \( P_0 \)?

SOLUTION: The direction is the unit vector pointing in the direction of the gradient vector at the point. Hence
\[
\text{direction } f \text{ change most rapidly at } P_0 \text{ is: } \frac{1}{\sqrt{\sqrt{76}}} < 2, -6, 6 >
\]

2. (20 pts) Consider the surface that is the graph of the equation
\[x^2 + 4y^2 + 3z^2 = 20\]

(a) Find an equation of the tangent plane to the surface at the point \( P_0 = (2, 1, 2) \).

SOLUTION: The above equation may be considered as the 20-level surface of the function \( f(x, y, z) = x^2 + 4y^2 + 3z^2 \), and the normal to the tangent plane at the point \( P_0 \) is the gradient of \( f \) at that point. Since \( \nabla f = \langle 2x, 8y, 6z \rangle \) we find \( \nabla f(P_0) = \langle 4, 8, 12 \rangle \). Hence the equation of the tangent plane is
\[
4(x - 2) + 8(y - 1) + 12(z - 2) = 0
\]

(b) Find parametric equations of the normal line to the surface at the point \( (2, 1, 2) \).

SOLUTION: The gradient vector at the point in question is also the direction vector for the normal line. Hence parametric equations of the normal line are:
\[
x = 2 + 4t, \ y = 1 + 8t, \ z = 2 + 12t
\]
3. (20 pts) Let \( f(x, y) = e^x \sin(y) \), and \( x = st^2 \), \( y = s^2t \).

(a) Compute \( \frac{\partial f}{\partial s} \), expressing your answers in terms of \( s \) and \( t \).

**SOLUTION:**

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin(y))(t^2) + (e^x \cos(y))(2st)
\]

\[
= (e^{st^2}(s^2t))(t^2) + (e^{st^2}\cos(s^2t))(2st)
\]

(b) Compute \( \frac{\partial f}{\partial t} \), expressing your answers in terms of \( s \) and \( t \).

**SOLUTION:**

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin(y))(2st) + (e^x \cos(y))(s^2)
\]

\[
= (e^{st^2}(s^2t))(2st) + (e^{st^2}\cos(s^2t))(s^2)
\]

4. (10 pts) Let \( f(x, y) = \cos\left(\frac{2x^2 + 3xy}{x^2 - 2xy + y^2}\right) \). Find the largest set on which \( f(x, y) \) is continuous. You must justify your work here by quoting appropriate facts about rational functions and the cosine function. [HINT: \( x^2 - 2xy + y^2 = (x - y)^2 \)]

**SOLUTION:** (In what follows the material that is in **boldfaced type** is the material that must be stated in some form in order to get full credit.)

This function is the composition of the rational function \( \frac{2x^2 + 3xy}{x^2 - 2xy + y^2} \) with the cosine function \( \cos(x) \). Since rational functions are continuous on their domains in the xy-plane, and since the denominator only vanishes when \( x^2 - 2xy + y^2 = (x - y)^2 = 0 \implies x = y \), the rational function \( \frac{2x^2 + 3xy}{x^2 - 2xy + y^2} \) is continuous on the entire xy-plane except along the line \( y = x \). Also, the cosine function is continuous on all of the real line. Hence the composition function \( f(x, y) = \cos\left(\frac{2x^2 + 3xy}{x^2 - 2xy + y^2}\right) \) is continuous at each point of the xy-plane except along the line \( y = x \).

5. (10 pts) The plane \( x = 2 \) intersects the graph of the function \( f(x, y) = x^2 + 2y^2 - 12 \) in a curve. Use the geometrical interpretation of partial derivatives to find parametric equations of the tangent line to this curve at the point \( P = (1, 2, -3) \).

**SOLUTION:** Since \( x \) is held constant, the slope of the line in the \( x = 2 \) plane is given by the y-partial derivative \( f_y(x, y) = 4y \) evaluated at the point \( (1, 2) \), so slope = \( f_y(1, 2) = 8 \). Hence the direction vector to the line is \( \vec{u} = \langle 0, 1, f_y(1, 2) \rangle = \langle 0, 1, 8 \rangle \), and the parametric equations of the line are

\[
x = 1, \quad y = 2 + t, \quad z = -3 + 8t
\]

6. (10 pts) You are sitting at a workstation with a Maple V worksheet on the computer screen. Assume that the command "with(plots):" has already been executed in your maple worksheet.
(a) Which Maple V command would you use, plot or implicitplot, if you want to plot the graph of \(x^2 + y^2 = 6\) in your Maple worksheet?
SOLUTION: \[\text{implicitplot}\]

(b) Which Maple V command would you use, plot or implicitplot, if you want to plot the graph of \(f(x) = x^2 - x + 5\) in your Maple worksheet?
SOLUTION: \[\text{plot}\]

(c) What is the "plot option" that you would include in a plot or implicitplot command if you want your plot to be displayed in green.
SOLUTION: \[\text{color = green}\]

7. **(10 pts Extra Credit)** Choose only ONE.

(a) **Explain** why the function \(f(x, y) = \frac{x + y}{x^2 + y^2 + 1}\) is differentiable on the entire xy-plane.

(b) **Prove** that the following limit does NOT exist:
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]