12.6 Surface Area

* Area of a surface $S$ with position vector $\mathbf{r}(u,v)$ is

$$\iint_S \left| \mathbf{r}_u \times \mathbf{r}_v \right| \, du \, dv$$

* If $S$ is given by $z = f(x,y)$ then the area of $S$ is

$$\iint_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA$$

Examples from Calculus: Early Transcendental Functions By Larson, Hostetler, Edwards

**Ex 1** Find the area of the portion of the paraboloid $z = 16 - x^2 - y^2$ in the 1st octant.

* Our surface is $z = 16 - x^2 - y^2$ so we can use our shortcut.

$$\left| \mathbf{r}_x \times \mathbf{r}_y \right| = \sqrt{1 + (-2x)^2 + (-2y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$z$ depends on $x$ and $y$, so our domain is given by what is going on in the $xy$ plane. Imagine taking the portion of our surface above the $xy$ plane and also in the 1st octant and smooshing it onto the $xy$-plane. This would give us $xy$-plane $\Rightarrow z = 0$ so $16 = x^2 + y^2 = r^2$

* Using polar coordinates we get

$$\text{Surface area} = \int_0^\pi \int_0^4 \sqrt{1 + 4r^2} \, r \, dr \, d\theta$$
Note! We could have described $z = 16 - x^2 - y^2$
with polar coordinates from the start.

$Z = 16 - r^2$

$x = r \cos \theta$

$y = r \sin \theta$

$\vec{r}_0 = \langle -r \sin \theta, r \cos \theta, 0 \rangle$

$\vec{r}_r = \langle \cos \theta, \sin \theta, -2r \rangle$

$\vec{r}_e \times \vec{r}_r = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-r \sin \theta & r \cos \theta & 0 \\
\cos \theta & \sin \theta & -2r
\end{vmatrix} = \langle -2r^2 \cos \theta, -2r^2 \sin \theta, -r^2 \rangle$

$|\vec{r}_e \times \vec{r}_r| = \sqrt{(-2r^2 \cos \theta)^2 + (-2r^2 \sin \theta)^2 + (-r)^2}$

$= \sqrt{4r^4 + r^2}$

$= r \sqrt{4r^2 + 1}$

$A(S) = \iint_{S_D} |\vec{r}_e \times \vec{r}_r| \, dr \, d\theta = \iint_{S_D} \left( 4\pi \int_{S_0} \frac{r}{\sqrt{4r^2 + 1}} \right) dr \, d\theta$

$\times \text{ We didn't need to put } r \, dr \, d\theta$

since by the definition we just need

$|\vec{r}_e \times \vec{r}_r| \, dr \, d\theta$. It is when we convert

to polar coordinates after the

cross product we need $r \, d\theta \, dr$, since

dy \, dx = r \, d\theta \, dr$. Obviously the 1st method

was much easier. $\times$
Ex2] Find the area of the surface of the cylinder \( f(x, y) = 9 - y^2 \) that lies above the triangle bounded by \( y = x, \ y = -x, \ \text{and} \ y = 3 \).

Surface is \( z = 9 - y^2 \). We can use the shortcut

\[
|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + (0)^2 + (-2y)^2} = \sqrt{1 + 4y^2}
\]

Now we need to find \( D \)

\[
\begin{align*}
D &= \{ (x, y) \mid y = x \text{ or } x = y, y = 3 \} \\
\end{align*}
\]

This is most easily described as a type II region

\[
\int_0^3 \int_{-y}^3 \sqrt{1 + 4y^2} \, dx \, dy = \int_0^3 2y \sqrt{1 + 4y^2} \, dy
\]

\[
U = 1 + 4y^2
\]

\[
\frac{1}{8} du = 8y \, dy
\]

\[
= \frac{1}{8} \int_1^{37} 2u^{1/2} \, du
\]

\[
= \frac{1}{4} \left[ \frac{2}{3} u^{3/2} \right]_1^{37}
\]

\[
= \frac{1}{6} \left[ \frac{2}{3} (37^{3/2} - 1) \right]
\]

\[
= \frac{1}{6} \left[ \frac{2}{3} (37^{3/2} - 1) \right]
\]
Ex 3: Find the area of the portion of the sphere $x^2+y^2+z^2=25$ inside the cylinder $x^2+y^2=9$

Surface: $x^2+y^2+z^2=25$

Solve for $z$ & use the shortcut $z = \sqrt{25-x^2-y^2}$

Note: This only gives us the surface above $z=0$. We will need to multiply our result by 2 to get the area of the whole region.

$$\left| \int x \times y \right| = \sqrt{1 + \left( \frac{-x}{\sqrt{25-x^2-y^2}} \right)^2 + \left( \frac{-y}{\sqrt{25-x^2-y^2}} \right)^2}$$

$$= \sqrt{\frac{25-x^2-y^2}{25-x^2-y^2} + \frac{x^2}{25-x^2-y^2} + \frac{y^2}{25-x^2-y^2}}$$

$$= \frac{5}{\sqrt{25-x^2-y^2}}$$

$D$ is again polar coordinate will make this problem easy (relatively)

$$SA = \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25-r^2}} r \, dr \, d\theta$$

$u = 25-r^2$, $u(0) = 25$, $u(3) = 16$

$du = -2r \, dr$

$\frac{1}{2} \, du = r \, dr$
\[ S(A) = 2 \int_S \int_{25}^\infty \frac{5}{\sqrt{u}} \, du \, d\theta \]
\[ = \int_0^{2\pi} \int_{16}^{25} 5u^{-1/2} \, du \, d\theta \]
\[ = 2\pi \left[ 10\sqrt{u} \right]_{16}^{25} = 20\pi \left[ 5 - 4 \right] = 20\pi \]

**EX4** From Stewart's Calculus p900

*38. Set up an integral for the area of the parametric surface given by the vector function

\[ \mathbf{r}(u, v) = v^2 \mathbf{i} - uv \mathbf{j} + u^2 \mathbf{k} \quad 0 \leq u \leq 3 \]
\[-3 \leq v \leq 3 \]

*Sadly our shortcut won't work here*

\[ \mathbf{r}_u = \langle 0, -v, 2u \rangle \]
\[ \mathbf{r}_v = \langle 2v, -u, 0 \rangle \]

\[ \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -v & 2u \\ 2v & -u & 0 \end{vmatrix} = \langle 2u^2, -4uv, 2v^2 \rangle \]

\[ \left| \mathbf{r}_u \times \mathbf{r}_v \right| = \sqrt{4u^4 + 16u^2v^2 + 4v^4} \]

If this were an 8 we could factor this—alas.

\[ S(A) = \int_0^3 \int_{-3}^3 \sqrt{4u^4 + 16u^2v^2 + 4v^4} \, dv \, du \]