2.9 Springs Revisited

The motion of an object with mass \( m \) on the end of a spring is given by
\[
m x'' + b x' + k x = F_{ext}
\]

Where
- \( m \) = mass of object
- \( b \) = damping constant
- \( k \) = spring constant
- \( x(t) \) = position of mass at time \( t \)
- \( F_{ext} \) = external force

If \( b = 0 \) : simple harmonic motion

If \( b^2 - 4mk > 0 \) : overdamping

If \( b^2 - 4mk = 0 \) : critical damping

If \( b^2 - 4mk < 0 \) : underdamping

Example from Advanced Engineering Mathematics by Zill & Cullen

Ex 1 A 24-pound weight attached to the end of a spring stretches it 4 inches. Find the equation of motion
We need to find mass $m$ and external force $F_{ext}$. This will give us $a = \frac{F_{ext}}{m}$.

2nd Law: $F = ma \Rightarrow F = mg$ (due to gravity).

We know $W = 24\text{ lb}$. From Newton's 2nd Law, $F = ma \Rightarrow W = mg$.

New equilibrium position from a point 3 inches above the rest position.
Since we have English units (ft, lbs, etc)

\[ g = 32 \text{ ft/s}^2 \] (I will give you gravity)

So \[ 24 = m \cdot 32 \]

\[ m = \frac{24}{32} = \frac{3}{4} \]

* Now we just need \( k \). Unless it is just given to you, we will find it using Hooke's Law \( F = kx \)

\( F \) is the force stretching the spring & \( x \) is how far it is stretched relative to the natural length.

* Weight of the object is our force.

\[ W = k \left( \frac{4}{12} \text{ ft} \right) \]

4 inches = \( \frac{4}{12} \text{ ft} = \frac{1}{3} \text{ ft} \)

* Remember our units need to agree lbs & ft or \( N \) & meters *

\[ 24 = k \left( \frac{1}{3} \right) \]

\[ k = 72 \]
\[ \frac{3}{4} x'' + 72x = 0 \]

* We need our initial conditions

\[ x(0) = -\frac{1}{4} \text{ ft} \]

Above equilibrium is negative

3 inches = \( \frac{3}{12} = \frac{1}{4} \text{ ft} \)

\[ x'(0) = 0 \]

released from rest means no initial velocity

If we were just asked to formulate the IVP

\[ \frac{3}{4} x'' + 72x = 0 \quad x(0) = -\frac{1}{4} \]

\[ x'(0) = 0 \]

For this first one we'll solve it all the way:

Characteristic equation

\[ \frac{3}{4} r^2 + 72 = 0 \]

\[ r^2 = -\frac{72}{\frac{3}{4}} \]

\[ r^2 = -24 \cdot 4 \]

\[ r = \pm \sqrt{24} \cdot 4 \]

\[ r = \pm 4 \sqrt{6} \]

Case 3 →
\[
\alpha = 0 \quad \beta = 4\sqrt{6}
\]

\[
x(t) = e^{\alpha t} \left[ c_1 \cos(\beta t) + c_2 \sin(\beta t) \right]
\]

\[
x(t) = c_1 \cos(4\sqrt{6} t) + c_2 \sin(4\sqrt{6} t)
\]

\[
x(0) = -\frac{1}{4} = c_1 \cos 0 + c_2 \sin 0
\]

\[
c_1 = -\frac{1}{4}
\]

\[
x(t) = -\frac{1}{4} \cos(4\sqrt{6} t) + c_2 \sin(4\sqrt{6} t)
\]

\[
x'(t) = -\frac{4}{\sqrt{6}} \sin(4\sqrt{6} t) \frac{1}{\sqrt{6}} + 4\sqrt{6} c_2 \cos(4\sqrt{6} t)
\]

\[
x'(0) = \sqrt{6} \sin 0 + 4\sqrt{6} c_2 \cos 0 = 0
\]

\[
c_2 = 0
\]

\[
x(t) = -\frac{1}{4} \cos(4\sqrt{6} t)
\]

Simple harmonic motion, \( b = 0 \)

Examples from Differential Equations by Brannan & Boyce

Ex 2 A mass of 0.1 kg stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if
there is no damping, determine the position \( x \) of the mass at time \( t \).

Again \( b = 0 \), \( F_{ext} = 0 \)

\[ m x'' + k x = 0 \]

We know \( m = 0.1 \text{ kg} \), we just need to find \( k \) using \( F = k x \).

The force that stretches the spring is the weight of the object.

\[ W = k x \]
\[ mg = k x \]
\[ 0.1 (9.8 \text{ m/s}^2) = k (0.05 \text{ m}) \]

\[ 0.98 = k \times 0.05 \]
\[ k = \frac{0.98}{0.05} \]

\[ x'' + \frac{9.8}{0.05} x = 0 \]
\[ x(0) = 0 \]
\[ x'(0) = 0.1 \text{ m/s} \]

1VP describing motion of mass: equilibrium

If we use the metric system, gravity if we are using the metric system.

\[ 10 \text{ cm/s} = 0.1 \text{ m/s} \]
\[ 0.1 \beta^2 + \frac{98}{3} = 0 \]
\[ \beta^2 = -\frac{98}{3} \]
\[ \beta = \pm \sqrt{-\frac{98}{3}} = \pm \frac{14}{3} i \quad \alpha = 0 \]

\[ x(t) = e^{\alpha t} \left[ C_1 \cos(14t) + C_2 \sin(14t) \right] \]

\[ x(t) = C_1 \cos(14t) + C_2 \sin(14t) \]

\[ x(0) = 0 = C_1 \cos 0 + C_2 \sin 0 \]
\[ = C_1 \cdot 1 \]
\[ C_1 = 0 \]

\[ x(t) = C_2 \sin(14t) \]

\[ x'(t) = 14C_2 \cos(14t) \]

\[ x'(0) = 14C_2 = 0.1 \]
\[ C_2 = 0.1 / 14 = 1 / 140 \]

\[ x(t) = \frac{1}{140} \sin(14t) \]
Ex 3 A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2. If the mass is set in motion from its equilibrium position with a downward velocity of 3 inches/sec find its position $x(t)$ at time $t$. Determine when the mass first returns to its equilibrium position.

$$mx''+bx'+kx = Fx +$$

$W = 16 \text{ lb} = mg \quad \text{they gave us weight we need mass}$

$16 = M(32)$

$$M = \frac{16}{32} = \frac{1}{2}$$

$b = 2 \quad \text{they gave this to us}$

*still need $k$  

$F = kx$  

$W = 16 = k \left( \frac{3}{4} \right) \quad \text{3 inches} = \frac{1}{4} \text{ft}$

$k = 64$
1VP describing motion of mass

\[ \frac{1}{2}x'' + 2x' + 64x = 0 \]
\[ x(0) = 0 \]
\[ x'(0) = \frac{1}{4} \text{ ft/s} \]

Remember downwards positive
upwards negative
likewise compression negative
stretch = positive

\[ \frac{1}{2} \tau^2 + 2 \tau + 64 = 0 \]

\[ \tau^2 + 4\tau + 128 = 0 \]
\[ \tau = \frac{-4 \pm \sqrt{16 - 4(128)}}{2} \]
\[ = \frac{-4 \pm \sqrt{16 - 4(16 \cdot 32)}}{2} \]
\[ = \frac{-4 \pm \sqrt{16 - 16 \cdot 32}}{2} \]
\[ = \frac{-4 \pm \sqrt{16 - 512}}{2} \]
\[ = \frac{-4 \pm \sqrt{16(1 - 32)}}{2} \]
\[ = \frac{-4 \pm 4\sqrt{-31}}{2} \]
\[ = -2 \pm 2\sqrt{31} \text{ i} \]

\[ b^2 - 4mk < 0 \text{ underdamping lots of oscillations} \]
\[x(t) = e^{-2t}\left[C_1 \cos(2\sqrt{3}t) + C_2 \sin(2\sqrt{3}t)\right]\]

\[x(0) = 0 \rightarrow C_1 = 0\]

\[x(t) = e^{-2t}C_2 \sin(2\sqrt{3}t)\]

\[x'(t) = -2e^{-2t}C_2 \sin(2\sqrt{3}t) + e^{-2t}C_2 2\sqrt{3} \cos(2\sqrt{3}t)\]

\[x'(0) = -2 \cdot \frac{1}{C_2} C_2 \cdot 0 + C_2 2\sqrt{3} = \frac{1}{4}\]

\[C_2 = \frac{1}{8\sqrt{3}}\]

\[x(t) = \frac{1}{8\sqrt{3}} e^{-2t} \sin(2\sqrt{3}t)\]

When will the mass return to equilibrium?

\[x(t) = 0 = \frac{1}{8\sqrt{3}} e^{-2t} \sin(2\sqrt{3}t)\]

All we need is to know when \(\sin(2\sqrt{3}t) = 0\) since the rest is never zero.

Obviously \(t=0\) works, this makes sense since we are starting from equilibrium.
We just want the next value of $t$ that will work:

\[ \sin 0 = 0, \sin \pi = 0, \sin 2\pi, \text{ etc.} \]

[\text{next value to give us zero}]

When does $2\sqrt{3}t = \pi$?

\[ \frac{\pi}{2\sqrt{3}} \]

or $t \approx 1.59$ seconds

\[ \text{time to return to equilibrium.} \]