MA 341 Test 3 Version 1 Show ALL of your work.

1. (10 points) Find the inverse of $A=\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$ using row operations. Show your work.

2. (10 points) Use the method of undetermined coefficients to find a particular solution to $\vec{x}'=\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\vec{x}+\begin{bmatrix} 3 \\ 2t \end{bmatrix}$

Hint: Don't find $\mathbf{x}_c$!

3. (8 points) The matrix $\begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ has the characteristic polynomial $(r+2)^2(r-7)=0$

Find the eigenvectors associated with $r=-2$

4. (15 points) Find the general solution to $\vec{x}'=A\,\vec{x}$, where $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{bmatrix}$

Hint: If $r=\alpha\pm\beta i$ with $\mathbf{u}=\mathbf{a}\pm\mathbf{b}i$, two linearly independent solutions are $e^{\alpha t}\cos(\beta t)\mathbf{a}-e^{\alpha t}\sin(\beta t)\mathbf{b}$ and $e^{\alpha t}\sin(\beta t)\mathbf{a}+e^{\alpha t}\cos(\beta t)\mathbf{b}$

5. (25 points) Find the general solution to $\vec{x}'=\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\vec{x}+\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ using the method of variation of parameters

Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Use the table given below to answer the following problems

$\mathcal{L}\{y''\}=s^2\mathcal{L}\{y\}-sy(0)-y'(0)$ $\quad$ $\mathcal{L}\{\cos bt\}=\dfrac{s}{s^2+b^2}$ $\quad$ $\mathcal{L}\{e^{at}\}=\dfrac{1}{s-a}$ $\quad$ $\mathcal{L}\{1\}=\dfrac{1}{s}$

$\mathcal{L}\{y'\}=s\mathcal{L}\{y\}-y(0)$ $\quad$ $\mathcal{L}\{\sin bt\}=\dfrac{b}{s^2+b^2}$ $\quad$ $\mathcal{L}\{t^n\}=\dfrac{n!}{s^{n+1}}$ $\quad$ $\mathcal{L}\{t^ne^{at}\}=\dfrac{n!}{(s-a)^{n+1}}$

$\mathcal{L}\{u(t-a)\}=\dfrac{e^{-as}}{s}$ $\quad$ $\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$ $\quad$ $\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)$

6. (18 points) Solve the IVP $y''-y'=u(t-2)$ $\quad y(0)=0 \quad y'(0)=0$ using the method of Laplace Transforms

7. (14 points) Express the given function using unit step functions and compute its Laplace transform.

$$g(t)=\begin{cases} t & \text{if } t<1 \\ 2 & \text{if } 1\le t<8 \\ 0 & \text{if } 8\le t \end{cases}$$

MA 341 T3 V1 Solutions

1. (10pts)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 + 2R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$$

2. (10pts) $\vec{X_p} = \vec{a}t + \vec{b}$ $\quad \vec{X_p}' = \vec{a}$

$$\vec{a} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\vec{a}t + \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\vec{b} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}t$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_2 \\ -2a_1 \end{bmatrix}t} + \begin{bmatrix} b_2 \\ -2b_1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}t}$$

$$\begin{bmatrix} a_2 \\ -2a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{l} a_2 = 0 \\ a_1 = 1 \end{array} \quad \vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_2 \\ -2b_1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad b_1 = 0 \quad b_2 = -2 \quad \vec{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\vec{X_p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}t + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} t \\ -2 \end{bmatrix}$$

3. (8pts)

$$(A+2I)\vec{u}=\vec{0}$$

$$\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2u_1-2u_2+u_3=0$$

$$u_3=-2u_1+2u_2$$

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

4. (15pts) $|A-rI| = (2-r)\left[(3-r)(1-r)+2\right]=0$

$$=(2-r)\left[3-4r+r^2+2\right]=0$$

$$=(2-r)\left[r^2-4r+5\right]=0 \qquad r=\frac{4\pm\sqrt{16-20}}{2}$$

$\boxed{r=2}$ $\qquad \boxed{r=2\pm i}$

$$(A-2I)\vec{u}=\vec{0}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \vec{u_1}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A-(2+i)I)\vec{u}=\vec{0}$$

$$\begin{pmatrix} 0-i & 0 & 0 \\ 0 & 1-i & 1 \\ 0 & -2 & -1-i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{aligned} &u_1=0 \\ &(1-i)u_2+u_3=0 \\ &u_3=(i-1)u_2 \end{aligned}$$

$$\vec{u}=\begin{bmatrix} 0 \\ 1 \\ i-1 \end{bmatrix}=\underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\vec{a}}+\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}} i$$

$$\vec{x} = c_1 e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\left(e^{2t}\cos t\begin{bmatrix}0\\1\\-1\end{bmatrix} - e^{2t}\sin t\begin{bmatrix}0\\0\\1\end{bmatrix}\right) + c_3\left(e^{2t}\sin t\begin{bmatrix}0\\1\\-1\end{bmatrix} + e^{2t}\cos t\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

5. (25pts)

$$|A-rI| = -r(3-r)+2 = 0$$
$$= -3r + r^2 + 2 = (r-2)(r-1) = 0$$

$$\boxed{r=2,\ r=1}$$

$$(A-2I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad u_2 = 2u_1$$

$$\boxed{\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$(A-I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad u_1 = u_2 \qquad \boxed{\vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\vec{X}_c = C_1 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbb{X} = \begin{bmatrix} e^{2t} & e^{t} \\ 2e^{2t} & e^{t} \end{bmatrix} \longrightarrow \mathbb{X}^{-1} = \frac{1}{e^{3t}-2e^{3t}}\begin{bmatrix} e^{t} & -e^{t} \\ -2e^{2t} & e^{2t} \end{bmatrix}$$
$$= \begin{bmatrix} -e^{-2t} & e^{-2t} \\ 2e^{-t} & -e^{-t} \end{bmatrix}$$

$$\vec{X}_p = \mathbb{X}\int \mathbb{X}^{-1}\vec{f}\,dt = \mathbb{X}\int \begin{bmatrix} -e^{-2t} & e^{-2t} \\ 2e^{-t} & -e^{-t} \end{bmatrix}\begin{bmatrix} e^{t} \\ 0 \end{bmatrix} dt$$
$$= \mathbb{X}\int \begin{bmatrix} -e^{-t} \\ 2 \end{bmatrix} dt = \mathbb{X}\begin{bmatrix} e^{-t} \\ 2t \end{bmatrix}$$

$$\vec{X_p} = \begin{pmatrix} e^{2t} & e^{t} \\ 2e^{2t} & e^{t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} + 2te^{t} \\ 2e^{t} + 2te^{t} \end{pmatrix}$$

$$\boxed{\vec{X} = C_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} e^{t} + 2te^{t} \\ 2e^{t} + 2te^{t} \end{bmatrix}}$$

6. (18pts)

$$s^2 \mathcal{L}\{y\} - \cancel{sy(0)} - \cancel{y'(0)} - \left[s\mathcal{L}\{y\} - \cancel{y(0)} \right] = \frac{e^{-2s}}{s}$$

$$\mathcal{L}\{y\} \left(s^2 - s \right) = \frac{e^{-2s}}{s}$$

$$\mathcal{L}\{y\} = \frac{e^{-2s}}{s^2(s-1)} = e^{-2s} \left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} \right]$$

$$A(s^2 - s) + B(s-1) + Cs^2 = 1$$

$$\underline{As^2} - \underline{As} + \underline{Bs} - B + \underline{Cs^2} = 1$$

$A + C = 0 \quad C = 1$

$-A + B = 0 \quad A = -1$

$-B = 1 \quad B = -1$

$$\mathcal{L}\{y\} = e^{-2s} \left[-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1} \right]$$

$$f(t) = -1 - t + e^{t} \rightarrow \boxed{y = u(t-2) \left[-1 + \cancel{-}(t+2) + e^{t-2} \right]}$$

7. (14pts)

$$g(t) = t + (2-t)\,u(t-1) - 2\,u(t-8)$$

$$\mathcal{L}\{g\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{\underbrace{2-(t+1)}_{1-t}\} - \frac{2e^{-8s}}{s}$$

$$\boxed{\mathcal{L}\{g\} = \frac{1}{s^2} + e^{-s}\left[\frac{1}{s} - \frac{1}{s^2}\right] - \frac{2e^{-8s}}{s}}$$