P1

12.6 Surface Area

* Area of a surface S with position vector $\vec{r}(u,v)$ is

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

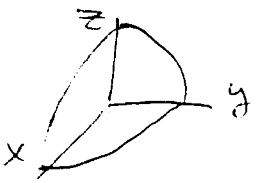
* If S is given by $z = f(x,y)$ then the area of S is $\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$

Examples from Calculus: Early Transcendental Functions By Larson, Hostetler, Edwards

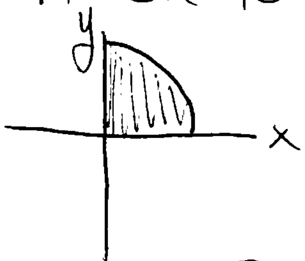
$\boxed{\text{Ex 1}}$ Find the area of the portion of the paraboloid $z = 16 - x^2 - y^2$ in the 1st octant

* Our surface is $z = 16 - x^2 - y^2$ so we can use our shortcut

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+(-2x)^2+(-2y)^2} = \sqrt{1+4x^2+4y^2}$$



z depends on x and y so our domain is given by what is going on in the xy plane. Imagine taking the portion of our surface above the xy plane & also in the 1st octant and smooshing it on to the xy-plane. This would give us



xy-plane $\to$ $z=0$ so $16 = x^2+y^2 = r^2$

Using polar coordinates we get

$$\text{Surface area} = \int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\, r\,dr\,d\theta$$

Note! We could have described $z = 16 - x^2 - y^2$ with polar coordinates from the start.

$z = 16 - r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & -2r \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, -r\sin^2\theta - r\cos^2\theta \rangle$$

$$= \langle -2r^2\cos\theta, -2r^2\sin\theta, -r \rangle$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(-2r^2\cos\theta)^2 + (-2r^2\sin\theta)^2 + (-r)^2}$$

$$= \sqrt{4r^4 + r^2}$$

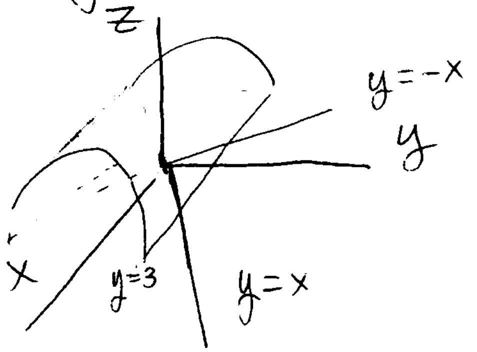
$$= r\sqrt{4r^2+1}$$

$$A(S) = \iint_D |\vec{r}_\theta \times \vec{r}_r| \, dr\,d\theta = \int_0^{2\pi} \int_0^4 \boxed{\sqrt{4r^2+1}\; r} \, dr\,d\theta$$

* We didn't need to put $r\,dr\,d\theta$ since by the definition we just need $|\vec{r}_\theta \times \vec{r}_r| \, dr\,d\theta$. It is when we convert to polar coordinates after the cross product we need $r\,dr\,d\theta$, since $dy\,dx = r\,dr\,d\theta$. Obviously the 1st method was much easier. *

P3

$\boxed{Ex2}$ Find the area of the surface of the cylinder $f(x,y) = 9-y^2$ that lies above the triangle bounded by $y=x$, $y=-x$ & $y=3$

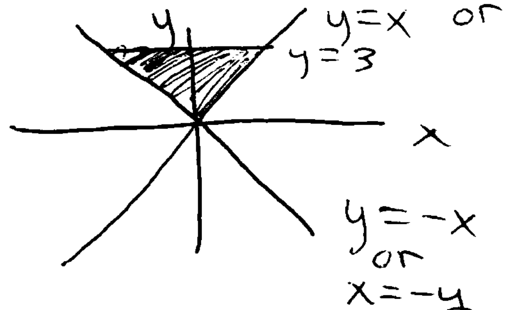


Surface is $z = 9-y^2$
we can use the shortcut

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+(0)^2+(-2y)^2}$$
$$= \sqrt{1+4y^2}$$

Now we need to find

D



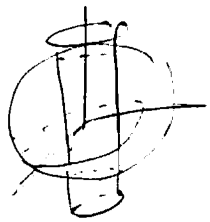
This is most easily described as a type II region

$$\int_0^3 \int_{-y}^{y} \sqrt{1+4y^2}\, dx\, dy = \int_0^3 2y\sqrt{1+4y^2}\, dy$$

$u = 1+4y^2$
$du = 8y\,dy$
$\frac{1}{8}du = y\,dy$
$u(0) = 1+4(0^2) = 1$
$u(3) = 1+4(3^2) = 37$

$$= \frac{1}{8}\int_1^{37} 2u^{1/2}\, du$$
$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^{37}$$
$$= \frac{1}{4}\left[\frac{2}{3}(37^{3/2}-1)\right]$$
$$= \frac{1}{6}\left[37^{3/2}-1\right]$$

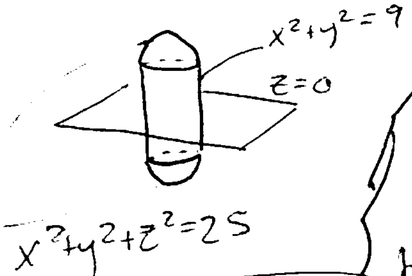
$\boxed{\text{EX 3}}$ Find the area of the portion of the sphere $x^2+y^2+z^2=25$ inside the cylinder $x^2+y^2=9$



surface $\Rightarrow x^2+y^2+z^2=25$

Solve for z & use the shortcut

$z=\sqrt{25-x^2-y^2}$

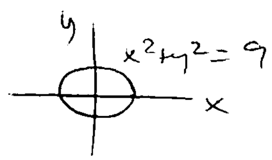


Note! This only gives us the surface above $z=0$. We will need to multiply our result by 2 to get the area of the whole region

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+\left(\frac{-x}{\sqrt{25-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{25-x^2-y^2}}\right)^2}$$

$$= \sqrt{\frac{25-x^2-y^2}{25-x^2-y^2}+\frac{x^2}{25-x^2-y^2}+\frac{y^2}{25-x^2-y^2}}$$

$$= \frac{5}{\sqrt{25-x^2-y^2}}$$

D is 

again polar coordinates will make this problem easy (relatively)

$$S(A)=2\int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25-r^2}}\, r\,dr\,d\theta$$

$u=25-r^2$ $\quad u(0)=25$

$du=-2r\,dr$ $\quad u(3)=16$

$-\frac{1}{2}du=r\,dr$

$$S(A) = \frac{2}{} \int_0^{2\pi} \int_{25}^{16} \frac{-1}{2} \frac{5}{\sqrt{u}}\, du\, d\theta$$

$$= \int_0^{2\pi} \int_{16}^{25} 5u^{-1/2}\, du\, d\theta$$

$$= 2\pi \left[10\sqrt{u} \right]_{16}^{25} = 20\pi[5-4] = \boxed{20\pi}$$

$\boxed{\text{EX4}}$ From Stewart's Calculus p900

* 38. Set up an integral for the area of the parametric surface given by the vector function

$\vec{r}(u,v) = v^2\hat{i} - uv\hat{j} + u^2\hat{k}$ $\quad 0 \le u \le 3$

$-3 \le v \le 3$

* Sadly our short cut won't work here

$\vec{r}_u = \langle 0, -v, 2u \rangle$ $\quad \vec{r}_v = \langle 2v, -u, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -v & 2u \\ 2v & -u & 0 \end{vmatrix} = \langle 2u^2, -4uv, 2v^2 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4 + 16u^2v^2 + 4v^4}$$

If this were an 8 we could factor this - alas.

$$S(A) = \int_0^3 \int_{-3}^3 \sqrt{4u^4 + 16u^2v^2 + 4v^4}\, dv\, du$$