1. (10 pts) Find the vector projection of the vector $\overrightarrow{P_0P_1}$ from the point $P_0 = (3, 1, -2)$ to the point $P_1 = (7, 0, 2)$ onto the direction of the line with parametric equations $x = 1 + 2t$, $y = -1 - t$, $z = 5t$.

**SOLUTION:** $\overrightarrow{P_0P_1} = <4, -1, 4>$ and the direction vector for the line is $\vec{v} = <2, -1, 5>$. The magnitude of $\vec{v}$ is $\sqrt{30}$. Hence $\text{proj}_{\vec{v}}(\overrightarrow{P_0P_1}) = \sum\frac{\vec{v} \cdot \overrightarrow{P_0P_1}}{\sqrt{30}} = \frac{29}{30} \vec{v}$

2. (20 pts) The pressure of hydrogen gas in a certain fuel cell is given by $P(x, y, z) = x^2y + y^2z$

(a) Find the gradient $\nabla P$ of the pressure function at the point $(2, 1, 0)$.

**SOLUTION:** $\nabla P(x, y, z) = <2xy, x^2 + 2yz, y^2> \implies \nabla P(2, 1, 0) = <4, 4, 1>$

(b) Find the greatest rate of increase of the pressure of the gas at the point $(2, 1, 0)$.

**SOLUTION:** $\nabla P(2, 1, 0) = <4, 4, 1> \implies ||\nabla P(2, 1, 0)|| = \sqrt{33}$

(c) Find the unit vector that points in the direction of the greatest increase of the pressure in the fuel cell at the point $(2, 1, 0)$.

**SOLUTION:** $\frac{1}{\sqrt{33}} <4, 4, 1>$

(d) Find the directional derivative of $P(x, y, z)$ at the point $(2, 1, 0)$ in the direction from $(2, 1, 0)$ to the point $(0, 2, 4)$.

**SOLUTION:** $\frac{1}{\sqrt{21}} <2, 1, 4> \cdot \nabla P(2, 1, 0) = 0$

(e) Find an equation for the tangent plane to the level surface of $P(x, y, z)$ at the point $(2, 1, 0)$.

**SOLUTION:** $4(x - 2) + 4(y - 1) + 1(z - 0) = 0$
3. **(15 pts)** Consider the function \( f(x, y) = x^3 - 3x + y^2 - 2y + 10 \). Find all critical points of the function, and then use the second derivative test to determine whether each critical point corresponds to a relative maximum, minimum or saddle point.

**SOLUTION:** The critical points are \((-1, 1) (2 \text{ points})\) and \((1, 1) (2 \text{ points})\). 
\((-1, 1)\) corresponds to a saddle point (5 points); \((1, 1)\) corresponds to a local minimum (6 points).

4. **(30 pts)** Consider the solid region in the **first octant** bounded on the top by \( z = 10 - x^2 - y^2 \), on the bottom by \( z = 0 \), and on the sides by the cylinder \( x^2 + y^2 = 4 \) and the planes \( x = 0 \) and \( y = 0 \). Then the volume of this region is given by the double integral

\[
\int_D \int (10 - x^2 - y^2) \, dA
\]

where \( D \) is the appropriate region in the \( xy \)-plane.

(a) Set up the above double integral as an explicit double iterated integral in **Cartesian coordinates** with \( dx \, dy \) as the order of integration. (Do not evaluate)

**SOLUTION:** limits 2.5 points each
\[
\int_0^2 \int_0^{\sqrt{4-y^2}} (10 - x^2 - y^2) \, dx \, dy
\]

(b) Set up the above double integral as an explicit double iterated integral in **Cartesian coordinates** with \( dy \, dx \) as the order of integration. (Do not evaluate)

**SOLUTION:** limits 2.5 points each
\[
\int_0^2 \int_0^{\sqrt{4-x^2}} (10 - x^2 - y^2) \, dy \, dx
\]

(c) Set up the above double integral as an explicit double iterated integral in **polar coordinates**. (Do not evaluate)

**SOLUTION:** limits 2 pts each; integrand 2 pts
There are two possible solutions:
\[
\int_0^{\pi/2} \int_0^2 r(10 - r^2) \, dr \, d\theta = \int_0^2 \int_0^{\pi/2} r(10 - r^2) \, d\theta \, dr
\]

5. **(30 pts)** Consider the triple integral

\[
\int\int\int_E (xyz) \, dV
\]

where \( E \) is the region that lies **above** the cone \( z = \sqrt{x^2 + y^2} \) and **below** the sphere \( x^2 + y^2 + z^2 = 18 \).

(a) Set up the explicit **triple** iterated integral in **Cartesian coordinates** needed to evaluate this triple integral, with \( dz \, dy \, dx \) as the order of integration (DO NOT EVALUATE).

**SOLUTION:**
\[
\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{18-x^2-y^2}}^{\sqrt{x^2+y^2}} (xyz) \, dz \, dy \, dx
\]
(b) Set up the explicit \textbf{triple} iterated integral in \textbf{cylindrical} coordinates needed to evaluate this triple integral, with \(dz \, dr \, d\theta\) as the order of integration \((DO \ NOT \ EVALUATE)\).

\[
\text{SOLUTION:} \quad \int_0^{2\pi} \int_0^3 \int_{r}^{\sqrt{18-r^2}} (27r^3 \cos \theta \sin \theta) \, dz \, dr \, d\theta
\]

(c) Set up the explicit \textbf{triple} iterated integral in \textbf{spherical} coordinates needed to evaluate this triple integral, with \(d\rho \, d\phi \, d\theta\) as the order of integration \((DO \ NOT \ EVALUATE)\).

\[
\text{SOLUTION:} \quad \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{18}} (\rho^5 \sin^3 \phi \cos \phi \cos \theta \sin \theta) \, d\rho \, d\phi \, d\theta
\]

6. (10 pts) One of the following two vector fields is conservative. Determine which is conservative, and then find all potential functions for that conservative vector field.

\[
\vec{F}(x, y, z) = (e^y)i + (y^2 - xe^y)j + (\cos(z) + 1)k \\
\vec{G}(x, y, z) = (e^y)i + (y^2 + xe^y)j + (\cos(z) + 1)k
\]

\[
\text{SOLUTION:} \quad \vec{G} \text{ is conservative (since } \nabla \times \vec{G} = 0\text{), and all potential functions for } \vec{G} \text{ are}
\]

\[
f(x, y, z) = xe^y + \frac{1}{3}y^3 + \sin(z) + z + C
\]

where \(C\) is an arbitrary constant.

---

**NOTE:** In the next three problems you are to \textbf{SET UP}, and then \textbf{EVALUATE} the appropriate integrals.

7. (15 pts)

(a) (5 pts) State Stokes’s Theorem. Be certain to state all assumptions and to define all symbols that occur in the formula(s) you write down.

\[
\text{SOLUTION:} \quad \text{Let } S \text{ be a piece-wise smooth surface (1 points) with piece-wise smooth boundary curve } C \text{ (1 points) that has positive orientation with respect to } S \text{ (1 points). Let } \vec{F} \text{ be a vector field with component functions that are differentiable on an open set containing } S \text{ (1 points). Then}
\]

\[
\int_C \vec{F} \cdot \, d\vec{r} = \int_s (\nabla \times \vec{F}) \cdot \, d\vec{S} \quad \text{(1 points)}
\]
(b) \textbf{(10 pts)} Use the Divergence Theorem to evaluate the flux integral

\[
\int_S \vec{F} \cdot \vec{n} \, dS
\]

where \( \vec{F} = (3x)\hat{i} + (-5y)\hat{j} + (+3z)\hat{k} \), and \( S \) is the closed surface bounded on the top by the upper hemisphere \( x^2 + y^2 + z^2 = 9 \) with \( z \geq 0 \), and bounded on the bottom by \( z = 0 \). [HINT: Use spherical coordinates.]

\textbf{SOLUTION:} Notice that the divergence of the vector field is \( \nabla \cdot \vec{F} = (3 - 5 + 3) = 1 \)

\[
\int_S \vec{F} \cdot \vec{n} \, dS = \int \int \int_D (\nabla \cdot \vec{F}) \, dV = \int \int \int_D 1 \, dV
\]

\[
= \int_0^\pi \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho
\]

\[
= \frac{1}{2} \text{ the volume of a sphere of radius 3}
\]

\[
= \frac{1}{2} \left( \frac{4}{3} \pi \cdot 3^3 \right) = 18 \pi
\]

8. \textbf{(15 pts)} Compute the integral \( \int_C \vec{F} \cdot d\vec{r} \) for the \textbf{work done} by the force \( \vec{F}(x, y, z) = (x^2 - y)\vec{i} + (x^2 + y)\vec{j} + 3e^{\cos(z)}\vec{k} \) in moving an object along the line segment from \((-1, 1, 1)\) to \((2, 4, 1)\).

\textbf{SOLUTION:} The parameterization of the curve is \( \vec{r}(t) = \langle -1 + 3t, 1 + 3t, 1 \rangle \) with \( 0 \leq t \leq 1 \). Then

\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle (-1 + 3t)^2 - (1 + 3t), (-1 + 3t)^2 + (1 + 3t), 3e^{\cos(1)} \rangle \cdot \langle 3, 0, 0 \rangle \, dt
\]

\[
= \int_0^1 6(-1 + 3t)^2 \, dt = \frac{1}{6}
\]

9. \textbf{(15 pts)} Let \( \vec{F} = 2x\vec{i} + 2y\vec{j} + z^2\vec{k} \), let \( S \) be the portion of the cone \( z = \sqrt{x^2 + y^2} \) that lies in the \textbf{first octant} below \( z = 1 \), and let the orientation of \( S \) be downward.

(a) Parameterize the surface \( S \).

\[ \vec{r}(x, y) = \langle x, y, \sqrt{x^2 + y^2} \rangle \text{ for } x^2 + y^2 \leq 1 \text{ in the first quadrant.} \]

(b) \textbf{Evaluate} the flux integral \( \int_S \vec{F} \cdot \vec{n} \, dS \).

\textbf{SOLUTION:} \( \vec{r}_x \times \vec{r}_y = \langle \frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 1 \rangle \). Then

\[
\int_S \vec{F} \cdot d\vec{S} = \pm \int_D \langle < 2x, 2y, x^2 + y^2 > \cdot \langle \frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 1 \rangle \, dA
\]

\[
= 3 \text{ points}
\]

\[
= 3 \text{ points}
\]

\[
= 3 \text{ points}
\]

4
\[ \begin{align*}
1 \text{ point} & \quad \iint_{D} \left( \frac{-2(x^2 + y^2)}{\sqrt{x^2 + y^2}} + x^2 + y^2 \right) dA \\
5 \text{ points} & \quad = - \int_{0}^{\pi/2} \int_{0}^{1} (-2r^2 + r^3) d\theta \, dr = \frac{5\pi}{24}
\end{align*} \]

10. (10 pts) Write the Maple V commands needed to:

(a) Define vectors \( \vec{A} = \langle 2, -1, 3 \rangle \) and \( \vec{B} = \langle 0, 6, 12 \rangle \) in a Maple worksheet, and then compute the cross product \( \vec{A} \times \vec{B} \).

**SOLUTION:**

1 point 1 point 3 points

\[
\text{A} := \langle 2, -1, 3 \rangle; \quad \text{B} := \langle 0, 6, 12 \rangle; \quad \text{crossprod(A, B)};
\]

(b) Plot the graph of the equation \( x^2 + y^2 - z^2 = 1 \) in yellow and the equation \( 2x + y - z = 1 \) in green on the same Maple plot, with \(-10 \leq x \leq 10, -5 \leq y \leq 5, \) and \(-3 \leq z \leq 3\).

**SOLUTION:**

\[
a := \text{implicitplot3d}(x^2 + y^2 - z^2 = 1, x = -10..10, y = -5..5, z = -3..3, \text{color=yellow}); \quad 2 \text{ points}
\]
\[
b := \text{implicitplot3d}(2x + y - z = 1, x = -10..10, y = -5..5, z = -3..3, \text{color=green}); \quad 2 \text{ points}
\]
\[
\text{display([a, b]); 1 point}
\]

11. (10 pts EXTRA CREDIT) Prove the fundamental theorem for line integrals, namely that if \( C \) is a piecewise smooth simple curve starting at \( Q \) and ending at \( P \), and if \( f(x, y, z) \) is a differentiable function, then

\[
\int_{C} \nabla f \cdot d\vec{r} = f(P) - f(Q)
\]

**SOLUTION:** Let the curve be parameterized by \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \) for \( a \leq t \leq b \), with \( P = \vec{r}(b) \) and \( Q = \vec{r}(a) \). Then:

\[
\int_{C} \nabla f \cdot d\vec{r} = \int_{a}^{b} < f_x, f_y, f_z > (x(t), y(t), z(t)) \cdot < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > dt
\]
\[
= \int_{a}^{b} \frac{d(f(x(t), y(t), z(t)))}{dt} dt
\]
\[
= f(x(t), y(t), z(t)) \Big|_{a}^{b}
\]
\[
= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))
\]
\[
= f(P) - f(Q)
\]