MA 405-003 **Test 4** **Spring 2008**

Name ____________________

Read each question carefully. You must SHOW ALL WORK for full credit.
NO CALCULATORS. Good luck.

1. (20%) For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_3 \end{pmatrix}$$

   Find the Range and Kernel of T.

2. (25%)

$$\text{For } A = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$$

   (a) Find the eigenvalues of A.

   (b) Find the corresponding eigenspaces of A.

   (c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

   (d) Find A^{100}.

   (e) Find e^A.

3. (10%) Determine the values of α for which

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

   is not diagonalizable. Justify.

4. (10%) Find the point on the line $y = 2x$ that is closest to the point $(1, 1)$.

5. (15%) Let $v_1 = \frac{1}{3}(2, -2, 1, 0)^T$ and $v_2 = \frac{1}{3}(0, 1, 2, 2)^T$

   (a) Show $\{v_1, v_2\}$ is an orthonormal set.

   (b) Write $v = (4, 0, 10, 8)^T$ as a linear combination of v_1 and v_2. Use dot products to solve, NOT a linear system.

6. (20%) Let $S = span\{v_1 = (1, 1, 1, 2)^T, v_2 = (1, 1, 2, 3)^T\}$. Find a basis for $S^{\perp}$.

BONUS: For $A = (a_{ij})_{n \times n}$, suppose α is not an eigenvalue of A.
Prove that X is an eigenvector of A if and only if X is an eigenvector of $(A - \alpha I)^{-1}$.

TEST 4 SOLUTIONS

1.

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$Range(T) = CS\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$$

$$Kernel(T) = N\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = span\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

2. (a)

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+1 & -1 \\ 2 & \lambda - 2 \end{pmatrix} = \lambda^2 - \lambda = \lambda(\lambda - 1)$$

Eigenvalues are 0 and 1.

(b)

$$\underline{\lambda = 0} \text{ eigenspace} = N\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow N(0I - A) = span\{(1,1)^T\}$

$$\underline{\lambda = 1} \text{ eigenspace} = N\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow N(1I - A) = span\{(1,2)^T\}$

(c) For example

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Note: Answer not unique.

(d) $P^{-1}AP = D$, therefore $P^{-1}A^kP = D^k$.

However $D^k = \begin{pmatrix} 0^k & 0 \\ 0 & 1^k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = D$. Therefore $A^k = PDP^{-1} = A$ for all k.

$$\text{So } A^{100} = A = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

(e) $A = PDP^{-1} \Rightarrow e^A = Pe^DP^{-1}$.

$$e^D = \begin{pmatrix} e^0 & 0 \\ 0 & e^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \Rightarrow A = P\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}P^{-1} = \begin{pmatrix} 2-e & e-1 \\ 2-2e & 2e-1 \end{pmatrix}$$

3. $\det(\lambda I - A) = (\lambda - \alpha)(\lambda)(\lambda - 2)$. So the eigenvalues are $\lambda = 0, 2, \alpha$. If $\alpha \neq 0, 2$ then we have three distinct eigenvalues, hence A is diagonalizable. If remains to consider what happens if α is 0 or 2.

   Case 1 $\alpha = 0$. The eigenvalues are $\lambda = 0, 0, 2$. The eigenvalue of 2 will correspond to one independent eigenvector since it's only a root of the characteristic polynomial once. It remains to consider $\lambda = 0$ which will correspond to either one or two independent eigenvectors.
$$0I - A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \to \ldots \to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
   which has nullity two, and hence corresponds to two independent eigenvector. This gives a total of 3 independent eigenvectors, hence A diagonalizable.

   Case 2 $\alpha = 2$. The eigenvalues are $\lambda = 0, 2, 2$. The eigenvalue of 0 will correspond to one independent eigenvector since it's only a root of the characteristic polynomial once. It remains to consider $\lambda = 2$ which will correspond to either one or two independent eigenvectors.
$$2I - A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \to \ldots \to \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
   which has nullity two, and hence corresponds to two independent eigenvectors. This gives a total of 3 independent eigenvectors, hence A diagonalizable.

   Therefore A is never defective, i.e. diagonalizable for all values of α.

4. Let X denote a vector along the line $y = 2x$, e.g. $X = (1, 2)^T$. Let Y denote the vector from the origin to the point of interest, i.e. $Y = (1, 1)^T$. Therefore we want to find $proj_X Y$.

   $proj_X Y = \frac{<X,Y>}{<X,X>} X = \frac{1+2}{1+4} X = \frac{3}{5} X = \frac{3}{5}(1, 2)^T = (\frac{3}{5}, \frac{6}{5})^T$.

5. (a) $< v_1, v_2 >= 0 \Rightarrow$ orthogonal.
   $< v_1, v_1 >= 1 =< v_2, v_2 >\Rightarrow$ orthonormal.

   (b) $< v, v_1 >= \frac{8}{3} + 0 + \frac{10}{3} + 0 = 6$
   $< v, v_2 >= 0 + 0 + \frac{20}{3} + \frac{16}{3} = 12$

   Therefore $v = 6v_1 + 12v_2$

6. If $x \in S^{\perp}$ then $v_1^T x = v_2^T x = 0$. That is if $x = (x_1, x_2, x_3, x_4)^T$ then for
$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \text{ we must have } Ax = \underline{0} \text{ or } \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In other words $S^{\perp} = N(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix} \to ... \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

so $S^{\perp} = N(A) = span\{(-1,1,0,0)^T, (-1,0,-1,1)^T\}$.

BONUS: <u>Proof</u>:

X is an eigenvector for $A \Leftrightarrow AX = \lambda X$ (for some λ) $\Leftrightarrow AX - \alpha X = \lambda X - \alpha X \Leftrightarrow$
$AX - \alpha IX = \lambda X - \alpha X \Leftrightarrow (A - \alpha I)X = (\lambda - \alpha)X \Leftrightarrow \frac{1}{\lambda - \alpha}X = (A - \alpha I)^{-1}X \Leftrightarrow$
X is an eigenvector of $(A - \alpha I)^{-1}$
Note: $A - \alpha I$ is invertible because α is not an eigenvalue of A.