MA 241 Practice on Series (page 1)

1. a) $3+\frac{3}{4}+\cdots+\frac{3}{4^{n-1}}+\cdots=\sum_{n=1}^{\infty}3\left(\frac{1}{4}\right)^{n-1}$ $\quad a=3$, $r=1/4$

<u>converges</u> b/c geometric w/ $|r|=1/4<1$.

Sum $=\frac{a}{1-r}=\frac{3}{1-1/4}=\frac{3}{3/4}=\underline{\underline{4}}$

b) $\sum_{n=1}^{\infty}2^{-n}3^{n-1}=\sum_{n=1}^{\infty}\frac{3^{n-1}}{2\cdot 2^{n-1}}=\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$ $\quad a=1/2$, $r=3/2$

<u>diverges</u> b/c geometric w/ $|r|=3/2>1$.

2. a) $\sum_{n=1}^{\infty}\frac{3n}{5n-1}$ $\quad \lim_{n\to\infty}\frac{3n}{5n-1}=\frac{3}{5}\neq 0$ $\therefore$ diverges (Divergence test)

b) $\sum_{n=1}^{\infty}\frac{1}{n^2+3}$ $\quad \lim_{n\to\infty}\frac{1}{n^2+3}=0$ $\therefore$ Div. Test inconclusive.

use comparison test $\to$ compare w/ $\frac{1}{n^2}$ (converging p-series)

$n^2+3>n^2 \Rightarrow \frac{1}{n^2}>\frac{1}{n^2+3}$

Since $\sum\frac{1}{n^2}$ converges (p-series w/ $p=2$) and $\frac{1}{n^2}>\frac{1}{n^2+3}$, $\sum\frac{1}{n^2+3}$ converges by the comparison test.

3. $\sum_{n=1}^{\infty}\frac{5}{(n+2)(n+3)}$ $\quad \frac{5}{(n+2)(n+3)}=\frac{A}{n+2}+\frac{B}{n+3}$ (partial fractions to decompose telescoping series)

$5=A(n+3)+B(n+2)$

$A+B=0$
$3A+2B=5$ $\to \boxed{A=5,\ B=-5}$

So our series is

$\sum_{n=1}^{\infty}\left(\frac{5}{n+2}-\frac{5}{n+3}\right)\to$ n^{th} partial sum $S_n=\underbrace{\left(\frac{5}{3}-\cancel{\frac{5}{4}}\right)}_{n=1}+\underbrace{\left(\cancel{\frac{5}{4}}-\cancel{\frac{5}{5}}\right)}_{n=2}+\cdots+\left(\cancel{\frac{5}{n+2}}-\frac{5}{n+3}\right)$

$S=\lim_{n\to\infty}\left(\frac{5}{3}-\cancel{\frac{5}{n+3}}\right)=\frac{5}{3}$ $\quad$ converges. sum $=5/3$.

4. a) $\sum_{n=1}^{\infty} \frac{1}{n^6}$ $p=6>1$ $\therefore$ converges

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ $p=\frac{2}{3} \leq 1$ $\therefore$ diverges

5. a) $\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$ $f(x) = \frac{1}{(3+2x)^2}$ is positive, decreasing, and continuous $\left(\begin{matrix}\text{since} \\ x \geq 1\end{matrix}\right)$

[verify decreasing w/ derivative]

$$\lim_{b\to\infty} \int_1^b (3+2x)^{-2}\,dx$$

$$= \lim_{b\to\infty} \left. \frac{-(3+2x)^{-1}}{2} \right|_1^b = \lim_{b\to\infty} \left[\cancelto{0}{\frac{-1}{2(3+2b)}} + \frac{1}{2(3+2)} \right] = \frac{1}{10}$$ nonzero & finite

the integral converges $\therefore$ the series converges.

b) $\sum_{n=1}^{\infty} \frac{1}{4n+7}$ $f(x) = \frac{1}{4x+7}$ is positive, decreasing, continuous (because $x \geq 1$)

[again - verify these]

$$\lim_{b\to\infty} \int_1^b \frac{1}{4x+7}\,dx = \lim_{b\to\infty} \left. \frac{\ln|4x+7|}{4} \right|_1^b = \lim_{b\to\infty} \left[\frac{\ln|4b+7|}{4} - \frac{\ln|4+7|}{4} \right] = \infty$$

the integral diverges $\therefore$ the series diverges.

6. a) $\sum_{n=1}^{\infty} \frac{1}{n^4+n^2+1}$ compare w/ $\sum_{n=1}^{\infty} \frac{1}{n^4}$ (converging p-series)

$n^4+n^2+1 > n^4$ (since $n \geq 1$)

$\Rightarrow \frac{1}{n^4} > \frac{1}{n^4+n^2+1}$ $\therefore$ converges by comparison test

b) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ compare w/ $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$ (converging geometric series)

$n3^n \geq 3^n$ for $n \geq 1$

$\Rightarrow \frac{1}{3^n} \geq \frac{1}{n3^n}$ $\therefore$ converges by comparison test

7. a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$ appears to behave like $\frac{1}{\sqrt{n}}$ for large n (diverging p-series)

try limit comparison

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+4}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+4} = 1 > 0 \text{ and finite}$$

$\therefore$ ~~converges~~ $\therefore$ diverges by limit comparison b/c $\frac{1}{\sqrt{n}}$ diverges.

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^3-5n}}$ appears to behave like $\frac{1}{2n^{3/2}}$ for large n (converging p-series)

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{4n^3-5n}}}{\frac{1}{2n^{3/2}}} = \lim_{n\to\infty} \frac{2n^{3/2}/n^{3/2}}{\sqrt{(4n^3-5n)/n^3}} = \lim_{n\to\infty} \frac{2}{\sqrt{4-5/n^2}} = 1$$

$\therefore$ converges by lim. comp b/c $\frac{1}{n^{3/2}}$ converges.

8. a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ $\lim_{n\to\infty} \frac{1}{2n-1} = 0$

decreasing: $2(n+1)-1 > 2n-1 \rightarrow \frac{1}{2n-1} > \frac{1}{2(n+1)-1}$

✓ converges by alternating series test.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$ $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

instead, look at $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1} \rightarrow$ limit dne $\therefore$ diverges by Div. Test.

9. a) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\,\cancel{3^n}}{\cancel{3^n}\cdot 3\,\cancel{n!}} \right| = \infty$

diverges by Ratio Test

b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ $\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| = \lim_{n\to\infty} \left| \frac{2^n \cdot 2 \cdot n^2}{(n+1)^2\, 2^n} \right| = 2\left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 = 2 > 1$

$\therefore$ diverges by Ratio Test

(page 2)

1. $\sum_{n=1}^{\infty} \frac{1}{10^n} = \sum_{n=1}^{\infty} \frac{1}{10}\left(\frac{1}{10}\right)^{n-1}$ $\quad a=\frac{1}{10},\ r=\frac{1}{10}$

geometric, $r=\frac{1}{10}<1$ $\therefore$ converges

2. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ compare w/ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (converging geometric series)

$\sin^2 n \le 1$ for all n

so $\frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$ $\therefore$ converges by comparison test.

3. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ limit compare w/ $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$\lim_{n\to\infty} \frac{n^3/2^n}{1/2^n} = \infty$ inconclusive

Ratio Test: $\lim_{n\to\infty} \left| \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2^n}}{2\cdot \cancel{2^n}} \left(\frac{n+1}{n}\right)^3 \right| = \frac{1}{2} < 1$

converges by Ratio test.

4. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\quad f(x) = \frac{\ln x}{x}$ $\quad$ positive, decreasing, continuous when $x>e$ so $x>3$.

$f'(x) = \frac{x\cdot\frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$ $\quad$ $1-\ln x<0$, $1<\ln x$, $e<x$

$\lim_{b\to\infty} \int_3^b \frac{\ln x}{x}\,dx$ $\quad$ let $u=\ln x$, $du = \frac{1}{x}dx$

$= \lim_{b\to\infty} \frac{(\ln x)^2}{2}\Big|_3^b = \lim_{b\to\infty} \left(\frac{(\ln b)^2}{2} - \frac{(\ln 3)^2}{2} \right) = \infty$ diverges by integral Test

5. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$ converges

geometric series, $r = \frac{2}{3} < 1$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}+1}{n+1} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+1}}_{1} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n+1}}_{2}$

look at them separately.

2 limit compare to harmonic series

$\displaystyle\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1 \therefore$ 2 diverges $\therefore$ the sum diverges

(doesn't matter what 1 does)

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^3+1}$ alternating series

$\displaystyle\lim_{n\to\infty} \frac{n}{n^3+1} = \lim_{n\to\infty} \frac{1/n^2}{1+1/n^3} = 0$

decreasing: $f(x) = \dfrac{x}{x^3+1}$ $\quad f'(x) = \dfrac{(x^3+1) - x(3x^2)}{(x^3+1)^2} = \dfrac{-2x^3+1}{(x^3+1)^2}$

$-2x^3+1 < 0$

$-2x^3 < -1$

$x^3 > 1/2$ ✓

converges by Alt. series test

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

Compare w/ $\dfrac{1}{n+1}$

$\ln(n+1) < n+1$

$\dfrac{1}{n+1} < \dfrac{1}{\ln(n+1)}$

diverges

diverges by comparison test.

(Anyone find a better way to show this? Let me know!)

9. $$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=0}^{\infty} 2\cdot\left(\frac{2}{5}\right)^n = \sum_{n=1}^{\infty} 2\cdot\left(\frac{2}{5}\right)^{n-1}$$

converges, geometric $r = \frac{2}{5} < 1$

10. $3.2\overline{394}$

$S = 3.2394394394\ldots$

$10S = 32.394394394\ldots$

$000S = 32394.394394394\ldots$

$0000S - 10S = 32394.\overline{394}\ldots - 32.\overline{394}$

$9990S = 32362$

$$\boxed{S = \frac{32362}{9990}}$$